# Posterior analysis of the compound truncated Weibull under different loss functions for censored data.

Khawla BOUDJERDA, Assia CHADLI, and Hocine FELLAG.

*Abstract*—In this paper, the Bayesian analysis of right truncated Weibull distribution is considered under type II censored data. Bayes estimators and corresponding risks have been derived using symmetric and asymmetric loss functions. Bayesian estimators of the parameters have not explicit forms, so we cannot solve analytically and that's why we applied the Monte-Carlo methods to find the results, especially the Metropolis-Hastings algorithm. Finally, we use Pitman closeness criterion and integrated mean square error (IMSE) to compare Bayesian and likelihood estimators (MLE).

*Keywords*—Bayes estimators**,** generalized quadratic loss function, Linex loss function, Risk loss function Pitman criterion, Metropolis-Hastings algorithm.

## I. Introduction

TRUNCATED statistical distributions arise when a random variable X follows a known distributional model, except that a portion of the sample space cannot be observed. If values of the random variable falling below a certain lower limit T are not observed at all, the distribution is said to be truncated on the left at T. A truncated distribution is defined as a conditional distribution that results from restricting the domain of the statistical distribution. Hence, truncated distributions are used in cases where occurrences are limited to values which lie above or below a given threshold or within a specified range. If occurrences are limited to values which lie below a given threshold, the lower (left) truncated distribution is obtained. Similarly, if occurrences are limited to values which lie above a given threshold, the upper (right) truncated distribution arises (see, e.g, Dusit and Cohen (1984)). Wingo (1988) proposed point estimation of parameters for a doubly truncated Weibull distribution. Mittaland (1989) investigated the problem of existence of the MLE for the truncated Weibull distribution, where the truncation point is assumed to be known. Martinez (1991) studied the MLE of parameters of the upper truncated Weibull distribution. Shalaby and El-Yousef (1993) presented bayesian estimates of parameters for a doubly truncated Weibull distribution and Shalaby (1993) discussed the Bayesian risk of the estimation. Seki and Yokoyama (1993) developed a simple and robust estimation method for the Weibull and truncated Weibull parameters. Balakrishnan and Mitra (2012) applied the EM algorithm to estimate the parameters of Weibull distribution when the model is truncated at left and data are right censored.

The Weibull distribution is a very popular distribution for modeling lifetime data. Indeed, left truncation and right censoring are often observed in lifetime data. For example, failures during the warranty period may not be counted. Items may also be replaced after certain timen following the replacement policy, so that failures of the item are ignored. Also, the truncated Weibull distribution can be used in several engineering fields (see Zutter et al, 1986, Maltamo et al, 2004, Palahi et al, 2007). Several authors have studied the parameter estimation of the truncated Weibull distribution and its application. Zhang and Xie (2011) studied the characteristics of the upper truncated Weibull distribution and parameter estimation by graphical approach. Kantar and Usta (2015) proposed, for the first time, the use of upper-truncated Weibull distribution, in modeling wind speed data and also in estimating wind power density. Using wind speed data measured in various regions of Turkey, upper-truncated Weibull distribution can be an alternative for use in the assessment of wind energy potential.

In this paper, we study the estimation of the right-truncated Weibull distribution which depends on three parameters. Two approaches are proposed. The first one is the classical maximum likelihood estimation (MLE). The second one is the Bayesian procedure performed under the generalized quadratic (GQ), the entropy and the linex loss functions. Using an exhaustive Monte- Carlo study, we compare the Bayesian estimators with respect the posterior risks (PR).

Then, we select the best estimator given by each loss function. These three bayesian estimators are compared to the maximum likelihood estimator (MLE) using Pitman closeness criterion and the integrated mean square error (IMSE).

The rest of this paper is organized as follows. In section 2, we present the model and the problematic. The section 3 deals with the maximum likelihood estimation. In section 4, we propose the Bayesian estimators under various loss functions and different prior distributions.

A Monte -Carlo study is proposed in section 5.

Khawla Boudjerda, PHD student , laboratory LaPS, Badji Mokhtar universityn ,Annaba, Algeria. E-mail : khawla18000@gmail.com.

Assia Chadli, Professor, Laboratory LaPS, Badji Mokhtar university, Annaba, Algeria. E-mail :chadliassia428@hotmail.com.

Hocine Fellag, Laboratory pf pure and applied mathematics, Mouloud Mammeri, University of Tizi-Ouzou, Algeria, e-mail : hfellag@yahoo.com.

## II. THE MODEL

Consider a random variable distributed according to the right truncated Weibull distribution with the parameters (α > 0, β> 0) and the density given by:

$$g(t;\alpha,\lambda,T)=\frac{(\frac{\lambda}{\alpha})(\frac{t}{\alpha})^{\lambda-1}e^{-(\frac{t}{\alpha})^{\lambda}}}{1-e^{-(\frac{T}{\alpha})^{\lambda}}},0\le t\le T. \quad (1)$$

The corresponding reliability function is defined as follows

$$R(t;\alpha,\lambda,T)=1-G(t;\alpha,\lambda,T) \quad (2)$$
$$=\frac{e^{-(\frac{t}{\alpha})^{\lambda}}-e^{-(\frac{T}{\alpha})^{\lambda}}}{1-e^{-(\frac{T}{\alpha})^{\lambda}}};0\le t\le T$$

the failure rate is

$$h(t;\alpha,\lambda,T)=\frac{g(t;\alpha,\lambda,T)}{R(t:\alpha,\lambda,T)} \quad (3)$$
$$=\frac{(\frac{\lambda}{\alpha})(\frac{t}{\alpha})^{\lambda-1}e^{-(\frac{t}{\alpha})^{\lambda}}}{e^{-(\frac{t}{\alpha})^{\lambda}}-e^{-(\frac{T}{\alpha})^{\lambda}}};0\le t\le T$$

The k[th] moment of the right truncated Weibull distribution is given by the following formula

$$E(t^k)=\int_0^{\infty}t^k g(t;\alpha,\lambda,T)=\frac{\alpha^k}{1-e^{-(\frac{t}{\alpha})^{\lambda}}}\Gamma(\frac{k}{\lambda}+1),k=1,2,3...$$

## III. LIKELIHOOD ESTIMATION

Consider a n-sample ($t_1$, $t_2$... $t_n$) generated from right-truncated Weibull model and a fixed constant m € {1, 2,. . . n}. The data are assumed to be censored with type II. i.e. The observations ($t_1$, $t_2$, ..., $t_m$) are observed only. The likelihood function is then

$$L(t|\alpha,\lambda,T)=\frac{n!}{(n-m)!}\lambda^m\alpha^{-m\lambda}\prod_{i=1}^{m}t_i^{\lambda-1}e^{-\sum_{i=1}^{m}(\frac{ti}{\alpha})^{\lambda}}(e^{-(\frac{tm}{\alpha})^{\lambda}}-e^{-(\frac{T}{\alpha})^{\lambda}})^{n-m}(1-e^{-(\frac{T}{\alpha})^{\lambda}})^{-n};0\le t\le T$$

We propose the following notations

$$l(t|\alpha,\lambda,t)=\ln L(t|\alpha,\lambda,T);S=\sum_{i=1}^{m}(\frac{t_i}{\alpha})^{\lambda};P=\prod_{i=1}^{m}t_i$$

And

$$\Phi=(e^{-(\frac{ti}{\alpha})^{\lambda}}-e^{-(\frac{T}{\alpha})^{\lambda}}),\Psi=(1-e^{-(\frac{T}{\alpha})^{\lambda}})$$

Then, we have

$$L(t|\alpha,\lambda,T)=\frac{n!}{(n-m)!}\lambda^m\alpha^{-m\lambda}Pe^{-s}\Phi^{n-m}\Psi^{N},0\le t\le T. \quad (4)$$

And the corresponding logarithm is

$$l(t|\alpha,\lambda,T)=\ln(n!)-\ln((n-m)!)+m\ln\lambda-m\lambda\ln\alpha+(\lambda-1)\sum_{i=1}^{m}\ln t_i-S+(n-m)\ln\Phi-n\ln\Psi \quad (5)$$

The MLE estimators of the three parameters α, λ and T are solutions of the vectorial equation

$$l(t|\alpha,\lambda,T)=0$$

Then, the likelihood equations are as follows

$$\frac{\partial l}{\partial\alpha}=-\frac{m\lambda}{\alpha}-\frac{\partial S}{\partial\alpha}+(n-m)\frac{\Phi_1}{\Phi}-n\frac{\Psi_1}{\Psi}=0$$
$$\frac{\partial l}{\partial\lambda}=\frac{m}{\lambda}+\sum_{i=1}^{m}\ln t_i-m\ln\alpha-\frac{\partial S}{\partial\lambda}+(n-m)\frac{\Phi_2}{\Phi}-n\frac{\Psi_2}{\Psi}=0$$
$$\frac{\partial l}{\partial T}=(n-m)\frac{\Phi_3}{\Phi}-n\frac{\Psi_3}{\Psi}=0$$

After algebraic transformations, we obtain the following non linear system

$$\frac{-m\lambda}{\alpha}+\frac{\lambda}{\alpha}S+(n-m)\frac{(\frac{t_m}{\alpha})^{\lambda}e^{-(\frac{t_m}{\alpha})^{\lambda}}-(\frac{T}{\alpha})^{\lambda}e^{-(\frac{T}{\alpha})^{\alpha}}}{\Phi}+n\frac{\lambda}{\alpha}\frac{(\frac{T}{\alpha})^{\lambda}e^{-(\frac{T}{\alpha})^{\lambda}}}{\Psi}=0$$
$$\frac{m}{\alpha}+\sum_{i=1}^{m}\ln(\frac{t_i}{\alpha})-\sum_{i=1}^{m}\ln(\frac{t_i}{\alpha})(\frac{t_i}{\alpha})^{\lambda}+(n-m)\frac{\ln(\frac{T}{\alpha})(\frac{T}{\alpha})^{\lambda}e^{-(\frac{T}{\alpha})^{\lambda}}-\ln(\frac{t_m}{\alpha})(\frac{t_u}{\alpha})^{\lambda}e^{-(\frac{t_m}{\alpha})^{\lambda}}}{\Phi}-n\frac{\ln(\frac{T}{\alpha})(\frac{T}{\alpha})^{\lambda}e^{-(\frac{T}{\alpha})^{\lambda}}}{\Psi}=0.$$
$$(n-m)\frac{\lambda}{T}\frac{(\frac{T}{\alpha})^{\lambda}e^{-(\frac{T}{\alpha})^{\lambda}}}{\Phi}-n\frac{\lambda}{T}\frac{(\frac{T}{\alpha})^{\lambda}e^{-(\frac{T}{\alpha})^{\lambda}}}{\Psi}=0:$$

There is no analytical solution of this system. Then, we need numerical methods to obtain approximate values of the maximum likelihood estimators' $\alpha_{MLE}$, $\lambda_{MLE}$, and $T_{MLE}$ of the parameters α, λ, and T respectively. In this paper, we will use the R package BB which has high capabilities for solving a nonlinear system of equations (Varadhan and Gilbert, 2009).

## III BAYESIAN ESTIMATION UNDER DIFFERENT LOSS FUNCTIONS

*Prior and posterior distributions*

In this section, we consider the prior distribution of (α, λ) (Shalaby and El-Youssef, 1993) defined as follows

$$\pi(\alpha,\lambda) \propto \lambda^{-a}\alpha^{-b}e^{-\frac{c}{\alpha}}, \alpha,\lambda \succ 0; a \succ 1; b,c \succ 0$$

Moreover, we choose the improper prior of T, which not depend on (α, λ) and given by π(T)=1/T. Then, the prior of (α, λ, T) Is

$$\pi(\alpha,\lambda,T) = \pi(\alpha,\lambda)\pi(T) = \frac{1}{T}\lambda^{-a}\alpha^{-b}e^{-\frac{c}{\alpha}}.$$

Moreover, we obtain the posterior density

$$\pi(\alpha,\lambda,T|t) = \frac{\frac{1}{T}\lambda^{m-a}\alpha^{-m\lambda-b}P^{\lambda-1}e^{-S-\frac{c}{\alpha}}\Phi^{n-m}\Psi^{-n}}{\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{T}\lambda^{m-a}\alpha^{-m\lambda-b}P^{\lambda-1}e^{-S-\frac{c}{\alpha}}\Phi^{n-m}\Psi^{-n}d\alpha d\lambda dT} \quad (6)$$

$$= K^{-1}\frac{1}{T}\lambda^{m-a}\alpha^{-m\lambda-b}P^{\lambda-1}e^{-S-\frac{c}{\alpha}}\Phi^{n-m}\Psi^{-m}$$

Where

$$K = \int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{T}\lambda^{m-a}\alpha^{-m\lambda-b}P^{\lambda-1}e^{-S-\frac{c}{\alpha}}\Phi^{n-m}\Psi^{-n}d\alpha d\lambda dT.$$

### A. *Loss functions*

We consider the generalized quadratic (GQ), the Linex and the entropy loss functions. The following table presents these loss functions and the expressions of the Bayesian estimators with The corresponding posterior risks (PR).

| Loss function | expression | Bayes estimator | posterior risk |
|---|---|---|---|
| Generalized quadratic | $L(\theta,\delta)=\tau(\theta)(\theta-\delta)^2$ | $\widehat{\delta}_{GQ}=\frac{E_\pi(\tau(\theta)\theta)}{E_\pi(\tau(\theta))}$ | $E_\pi(\tau(\theta)(\theta-\widehat{\delta}_{GQ})^2)$ |
| Entropy | $L(\theta,\delta)=(\frac{\delta}{\theta})^p-pln(\frac{\delta}{\theta})-1$ | $\widehat{\delta}_E=[E_\pi(\theta)^{-p}]^{-1/p}$ | $p[E_\pi(ln(\theta)-ln(\widehat{\delta}_E))]$ |
| Linex | $L(\theta,\delta)=e^{r(\delta-\theta)}-r(\delta-\theta)-1$ | $\widehat{\delta}_L=\frac{-1}{r}ln(E_\pi(e^{-r\theta}))$ | $r(\widehat{\delta}_Q-\widehat{\delta}_L)$ |

TAB. 1: **The loss functions and the corresponding bayesian estimators and posterior risk of the parameters**

Under the generalized quadratic loss function assuming $\tau(\theta)=\theta^{\beta-1}$, the Bayes estimators are given by the formulas

$$\alpha_{CQ} = \frac{\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{T}\alpha^{-m\lambda-b+\beta}\lambda^{m-a}P^{\lambda-1}e^{-S-\frac{c}{\alpha}}\Phi^{n-m}\Psi^{-n}d\alpha d\lambda dT}{\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{T}\alpha^{-m\lambda+b-\beta-1}\lambda^{m-a}P^{\lambda-1}e^{-S-\frac{c}{\alpha}}\Phi^{n-m}\Psi^{-n}d\alpha d\lambda dT}$$

$$\alpha_{CQ} = \frac{\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{T}\alpha^{-m\lambda+b+\beta}\lambda^{m-a}P^{\lambda-1}e^{-S-\frac{c}{\alpha}}\Phi^{n-m}\Psi^{-n}d\alpha d\lambda dT}{\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{T}\alpha^{-m\lambda+b+\beta-1}\lambda^{m-a}P^{\lambda-1}e^{-S-\frac{c}{\alpha}}\Phi^{n-m}\Psi^{-n}d\alpha d\lambda dT}$$

$$T_{CQ} = \frac{\int_0^\infty\int_0^\infty\int_0^\infty T^{\beta-1}\alpha^{-m\lambda-b}\lambda^{m-a}P^{\lambda-1}e^{-S-\frac{c}{\alpha}}\Phi^{n-m}\Psi^{-n}d\alpha d\lambda dT}{\int_0^\infty\int_0^\infty\int_0^\infty T^{\beta-2}\alpha^{-m\lambda-b}\lambda^{m-a}P^{\lambda-1}e^{-S-\frac{c}{\alpha}}\Phi^{n-m}\Psi^{-n}d\alpha d\lambda dT}$$

The corresponding posterior risks are then

$$PR(\alpha_{GQ}) = E_\pi(\alpha^{\beta+1}) - 2\alpha_{GQ}E_\pi(\alpha^\beta) + \alpha_{GQ}^2 E_\pi(\alpha^{\beta-1})$$

$$PR(\lambda_{GQ}) = E_\pi(\lambda^{\beta+1}) - 2\lambda_{GQ}E_\pi(\lambda^\beta) + \lambda_{GQ}^2 E_\pi(\lambda^{\beta-1})$$

$$PR(T_{GQ}) = E_\pi(T^{\beta+1}) - 2T_{GQ}E_\pi(T^\beta) + T_{GQ}^2 E_\pi(T^{\beta-1})$$

Notice that, when β= 1, we have the basic quadratic loss. Under the entropy loss function, we obtain the following estimators

$$\alpha_e = k^{\frac{1}{p}}[\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{T}\alpha^{-m\lambda-b-p}\lambda^{m-a}P^{\lambda-1}e^{-S-\frac{c}{\alpha}}\Phi^{n-m}\Psi^{-n}d\alpha d\lambda dT]^{\frac{-1}{p}}$$

$$\lambda_e = k^{\frac{1}{p}}[\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{T}\alpha^{-m\lambda-b}\lambda^{m-a-p}P^{\lambda-1}e^{-S-\frac{c}{\alpha}}\Phi^{n-m}\Psi^{-n}d\alpha d\lambda dT]^{\frac{-1}{p}}$$

$$T_e = k^{\frac{1}{p}}[\int_0^\infty\int_0^\infty\int_0^\infty T^{-p-1}\alpha^{-m\lambda-b}\lambda^{m-a}P^{\lambda-1}e^{-S-\frac{c}{\alpha}}\Phi^{n-m}\Psi^{-n}d\alpha d\lambda dT]^{\frac{-1}{p}}$$

The posterior risks are

$$PR(\alpha_E) = pE_\pi(\ln(\alpha) - \ln(\alpha_E));$$
$$PR(\lambda_E) = pE_\pi(\ln(\lambda) - \ln(\lambda_E));$$
$$PR(T_E) = pE_\pi(\ln(T) - \ln(T_E))$$

following estimators Under the Linex loss function, we obtain the

$$\alpha_L = -\frac{1}{r}[K^{-1}\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{T}\alpha^{-m\lambda-b}\lambda^{m-a}P^{\lambda-1}e^{-S-\frac{c}{\alpha}-r\alpha}\Phi^{n-m}\Psi^{-n}d\alpha d\lambda dT]$$

$$\lambda_L = -\frac{1}{r}[K^{-1}\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{T}\alpha^{-m\lambda-b}\lambda^{m-a}P^{\lambda-1}e^{-S-\frac{c}{\alpha}-r\lambda}\Phi^{n-m}\Psi^{-n}d\alpha d\lambda dT]$$

$$T_L = -\frac{1}{r}[K^{-1}\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{T}\alpha^{-m\lambda-b}\lambda^{m-a}P^{\lambda-1}e^{-S-\frac{c}{\alpha}-rT}\Phi^{n-m}\Psi^{-n}d\alpha d\lambda dT]$$

The corresponding posterior risks are

$$PR(\alpha_L) = r(\alpha_Q - \alpha_L); PR(\lambda_L) = r(\lambda_Q - \lambda_L); PR(T_L) = r(T_Q - T_L)$$

Since, we cannot calculate the analytical expressions of all these estimators; we will use MCMC procedures as Metropolis-Hastings algorithm in the following Monte Carlo section.

## IV MONTE CARLO STUDY

In this section, we propose to perform a Monte Carlo study assuming that a = 2, b = c = 1, α = 1, λ= 2.0 and T = 1.5. Then, using N = 1000 samples of the right truncated model, we obtain the following results.

### A. Likelihood estimation

Since analytical formulas are not available, to obtain the maximum likelihood estimators, we need to use numerical procedures. In this section we will use the R package BB which presents very high performances for non linear systems. In particular, we need the function BB solves in BB for our problem. This function offers a reliable, low-cost method to solve large-scale nonlinear systems of equations (Varadhan and Gilbert, 2009). The results are as follows:

| n | m | parameter | mle |
|---|---|---|---|
| 10 | 7 | α<br>λ<br>T | **0.6951(0.0929)**<br>**1.9800(0.0003)**<br>**1.9452(0.1660)** |
| 30 | 21 | α<br>λ<br>T | **0.9036(0.0092)**<br>**2,0003(1,07.10$^{-7}$)**<br>**1.6203(0.0144)** |
| 50 | 35 | α<br>λ<br>T | **0.9364(0.0040)**<br>**2.0367(0.0013)**<br>**1.5945(0.0089)** |
| 100 | 70 | α<br>λ<br>T | **0.9943(3,22 10-5)**<br>**2.0368(0.0013)**<br>**1.5149(0.0002)** |
| 200 | 140 | α<br>λ<br>T | **0.9268(0.0053)**<br>**2.0957(0.0091)**<br>**1.6457(0.0212)** |

**Tab. 2**: The MLE of the parameters with quadratic error (in brackets)

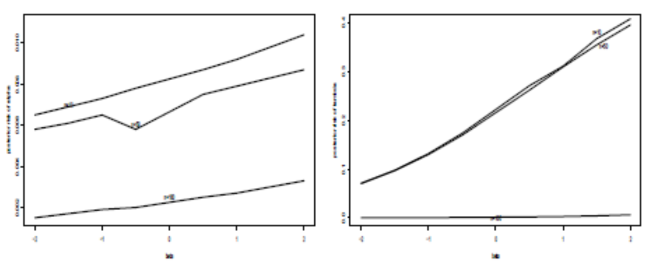
We remark that the estimated values of α and λ are close to the true values of parameters. However, the estimation of T Is not close to the true value.

### B. Bayesian estimation

The bayesian estimators are obtained with performing the MCMC methods. The table 1 presents the bayesian estimations and the corresponding posterior risks, in brackets, under the generalized quadratic loss function. We remark that the value β= −2 gives us the best posterior risk and then improve the basic quadratic case. Also, we obtain the smallest suitable posterior risk when n is high. This is illustrated by the figure 1.

| $n$ | $m$ | parameter | $\beta$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | -2 | -1.5 | -1 | -0.5 | 0.5 | 1 | 1.5 | 2 |
| 10 | 7 | $\alpha$ | 1.0902 (0.0067) | 1.0943 (0.0071) | 1.0981 (0.0076) | 1.1026 (0.0081) | 1.1111 (0.0091) | 1.1155 (0.0096) | 1.1198 (0.0102) | 1.1242 (0.0108) |
| | | $\lambda$ | 1.1873 (0.0720) | 1.2534 (0.0982) | 1.3350 (0.1309) | 1.4307 (0.1696) | 1.6448 (0.2583) | 1.7491 (0.3032) | 1.8430 (0.3453) | 1.9225 (0.3835) |
| | | T | 1.1928 (0.0436) | 1.2286 (0.0520) | 1.2675 (0.0614) | 1.3085 (0.0714) | 1.3921 (0.0924) | 1.4323 (0.1030) | 1.4699 (0.1134) | 1.5043 (0.1234) |
| 30 | 21 | $\alpha$ | 1.0784 (0.0055) | 1.0816 (0.0058) | 1.0849 (0.0061) | 1.0882 (0.0064) | 1.0950 (0.0071) | 1.0984 0.0074 | 1.1018 (0.0078) | 1.1052 (0.0082) |
| | | $\lambda$ | 1.1633 (0.0708) | 1.2244 (0.0974) | 1.3014 (0.1310) | 1.3936 (0.1713) | 1.6072 (0.2651) | 1.7146 (0.3127) | 1.8128 (0.3567) | 1.8970 0.3951) |
| | | T | 1.1737 (0.0429) | 1.2075 (0.0514) | 1.2446 (0.0608) | 1.2842 (0.0708) | 1.3661 (0.0918) | 1.4065 (0.1023) | 1.4444 (0.1124) | 1.4793 (0.1218) |
| 50 | 35 | $\alpha$ | 1.0750 (0.0057) | 1.0783 (0.0060) | 1.0817 (0.0063) | 1.0852 (0.0067) | 1.0922 (0.0074) | 1.0957 (0.0078) | 1.0993 (0.0082) | 1.1029 (0.0085) |
| | | $\lambda$ | 1.1440 (0.0685) | 1.1997 (0.0951) | 1.2713 (0.1239) | 1.3590 (0.1713) | 1.5699 (0.2724) | 1.6802 (0.3256) | 1.7835 (0.3758) | 1.8740 (0.4201) |
| | | T | 1.1565 (0.0425) | 1.1886 (0.0514) | 1.2245 (0.0612) | 1.2634 (0.0720) | 1.3464 (0.0951) | 1.3880 (0.1069) | 1.4278 (0.1182) | 1.4650 (0.1290) |
| 100 | 70 | $\alpha$ | 1.2021 (0.0015) | 1.2032 (0.0017) | 1.2044 (0.0018) | 1.2055 (0.0020) | 1.2078 (0.0025) | 1.2089 (0.0027) | 1.2100 (0.0030) | 1.2112 (0.0033) |
| | | $\lambda$ | 2.2002 (0.0002) | 2.2008 (0.0003) | 2.2015 (0.0005) | 2.2021 (0.0008) | 2.2034 (0.0019) | 2.2041 (0.0028) | 2.2047 (0.0042) | 2.2053 (0.0062) |
| | | T | 1.7016 (0.0005) | 1.7024 (0.0007) | 1.7032 (0.0009) | 1.7040 (0.0012) | 1.7057 (0.0021) | 1.7065 (0.0027) | 1.7073 (0.0036) | 1.7081 (0.0047) |
| 200 | 140 | $\alpha$ | 1.2109 (0.0013) | 1.2118 (0.0014) | 1.2128 (0.0015) | 1.2138 (0.0017) | 1.2157 (0.0021) | 1.2167 (0.0023) | 1.2176 (0.0025) | 1.2186 (0.0028) |
| | | $\lambda$ | 2.1894 (0.0002) | 2.1901 (0.0004) | 2.1908 (0.0006) | 1.1915 (0.0009) | 2.1929 (0.0020) | 2.1936 (0.0030) | 2.1941 (0.0044) | 2.1949 (0.0065) |
| | | T | 1.7005 (0.0005) | 1.7013 (0.0006) | 1.7021 (0.0008) | 1.7028 (0.0011) | 1.7043 (0.0019) | 1.7051 (0.0025) | 1.7059 (0.0033) | 1.7066 (0.0044) |

TAB. 3: **Bayes estimators and PR (in brackets) under generalized quadratic loss function**



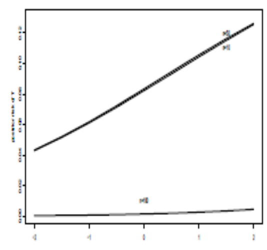


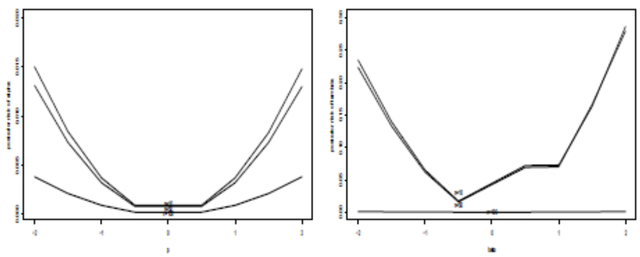
Figure1. **Posterior risks of α, λ and T under generalized quadratic loss function**

With the entropy loss function, we obtain the following table where we can notice that the value p = −0.5 and the cases n = 100 and n = 200 provide the best posterior risk.

| $n$ | $m$ | parameter | $p$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | -2 | -1.5 | -1 | -0.5 | 0.5 | 1 | 1.5 | 2 |
| 10 | 7 | $\alpha$ | 1.1198 (0.0155) | 1.1176 (0.0087) | 1.1155 (0.0038) | 1.1133 (0.0009) | 1.1090 (0.0009) | 1.1069 (0.0038) | 1.1047 (0.0086) | 1.1026 (0.0154) |
| | | $\lambda$ | 1.8338 (0.2151) | 1.7939 (0.1284) | 1.7491 (0.0603) | 1.6998 (0.0158) | 1.5916 (0.0681) | 1.5362 (0.0694) | 1.4825 (0.1576) | 1.4321 (0.2793) |
| | | T | 1.4678 (0.1052) | 1.4507 (0.0612) | 1.4323 (0.0281) | 1.4129 (0.0072) | 1.3717 (0.0075) | 1.3504 (0.0307) | 1.3292 (0.0698) | 1.3083 (0.1248) |
| 30 | 21 | $\alpha$ | 1.1018 (0.0123) | 1.1001 (0.0069) | 1.0984 (0.0031) | 1.0967 (0.0007) | 1.0933 (0.0007) | 1.0916 (0.0030) | 1.0899 (0.0069) | 1.0882 (0.0123) |
| | | $\lambda$ | 1.8035 (0.2284) | 1.7616 (0.1359) | 1.7146 (0.0636) | 1.6633 (0.0166) | 1.5530 (0.0707) | 1.4977 (0.0716) | 1.4449 (0.1612) | 1.3961 (0.2837) |
| | | T | 1.4424 (0.1077) | 1.4250 (0.0625) | 1.4065 (0.0286) | 1.3870 (0.0073) | 1.3461 (0.0076) | 1.3252 (0.0308) | 1.3045 (0.0699) | 1.2843 (0.1245) |
| 50 | 35 | $\alpha$ | 1.0993 (0.0129) | 1.0975 (0.0073) | 1.0957 (0.0032) | 1.0940 (0.0008) | 1.0904 (0.0008) | 1.0887 (0.0032) | 1.0869 (0.0072) | 1.0852 (0.0128) |
| | | $\lambda$ | 1.7744 (0.2434) | 1.7297 (0.1442) | 1.6802 (0.0671) | 1.6268 (0.0174) | 1.5149 (0.0728) | 1.4603 (0.0731) | 1.4091 (0.1632) | 1.3625 (0.2848) |
| | | T | 1.4259 (0.1143) | 1.4075 (0.0661) | 1.3880 (0.0301) | 1.3676 (0.0077) | 1.3256 (0.0079) | 1.3045 (0.0318) | 1.2838 (0.0717) | 1.2639 (0.1269) |
| 100 | 70 | $\alpha$ | 1.2100 (0.0037) | 1.2095 (0.0021) | 1.2089 (0.0009) | 1.2083 (0.0002) | 1.2072 (0.0002) | 1.2066 (0.0009 | 1.2061 (0.0021) | 1.2055 (0.0037) |
| | | $\lambda$ | 2.2047 (0.0011) | 2.2044 (0.0006) | 2.2041 (0.0002) | 2.2037 (0.00007) | 2.2031 (0.0002) | 2.2028 (0.0002) | 2.2024 (0.0006) | 2.2021 (0.0011) |
| | | T | 1.7073 (0.0019) | 1.7069 (0.0010) | 1.7065 (0.0004) | 1.7061 (0.0001) | 1.7053 (0.0001) | 1.7048 (0.0004) | 4.7044 (0.0010) | 1.7040 (0.0019) |
| 200 | 140 | $\alpha$ | 1.2176 (0.0031) | 1.2171 (0.0017) | 1.2167 (0.0007) | 1.2162 (0.0001) | 1.2152 (0.0001) | 1.2147 (0.0007) | 1.2143 (0.0017) | 1.2138 (0.0031) |
| | | $\lambda$ | 2.1942 (0.0012) | 2.1939 (0.0007) | 1.1936 (0.0003) | 1.1932 (0.00007) | 1.1925 (0.0003) | 1.1922 (0.0003) | 1.1918 (0.0007) | 1.1915 (0.0012) |
| | | T | 1.7059 (0.0017) | 1.7055 (0.0010) | 1.7051 (0.0004) | 1.7047 (0.0001) | 1.7040 (0.0001) | 1.7036 (0.0004) | 1.7032 (0.0010) | 1.7028 (0.0017) |

TAB. 4: **Bayes estimators and PR (in brackets) under entropy loss function**

The following figure illustrates well this situation.



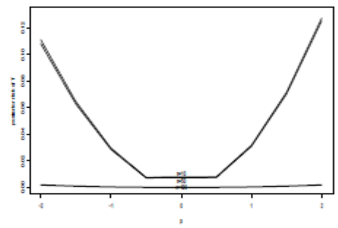


Figure2. **Posterior risks of α, λ and T under Linex loss function**

Under the linex loss function, the results are given in the following table.

| $n$ | $m$ | parameter | $r$ = -2 | -1.5 | -1 | -0.5 | 0.5 | 1 | 1.5 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 7 | $\alpha$ | 1.1252 (0.0195) | 1.1228 (0.0109) | 1.1203 (0.0048) | 1.1179 (0.0012) | 1.1131 (0.0012) | 1.1107 (0.0048) | 1.1083 (0.0107) | 1.1059 (0.0191) |
| | | $\lambda$ | 1.9693 (0.4403) | 1.9301 (0.2714) | 1.8809 (0.1317) | 1.8204 (0.0356) | 1.6703 (0.0394) | 1.5892 (0.1599) | 1.5120 (0.3556) | 1.4431 (0.4303) |
| | | T | 1.5205 (0.1764) | 1.5016 (0.1038) | 1.4805 (0.0481) | 1.4573 (0.0124) | 1.4059 (0.0131) | 1.3788 (0.0535) | 1.3515 (0.1212) | 1.3248 (0.2150) |
| 30 | 21 | $\alpha$ | 1.1059 (0.0150) | 1.1040 (0.0084) | 1.1021 (0.0037) | 1.1002 (0.0009) | 1.0965 (0.0009) | 1.0946 (0.0037) | 1.0928 (0.3576) | 1.0909 (0.6106) |
| | | $\lambda$ | 1.9446 (0.4599) | 1.9035 (0.2833) | 1.8518 (0.1372) | 1.7885 (0.0369) | 1.6339 (0.0403) | 1.5524 (0.1622) | 1.4762 (0.3576) | 1.4093 (0.6106) |
| | | T | 1.4949 (0.1767) | 1.4758 (0.1039) | 1.4546 (0.0480) | 1.4314 (0.0124) | 1.3814 (0.0130) | 1.3537 (0.00527) | 1.3271 (001190) | 1.3013 (0.2104) |
| 50 | 35 | $\alpha$ | 1.1036 (0.0156) | 1.1016 (0.0088) | 1.0997 (0.0039) | 1.0977 (0.0009) | 1.0938 (0.0009) | 1.0918 (0.0038) | 1.0899 (0.0087) | 1.0880 (0.0154) |
| | | $\lambda$ | 1.9271 (0.4939) | 1.8821 (0.3029) | 1.8252 (0.1457) | 1.7580 (0.0389) | 1.5972 (0.0414) | 1.5155 (0.1646) | 1.4411 (0.3586) | 1.3770 (0.6063) |
| | | T | 1.4822 (0.1884) | 1.4615 (0.1103) | 1.4388 (0.0508) | 1.4141 (0.0130) | 1.3609 (0.0135) | 1.3336 (0.0543) | 1.3068 (0.1217) | 1.2811 (0.2136) |
| 100 | 70 | $\alpha$ | 1.2117 (0.0055) | 1.2110 (0.0031) | 1.2103 (0.0013) | 1.2096 (0.0003) | 1.2082 (0.0003) | 1.2075 (0.0014) | 1.2068 (0.0031) | 1.2062 (0.0056) |
| | | $\lambda$ | 2.2069 (0.0056) | 2.2062 (0.0031) | 2.2055 (0.0014) | 2.2048 (0.0003) | 2.2034 (0.0003) | 2.2026 (0.0014) | 2.2019 (0.0031) | 2.2012 (0.0055) |
| | | T | 1.7093 (0.0056) | 1.7086 (0.0031) | 1.7079 (0.0013) | 1.7072 (0.0003) | 1.7058 (0.0003) | 1.7051 (0.0013) | 1.7044 (0.0031) | 1.7037 (0.0055) |
| 200 | 140 | $\alpha$ | 1.2190 (0.0047) | 1.2184 (0.0026) | 1.2178 (0.0011) | 1.2173 (0.0002) | 1.2161 (0.0002) | 1.2155 (0.0011) | 1.2143 (0.0026) | 1.2143 (0.0046) |
| | | $\lambda$ | 2.1965 (0.0059) | 2.1958 (0.0033) | 2.1951 (0.0014) | 2.1643 (0.0003) | 2.1928 (0.0003) | 2.1920 (0.0015) | 2.1913 (0.0034) | 2.1905 (0.0060) |
| | | T | 1.7077 (0.0051) | 1.7071 (0.0029) | 1.7064 (0.0012) | 1.7058 (0.0003) | 1.7015 (0.0003) | 1.7038 (0.0012) | 1.7031 (0.0029) | 1.7025 (0.0051) |

TAB. 5: **Bayes estimators and PR (in brackets) under Linex loss function**

One can notice that the value r = −0.5 provides the best PR (see figure 3).

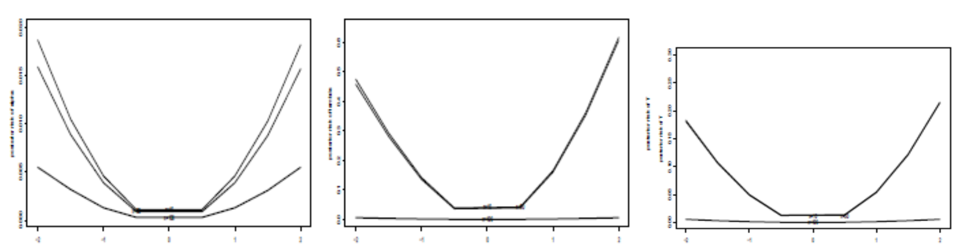


Figure2. **Posterior risks of α, λ and T under the Entropy loss function**

If we compare the three loss functions, we notice that the entropy loss function provides the best Bayesian estimator of α, λ and T. This Is illustrated by the following table

| n | m | paramater | generalized quadratic($\beta = -2$) | Entropy(p=-0.5) | Linex(r=-0.5) |
|---|---|---|---|---|---|
| 10 | 7 | $\alpha$ | 1.0902 (0.0067) | 1.1133 (0.0009) | 1.1179 (0.0012) |
| | | $\lambda$ | 1.1873 (0.0720) | 1.6998 (0.0158) | 1.8204 (0.0356) |
| | | T | 1.1928 (0.0436) | 1.4129 (0.0072) | 1.4573 (0.0124) |
| 30 | 21 | $\alpha$ | 1.0784 (0.0055) | 1.0969 (0.0007) | 1.1002 (0.0369) |
| | | $\lambda$ | 1.1633 (0.0429) | 1.6633 (0.0073) | 1.7885 (0.0124) |
| | | T | 1.1737 (0.0429) | 1.3870 (0.0073) | 1.4314 (0.0124) |
| 50 | 35 | $\alpha$ | 1.0750 (0.0057) | 1.0940 (0.0008) | 1.0977 (0.0009) |
| | | $\lambda$ | 1.1440 (0.0685) | 1.6368 (0.0174) | 1.7580 (0.0389) |
| | | T | 1.1565 (0.0425) | 1.3676 (0.0077) | 1.4141 (0.0130) |
| 100 | 70 | $\alpha$ | 1.2021 (0.0015) | 1.2083 (0.0002) | 1.2048 (0.0003) |
| | | $\lambda$ | 2.2002 (0.0002) | 2.2037 (0.00007) | 2.2048 (0.0003) |
| | | T | 1.7016 (0.0001) | 1.7061 (0.0001) | 1.7072 (0.0003) |
| 200 | 140 | $\alpha$ | 1.2109 (0.0013) | 1.2162 (0.0001) | 1.2173 (0.0002) |
| | | $\lambda$ | 2.1894 (0.0002) | 2.1932 (0.00007) | 2.1643 (0.0003) |
| | | T | 1.7005 (0.0005) | 1.7047 (0.0001) | 1.7058 (0.0003) |

TAB. 6: **Bayes estimators and PR (in brackets) under the tree loss functions**

### *A. Comparison with the likelihood estimators*

In this section, we propose to compare the best Bayesian estimators obtained above with the maximum likelihood estimator. For this, we propose to use the following criterions: the Pitman closeness (Pitman, 1937, Fuller, 1982 and Jozani, 2012) and the integrated mean square error (IMSE) defined as follows:

*Definition 5.1* An estimator $\theta_1$ of a parameter $\theta$ dominates in the sense of Pitman closeness criterion another estimator $\theta_2$, if for all $\theta \in \Theta$

$$P_\theta\left[|\theta_1 ... \theta| \prec |\theta_2 - \theta|\right] \succ 0.5$$

Consider the estimates $\theta_i$ (i=1… N) Obtained with N samples of the model.

*Definition 5.2 the* integrated mean square error is defined as

$$IMSE = \frac{\sum_{i=1}^{N}(\theta_i - \theta)^2}{N}$$

In the following, we present the values of the Pitman probabilities which allows us to compare the bayesian estimators with the MLE under the tree loss functions where $\beta = -2$, $p = -0.5$ and $r = -0.5$. The Table 5 should be read as follows : when the probability Is greater than 0.5, the bayesian estimator Is better than the MLE estimator. Then, we notice that, according to this criterion

- When n is not high, the Bayesian estimators $\alpha_B$ and $T_B$ of $\alpha$ and T are better than the MLE's $\alpha_{MLE}$ and $T_{MLE}$. The generalized quadratic loss function provides the best values. However, $\lambda_{MLE}$ is closer to the true value than all the bayesian estimators.
- When n is high, the MLE of the three parameters perform better than the Bayesian estimators.

| n | m | par | GQ($\beta = -2$) | Entropie(p=-0.5) | Linex(r=-0.5) |
|---|---|---|---|---|---|
| 10 | 7 | $\alpha$ | 0.699 | 0.638 | 0.654 |
| | | $\lambda$ | 0.249 | 0.130 | 0.159 |
| | | T | 0.654 | 0.585 | 0.609 |
| 30 | 21 | $\alpha$ | 0.575 | 0.556 | 0.559 |
| | | $\lambda$ | 0.273 | 0.109 | 0.151 |
| | | T | 0.532 | 0.489 | 0.501 |
| 50 | 35 | $\alpha$ | 0.535 | 0.514 | 0.518 |
| | | $\lambda$ | 0.231 | 0.087 | 0.117 |
| | | T | 0.423 | 0.393 | 0.396 |
| 100 | 70 | $\alpha$ | 0.152 | 0.146 | 0.146 |
| | | $\lambda$ | 0.130 | 0.136 | 0.161 |
| | | T | 0.413 | 0.376 | 0.383 |
| 200 | 140 | $\alpha$ | 0.097 | 0.095 | 0.095 |
| | | $\lambda$ | 0.221 | 0.077 | 0.086 |
| | | T | 0.278 | 0.250 | 0.257 |

TAB. 7: **Pitman comparaison of the estimators of $\alpha$,$\lambda$ and $T$**

The Table 6 presents the values of integrated mean square error (IMSE) of the estimators Bayesian of the parameters under the tree loss function, and the maximum likelihood estimators.

According to this criterion, when n Is small, the Bayesian estimators $\alpha_B$ and $T_B$ provide the smallest IMSE for $\alpha$ and T comparatively to $\alpha_{MLE}$ and $T_{MLE}$. Also, the values provided by the generalized quadratic loss function are relatively equivalent to the entropy and linex. But, the IMSE of $\lambda_{MLE}$ is smaller than the IMSE of the Bayesian estimators. If n Is high, then, all the Bayesian estimators perform better than the MLE estimators and the generalized quadratic loss function provides the best values of the IMSE.

| n | m | parameter | mle | QG | Entropie | Linex |
|---|---|---|---|---|---|---|
| 10 | 7 | $\alpha$ | 0.2695 | 0.0080 | 0.0221 | 0.0190 |
| | | $\lambda$ | 0.1184 | 0.4333 | 0.3666 | 0.3696 |
| | | T | 0.3278 | 0.1034 | 0.1070 | 0.7038 |
| 30 | 21 | $\alpha$ | 0.1031 | 0.0098 | 0.0168 | 0.0154 |
| | | $\lambda$ | 0.0686 | 0.3922 | 0.3938 | 0.3905 |
| | | T | 0.2579 | 0.1036 | 0.1107 | 0.1087 |
| 50 | 35 | $\alpha$ | 0.0702 | 0.0114 | 0.0167 | 0.157 |
| | | $\lambda$ | 0.0551 | 0.4181 | 0.4275 | 0.4239 |
| | | T | 0.2437 | 0.1125 | 0.1191 | 0.1175 |
| 100 | 70 | $\alpha$ | 0.0872 | 0.0380 | 0.0460 | 0.0445 |
| | | $\lambda$ | 0.0529 | 0.0191 | 0.0438 | 0.0362 |
| | | T | 0.2840 | 0.0321 | 0.0449 | 0.0421 |
| 200 | 140 | $\alpha$ | 0.1908 | 0.1624 | 0.1672 | 0.1663 |
| | | $\lambda$ | 0.5467 | 0.5287 | 0.5493 | 0.5432 |
| | | T | 0.4838 | 0.3167 | 0.3263 | 0.3242 |

TAB. 8: **The IMSE of the estimators of $\alpha$,$\lambda$ and $T$**

## IV. CONCLUSION

In this paper, we compared bayesian estimators of the right-truncated Weibull distribution under different loss functions. In the Bayesian estimation, for each loss function, we obtained the suitable parameter which optimize the bayesian estimation. Then, our Monte Carlo study showed that the entropy loss function provides the smallest posterior risks. These selected bayesian estimators are compared to the maximum likelihood estimators of the parameters using the Pitman closeness criterion and the integrated mean square error. Then, using our exhaustive Monte Carlo procedure, we showed that when n Is small, the Bayesian estimators are better for α and T and not for λ. If n is enough high, the MLE's are closer to the true values but provide the highest IMSE than the bayesian estimators. To improve this paper, one can study the robustness aspects of these estimators and complete this comparison.

## REFERENCES


[1] E.J.G. Pitman, "The closest estimates of statistical parameters," Proc. Cambridge Phil. Soc, vo.l33,1937,pp.212-222.

[2] W.A Fuller, "Closeness of estimators," Encyclopedia of statistical, vol 2,19821.

[3] C. Dusit and A.C Cohen, "Estimation in the singly truncated Weibull distribution with an unknown truncation point," Co,,unications in statistics-Theory and methods; vol.13, 1984, Issue.7.

[4] R.B. Zutter and all, "Characterizindg diameter distributions with modified datatypes and forms of the Weibull distributions," Forest science, vol.32, 1986, pp.37-48.

[5] D.R Wingo, "Parametric point estimator for a doubly truncated Weibull distribution," Microelectron. Reliab, vol. 28, 1988,pp 613-617.

[6] D.R Wingo, "The left-truncated Weibull distribution," Theory and computations. Statistical papers, vol30, 1989, pp39-48.

[7] M.M Mittal and R.C Dahiya,"Estimating of truncated Weibull distribution," Co,,un Stat-Theory and Method, vol.18, 1989, 2027-2042.

[8] S. Martinez,"On a test for generalized upper truncated Weibull distributions," Stat. proab. Lett, vol.12, 1991, pp 27-39.

[9] T. Seki and S. Yokoyama, "Simple and robust estimation of the Weibull distribution ," Microelectron. Reliab, 1993, vol.33, pp 45-52.

[10] O.A Shalaby, "The Bayes risk for the doubly truncated Weibull distribution," Microelectron. Reliab, vol 33, 1993, pp 2189-2192.

[11] O.A Shalaby and El youcef,"Bayesian analysis of the parameters of doubly truncated Weibull distribution," Microelectron .Reliab, vol.33, 1993, pp 1199-1211.

[12] M. Maltamo and all, "Estimation of timber and stem density based scanning laser altimetry and expected tree size distribution functions," RemSens Environ, vol. 90, 2004,pp 319-330.

[13] M. Palahi and all, "Comaparaison of beta, Johnson's Weibull and truncated Weibull functions for modeling the diameter distribution of forest stands in Catalonia (north –east of spain)," Eur J Forest Res ,vol.4, pp 212-222.

[14] R. VAradhan and all, "An R package for solving a large system of nonlinear equations and for optimazing a high-dimensional Nonlinear objective objective function," Journal of statistical software, vol. 32, 2009.

[15] T. Zhang and M. Xie, "ON the upper truncated weibull distribution and its reliability implications," Reliability Engineering and system safety, vol 96, 2011, pp 194-200.

[16] Balakrishnan,N. and Mitra,"Left truncated and right censored Weibull data and likelihood inference with an illustration," computation statistics and data analysis, vol. 56, 2012 , pp 613-617.

[17] M.J. Jozani, "A note on Pitman's measure of closeness with balanced loss function," Statistics, 2012, pp 1-6.

[18] Y.M Kantar and I. Usta, "Analysis of the upper truncated Weibull distribution for wind speed, " Energy conversion and management, vol. 96, 2015, pp 81-88.