The effects of approximating a probabilistic process using the renewal reward theorem and the optimal solution of an EOQ model with quality

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Abstract—The renewal theory has been applied in various areas describing stochastic processes such as queueing systems and probabilistic inventory models. Numerous research studies incorporating quality into classical inventory models have made use of the renewal reward theorem to approximate the optimal solution. Application of the renewal reward theorem to an economic order quantity (EOQ) model with quality is examined by developing and comparing the exact and an approximate optimal solution. The EOQ model considered in this paper examines an inventory situation where items received are of two types of quality, perfect and imperfect. Moreover, the case when the supplier of these items offers a delay in payment is presented. The percentage of perfect quality items is assumed to be a random variable having a known probability distribution. Both types of items have continuous demand, and items of imperfect quality are sold at a discounted price. A mathematical model is developed and closed form formulas for the exact and the approximate optimal order quantity based on the renewal reward theorem are obtained. A numerical example is provided to illustrate the model and to compare the exact and approximate optimal solutions.

Keywords—Probabilistic process, renewal reward theorem, economic order quantity, imperfect quality items.

I. Introduction

A renewal process describes a stochastic model in which random events occur successively over time intervals. The times between arrivals are assumed to be independent and have identical probability distributions. The mathematical structure for a renewal process can be used to describe many realistic models.

The renewal theory has been applied in various areas describing stochastic processes such as queueing systems and probabilistic inventory models. Recently, numerous research studies incorporating quality into classical inventory models have made use of the renewal reward theorem to approximate the optimal solution, see for instance [1]-[3]. Applying the renewal reward theorem provides a simplified manner to calculate the expected value of the total cost (total profit) per unit time function. The purpose of this paper is to assess the approximation of the optimal solution for an economic order quantity model with quality by developing a mathematical model and comparing the approximate and exact solution.

The remainder of this paper is organized as follows. In section two, a review of the related literature is provided. The mathematical model is developed in section three. The approximate solution is obtained in section four by applying the renewal reward theorem and a numerical example is presented in section five. The exact solution is developed in section six. Numerical examples comparing the exact and approximate solutions are found in section seven. The paper is concluded in section eight whereby a discussion of the results obtained along with suggestions for future research are presented.

II. Literature Review

The classical economic order quantity (EOQ) model describes an inventory situation where the demand rate for a certain item is D. At the beginning of each inventory cycle, an order of size Q is received from a supplier at a unit cost C to meet the demand and an ordering cost of K. Let h be the holding cost per unit per unit time. Then the total inventory cost per unit time function is given by

$$TCU(Q) = KD/Q + CD + hQ/2,$$

and the optimal order quantity or the economic order quantity is

$$Q^* = \sqrt{\frac{2KD}{h}}. \tag{1}$$

This model is established based on several basic assumptions that are seldom encountered in practice. Ever since the EOQ model and the closely related economic production quantity (EPQ) model were introduced in the early 20th century, researchers have been studying these models extensively under real-life situations [4].

A vast literature on inventory and production models extended the EOQ/EPQ models in numerous directions by modifying or relaxing the underlying assumptions of the models. One of the assumptions of the classical EOQ model is that all items received from a supplier are of the perfect quality type. Due to deterioration, shifting production process, or other factors, some items may be of imperfect quality. Good examples of such situations are found in the electronics industry. Another basic assumption is that payment for the items is made at the beginning of the inventory cycle when the order is received. A survey of papers from literature tackling inventory models in which the above mentioned assumptions are considered is presented below.

In [5], an EPQ model with imperfect production process and demand dependent unit production cost is presented. The effects of defective items on the basic EOQ model is investigated [6]. An EOQ model with fixed and variable costs for screening and removing defective items is proposed [7]. In [8], an EPQ model in which defective items can be reworked instantaneously at a cost was presented. It was concluded that the presence of defective products motivates smaller lot sizes.

In [1], a new line of research in the field of inventory management that ensures quality was initiated. An EOQ model was proposed where items received from the supplier contain imperfect items that are not necessarily defective and can be salvaged at a discounted price. The production lot sizing with the reworking of imperfect quality items was studied in [9]. The optimal lot size for an EPQ model with random defective rate, rework, and backlogging was determined in [10]. In [11], an EPQ model in which defective items are reworked or scraped is developed and a numerical method for determining the optimal lot size is presented. An EPQ model with different qualities items having continuous demands is developed in [12]. An extensive survey of such articles is given in [3].

In a different direction, numerous papers incorporating financial factors, such as time value of money, inflation, and credit facilities, into the classical inventory models can be found in the literature. Salameh and El-Kassar [13] investigated the effects of time value of money and credit facility on the optimality of the single period inventory model. Salameh et al. [14] presented a continuous review inventory model with delay in payments. The effects of time discounting on the EOQ and EPQ models were examined in [15] and [16]. In [17], a uniform replenishment inventory model with payment credit facilities is developed.

El-Kassar and Salameh [18] introduced an EPQ model that accounts for the cost of raw material. El-Kassar and Dah [19] extended that model to include time value of money. El-Kassar et al. [20] generalized the model to a multi-stage production process. Time discounting was incorporated into the multi-stage model [21]. El-Kassar et al. [22] investigated the effects of imperfect quality items of raw material on the EPQ model. El-Kassar et al. [23] extended the model to account for time value of money. In [24], the effects of having a probabilistic percentage of imperfect quality items of raw material on the optimal production lot size is studied. In [25] the joint probability distribution and the minimum of a set of normalized random variables is examined. In [26], the results are generalized and applied to investigate an EPQ model that accounts for the quality of various types of raw materials/components used in the production process.

This paper considers an inventory situation where items received from a supplier are of two types of quality, perfect and imperfect. The percentage of perfect quality items is a random variable having a known probability distribution. It is assumed that demands of both types of items occur continuously. A screening process conducted at the beginning of the inventory cycle is used to detect the imperfect quality items. Items of perfect quality are sold at a regular price while the imperfect ones are sold at a discount. Also, this paper considers the supplier is offering a delay of payment for the items received at the beginning of the cycle. In each case, an approximate solution using the renewal reward theorem is obtained along with the exact solution. The solution solutions are compared using numerical examples.

III. The Mathematical Model

To develop the proposed model, the following notation will be used throughout this paper:

- Q Quantity ordered
- D_p Demand rate of items that are of perfect quality
- D_i Demand rate of items that are of imperfect quality
- D Demand rate of both perfect and imperfect quality items, $D = D_p + D_i$
- q Percentage of perfect quality items received
- $f(q)$ Probability density function for q
- μ Expected percentage of perfect quality items received
- σ Standard deviation of the percentage of perfect quality items received
- K Ordering cost per inventory cycle
- h Holding cost per unit per unit time
- C Unit purchasing cost
- C_s Unit screening cost
- S_p Regular selling price of one perfect quality item
- S_i Discounted selling price of one imperfect quality item
- T Total inventory cycle length
- T_p Perfect quality items inventory cycle length
- T_i Imperfect quality items inventory cycle length
- i Interest rate
- M Delay in payment period offered by the supplier
- $E[.]$ Expected value of an expression

Consider the inventory situation where an order of size Q is received from a supplier at a unit purchasing price C and an ordering cost K. It is assumed that each order received, at the beginning of the inventory cycle, contains both perfect and imperfect quality items. The percentage of perfect quality items is a random variable q having a known probability density function $f(q)$. Items of imperfect quality are detected through a screening process conducted at the beginning of the inventory cycle. The perfect quality items are sold at the regular unit price S_p and the imperfect quality items at a discounted unit price of S_i, where $S_i < S_p$. Most EOQ models with quality assume that the imperfect quality items are sold as a single batch at the end of the screening period. As in [27], we assume that both perfect and imperfect items have continuous demand, of rates D_p and D_i, respectively. In order to meet the demands of both types of items, we assume that the screening rate is greater than both D_p and D_i. Fig. 1 illustrates the behavior of the inventory level.

The order of size Q received from the supplier contains qQ perfect quality items and $(1-q)Q$ imperfect quality items, we have that the cycle lengths for the two types of items are given by

$$T_p = qQ / D_p , \quad (2)$$

and

$$T_i = (1-q)Q / D_i . \quad (3)$$

The combined inventory cycle length for the is $T = \max\{T_p, T_i\}$. During the time period from $t = 0$ until $T_m = \min\{T_p, T_i\}$, the combined inventory level decreases at a combined demand rate of $D = D_p + D_i$ until either type of items is sold out. After that period, the inventory level decreases at a rate of D_p or D_i. In order to avoid shortages of perfect quality items, we assume that $T_p \geq T_i$ so that $qQ/D_p \geq (1-q)Q/D_i$. This is equivalent to $qD \geq D_p$. In this case,

$$T = \max\{T_p, T_i\} = T_p , \quad (4)$$

and the imperfect quality items will be sold out at time $T_m = \min\{T_p, T_i\} = T_i$ when the combined inventory is at a level of

$$Q - T_i D = (T_p - T_i)D_p = (qD - D_p)Q / D_i , \quad (5)$$

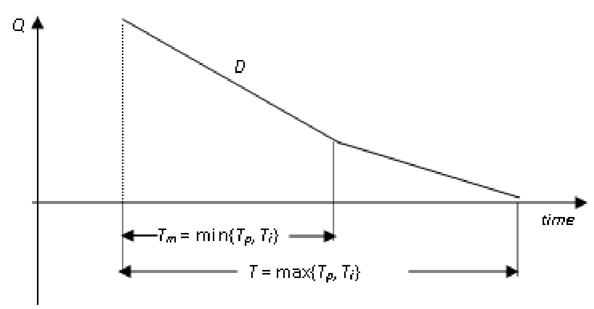

Fig. 1 combined inventory level of perfect and imperfect quality items

To find the optimal order quantity, the total profit per unit time function $TPU(Q)$ is maximized. This function is determined by accounting for the cost and revenue components incurred or received during the inventory cycle. During the inventory cycle, the revenue components received are:

Perfect quality items sold at regular price $= S_p qQ$
Imperfect quality items sold at discounted price $= S_i(1-q)Q$
Interest earned on sales during credit facility period

$$= \int_0^M i(M-t)\left(SpD_p + S_i D_i\right)dt = i\left(S_p D_p + S_i D_i\right)M^2 / 2 .$$

Hence, he total revenue function per inventory cycle is given by

$$TR(Q) = S_p qQ + S_i(1-q)Q + (S_p D_p + S_i D_i)iM^2 / 2 . \quad (6)$$

The total cost function per inventory cycle, $TC(Q)$, consists of four the following components:

Purchasing cost = CQ
Screening cost = $C_s Q$
Setup cost = K
Holding cost = $h\times$(Area under the curve in Fig. 1)

From (2)-(5), the area under the curve in Fig. 1 is

$$\begin{aligned} Area &= \tfrac{1}{2} T_p qQ + \tfrac{1}{2} T_i (1-q)Q \\ &= \tfrac{1}{2}(qQ/D_p)qQ + \tfrac{1}{2}((1-q)Q/D_i)(1-q)Q \\ &= \tfrac{1}{2}Q^2(q^2 D_i + (1-q)^2 D_p)/(D_p D_i). \end{aligned} \quad (7)$$

Hence, the total cost function per inventory cycle is given by

$$TC(Q) = (C + C_s)Q + K + \frac{Q^2 h(q^2 D_i + (1-q)^2 D_p)}{2D_p D_i}. \quad (8)$$

From (6) and (8), we have that the total profit function per inventory cycle is

$$\begin{aligned} TP(Q) &= S_p qQ + S_i(1-q)Q + (S_p D_p + S_i D_i)iM^2 / 2 \\ &\quad - (C + C_s)Q - K - \frac{Q^2 h(q^2 D_i + (1-q)^2 D_p)}{2D_p D_i}. \end{aligned} \quad (9)$$

IV. Approximate Solution

The total profit per unit time function is obtained by dividing the total profit function per inventory cycle by the cycle length; i.e., $TPU(Q) = TP(Q)/T$. By the renewal reward theorem, the expected profit per unit time is approximated by

$$E[TPU(Q)] = E[TP(Q)] / E[T] . \quad (10)$$

To evaluate $E[TP(Q)]$, we need the following:

$$E[q] = \int_{-\infty}^{\infty} qf(q)dq = \mu, \quad (11)$$

$$\begin{aligned} E[q^2] &= \int_{-\infty}^{\infty} q^2 f(q)dq \\ &= \int_{-\infty}^{\infty} (q^2 - 2\mu q + \mu^2 + 2\mu q - \mu^2) f(q)dq \\ &= \int_{-\infty}^{\infty} (q-\mu)^2 f(q)dq + \int_{-\infty}^{\infty} (2\mu q - \mu^2) f(q)dq \\ &= \sigma^2 + 2\mu^2 - \mu^2 = \sigma^2 + \mu^2, \end{aligned} \quad (12)$$

and

$$E[(1-q)^2] = \int_{-\infty}^{\infty} (1-q)^2 f(q)dq$$

$$= \int_{-\infty}^{\infty}(1-2q+q^2)f(q)dq$$
$$= 1-2\mu+\sigma^2+\mu^2= (1-\mu)^2+\sigma^2 \quad (13)$$

From (2), (4) and (11), we have

$$E[T] = E[T_p] = E[qQ/D_p] = E[q]Q/D_p = \mu Q/D_p\,, \quad (14)$$

and from (9), (11) and (12), we get

$$\begin{aligned} E[TP(Q)] &= E\Big[S_p qQ + S_i(1-q)Q - (C+C_s)Q - \\ &\quad - K + (S_pD_p + S_iD_i)iM^2/2 \\ &\quad - Q^2h(q^2D_i + (1-q)^2D_p)/(2D_pD_i)\Big]. \\ &= S_pE[q]Q + S_i(1-E[q])Q - (C+C_s)Q \\ &\quad - K + (S_pD_p + S_iD_i)iM^2/2 \\ &\quad - Q^2h(E[q^2]D_i + E[(1-q)^2]D_p)/(2D_pD_i). \\ &= S_p\mu Q + S_i(1-\mu)Q - (C+C_s)Q \\ &\quad - K + (S_pD_p + S_iD_i)iM^2/2 \\ &\quad - \frac{Q^2h((\sigma^2+\mu^2)D_i + (\sigma^2+(1-\mu^2))D_p)}{2D_pD_i}. \end{aligned} \quad (15)$$

The expected profit per unit time is obtained from (14) and (15) as

$$\begin{aligned} E[TPU(Q)] &= (D_p/\mu)\Big[S_p\mu + S_i(1-\mu) - (C+C_s) \\ &\quad - K/Q + (S_pD_p + S_iD_i)iM^2/(2Q) \\ &\quad - \frac{Qh((\sigma^2+\mu^2)D_i + (\sigma^2+(1-\mu^2))D_p)}{2D_pD_i}\Big]. \end{aligned} \quad (16)$$

Differentiating the expression in (16), we have

$$\begin{aligned} \frac{dE[TPU(Q)]}{dQ} &= \frac{D_p}{\mu}\Big[\frac{K}{Q^2} - \frac{(S_pD_p + S_iD_i)iM^2}{2Q^2} \\ &\quad - \frac{h((\sigma^2+\mu^2)D_i + (\sigma^2+(1-\mu^2))D_p)}{2D_pD_i}\Big]. \end{aligned} \quad (17)$$

The economic order quantity is obtained by setting the derivative in (17) equal to zero and solving for Q. The economic order quantity is

$$Q^* = \sqrt{\frac{D_pD_i(2K - iM^2(S_pD_p + S_iD_i))}{h((\sigma^2+\mu^2)D_i + ((1-\mu)^2+\sigma^2)D_p)}}\,. \quad (18)$$

Note that the condition $2K-(S_pD_p+ S_iD_i)iM^2 > 0$ is needed to calculate Q^*. Also note that the second derivative of the $E[TPU(Q)]$ function $((S_pD_p+S_iD_i)iM^2-2K)D_p/(qQ^3)$ is negative so that $E[TPU(Q)]$ is a convex function and Q^* is unique. In the case when the supplier does not offer a delay in payment, i.e., $M = 0$, (16) and (18) reduce to

$$\begin{aligned} E[TPU(Q)] &= \frac{D_p}{\mu}\Big[S_p\mu + S_i(1-\mu) - (C+C_s) - \frac{K}{Q} \\ &\quad - \frac{Qh((\sigma^2+\mu^2)D_i + (\sigma^2+(1-\mu^2))D_p)}{2D_pD_i}\Big]. \end{aligned} \quad (19)$$

and

$$Q^* = \sqrt{\frac{2KD_pD_i}{h((\sigma^2+\mu^2)D_i + ((1-\mu)^2+\sigma^2)D_p)}}\,. \quad (20)$$

Similar analysis can be conducted for the case when $T_i > T_p$. This case will not be considered since it implies that perfect quality items would be out of stock over a period of length $T_i - T_p$. During this period only imperfect quality items are sold at a discounted price. This may result in considerably less profit. In such a case, the model can be modified to allow backorder or to consider selling the remaining quantity of imperfect quality items at even a lower price than the already discounted unit selling price S_i.

V. Numerical Example

Consider the problem with parameters: D_p = 40 units/day, D_i = 20 units/day, C = \$250, C_s = \$5, K = \$1825, h = \$0.2/unit/day, S_p = \$350, S_i = \$300. Since the combined daily demand rate is D = 400 + 200 = 600 units, the ratio $D_p/D = 2/3 < q$. This guaranties that $T_p \geq T_i$. For a uniformly distributed random variable q over the interval $[a, b]$, the expected value is $E[q] = \mu = (a+b)/2$ and the variance is $\sigma^2 = (b-a)^2/12$. Thus for a percentage of perfect quality items uniformly distributed over [70%, 90%], we have that μ = 80%, $\sigma^2 = (0.9-0.7)^2/12 = 0.00333$ and σ = 0.057735. Using the equations (17) and (18), the optimal order quantity and the corresponding maximum daily profit were found to be Q^* = 1000 units and $E[TPU(Q^*)]$ = \$4067.5. Fig. 2 illustrates the behavior of the expected daily profit function $E[TPU(Q)]$.

Analyzing the optimal solution, for an order size of Q^* = 1000 units, one expects μQ^* = 800 units of perfect quality and $(1-\mu)Q^*$ = 200 units of imperfect quality. The expected length of the perfect quality items inventory cycle is $\mu Q^*/D_p$ = 20 days, and that of the imperfect quality items is $(1-\mu)Q^*/D_i$ = 10 days. After 10 days, all imperfect quality items will be sold out and 400 units of perfect quality items will remain in stock to be sold over the rest of the inventory cycle.

The expected revenues from selling perfect quality items at regular price is S_pqQ^* = \$280,000. As for the imperfect quality items, sales at discounted price is $S_i(1-\mu)Q^*$ = \$60,000. The purchasing cost per cycle is CQ^* = \$250,000 and the screening cost is C_sQ^* = \$5,000. Hence, the expected total revenue per inventory cycle is \$340,000.

The holding cost per inventory cycle is calculated by multiplying the area in Fig. 1 by the holding cost per unit per unit time.

The expected area is $Q^2((\sigma^2+\mu^2)Di+(1-\mu^2+\sigma^2)Dp)/(2D_pD_i)$ = 9,125 and the expected holding cost is \$0.2(9,125) = \$1,825. Thus, the total cost per inventory cycle is \$1,825 + \$250,000 + \$5,000 + \$1,825 = \$258,650 and the expected total profit per inventory cycle is \$340,000. By dividing by the inventory cycle length, the expected daily profit is found to be \$4067.5.

Note that the order quantity (1) based on the classical EOQ model results in an order quantity of Q = 1046.42 units.

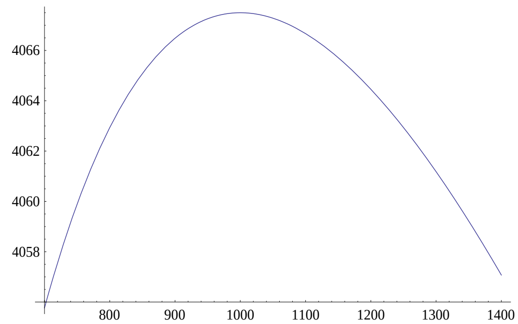

Fig. 2 The expected total annual profit function $E[TPU(Q)]$

Now we consider the case where the supplier offers a new policy that allows a 15 days delay in payment. Based on a 7.7% interest rate and using (16) and (17) to calculate the optimal order quantity and the daily profit, the optimal order quantity was calculated and found to be 860 units with a corresponding daily profit of \$4,093. This results in an additional annual profit of \$9,314.8 over the policy with no delay in payment.

VI. Exact Solution

The total profit per unit time function is obtained by dividing the total profit function per inventory cycle by the cycle length; i.e., $TPU(Q) = TP(Q)/T$. The exact expression for the expected profit per unit time is obtained by

$$E[TPU(Q)] = E[TP(Q)/T]. \tag{21}$$

Define γ by

$$\gamma = E[1/q] = \int_{-\infty}^{\infty} (1/q)dq . \tag{22}$$

When q is uniformly distributed over $[a, b]$, (22) becomes

$$\gamma = E[1/q] = \int_{-\infty}^{\infty} (1/q)dq = \int_{-\infty}^{\infty} \frac{1/(b-a)}{q} dq = \frac{\ln(b/a)}{b-a} . \tag{23}$$

From (2) and (9), we have

$$\begin{aligned} TPU(Q) = \frac{D_p}{qQ}\Bigg[& S_p qQ + S_i(1-q)Q - (C+C_s)Q \\ & - K + (S_pD_p + S_iD_i)iM^2/2 \\ & - \frac{Q^2 h(q^2D_i + (1-q)^2 D_p)}{2D_pD_i} \Bigg]. \end{aligned}$$

$$\begin{aligned} &= Dp(S_p + S_i(1/q-1)) \\ &\left(\frac{iM^2(S_pD_p+S_iD_i)}{2Q} - C - C_s - \frac{K}{Q}\right)/q \\ &- hQ(qD_i + (1/q-2+q)D_p)/(2D_pD_i)). \end{aligned} \tag{24}$$

From (11) and (22), the expected profit per unit time is

$$\begin{aligned} E[TPU(Q)] &= D_p(S_p + S_i(\gamma-1)) \\ &+ D_p\gamma\left(\frac{iM^2(S_pD_p+S_iD_i)}{2Q} - C - C_s - \frac{K}{Q}\right) \\ &- hQ(\mu D_i + (\gamma-2+\mu)D_p)/(2D_i)) \end{aligned} \tag{25}$$

Differentiating the expression in (25), we have

$$\begin{aligned} \frac{dE[TPU(Q)]}{dQ} &= D_p(\gamma(-iM^2(S_pD_p+S_iD_i)/(2Q^2) \\ &+ K/Q^2) - h(\mu D_i + (\gamma-2+\mu)D_p)/(2D_pD_i)) \end{aligned} \tag{26}$$

The economic order quantity is obtained by setting the derivative in (26) equal to zero and solving for Q. The economic order quantity is

$$Q^* = \sqrt{\frac{\gamma D_pD_i(2K - iM^2(S_pD_p+S_iD_i))}{h(\mu D_i + (\gamma-2+\mu)D_p)}} . \tag{27}$$

Note that the expression $(\gamma-2+\mu)$ is the expected value of $(1-q)^2/q$, which is positive. Hence, $2K-(S_pD_p+ S_iD_i)iM^2$ need to be positive in order to calculate Q^*. Also note that the second derivative of the $E[TPU(Q)]$ function $((S_pD_p+S_iD_i)iM^2-2K)\gamma D_p/(qQ^3)$ is negative so that $E[TPU(Q)]$ is a convex function and Q^* is unique.

It is worth noting that when q is constant, we have that $\mu = q$, $\sigma = 0$ and $\gamma = 1/q$ and the two expressions (18) and (27) become identical.

In the case when the supplier does not offer a delay in payment, i.e., $M = 0$, (25) and (27) reduce to

$$\begin{aligned} E[TPU(Q)] &= D_p(S_p + S_i(\gamma-1)) \\ &+ D_p\gamma(-C - C_s - K/Q) \\ &- hQ(\mu D_i + (\gamma-2+\mu)D_p)/(2D_i)). \end{aligned} \tag{28}$$

and

$$Q^* = \sqrt{\frac{2K\gamma D_pD_i}{h(\mu D_i + (\gamma-2+\mu)D_p)}} . \tag{29}$$

VII. Comparing The Exact And Approximate Solutions

To compare the exact and approximate solutions, we begin by noting that the approximate optimal solution based on the renewal reward theorem uses $1/\mu$ as an approximation for γ. This approximation is examined in Table 1, where μ is the expected value of a uniformly distributed random variable q over the interval $[a, b]$. The results show that the error can become very large. Hence, approximating the optimal solution of (27) by (18) must be examined further.

Table 1: Exact and approximate values of γ

	a	*b*	Exact	Approximate	%Error
1	0.9	1	1.05361	1.05263	0.09%
2	0.8	1	1.11572	1.11111	0.41%
3	0.7	1	1.18892	1.17647	1.05%
4	0.6	1	1.27706	1.25	2.12%
5	0.4	1	1.52715	1.42857	6.46%
6	0.3	1	1.71996	1.53846	10.55%
7	0.2	1	2.0118	1.66667	17.16%
8	0.1	1	2.55843	1.81818	28.93%
9	0.09	1	2.64609	1.83486	30.66%
10	0.05	1	3.1534	1.90476	39.60%
11	0.04	1	3.353	1.92308	42.65%
12	0.03	1	3.61501	1.94175	46.29%
13	0.02	1	3.99186	1.96078	50.88%
14	0.007	1	4.99682	1.9861	60.25%
15	0.006	1	5.14688	1.98807	61.37%
16	0.003	1	5.82662	1.99402	65.78%
17	0.002	1	6.22706	1.99601	67.95%
18	0.001	1	6.91467	1.998	71.10%

Now we consider the numerical example of section 4. The exact solution calculated using (26) results in an optimal order quantity of Q^* = 1002 units when no delay is offered. The corresponding maximum daily profit were found to be and $E[TPU(Q^*)]$ = \$4078.8. In the case when a delay of period of M = 15 days is offered, (26) and (27) showed an optimal order quantity of Q^* = 862 units and a corresponding maximum daily profit of $E[TPU(Q^*)]$ = \$4,104.38. Accordingly, the renewal reward theorem resulted in a very good approximation for the optimal order quantity and a relatively good approximation for the total daily profit. However, the difference in the profit can accumulate and become larger when the annual profit is calculated.

In the following, we compare the exact and the approximate solution by considering the percentage of perfect quality items to be uniformly distributed over varying intervals $[a,b]$. Using the parameters of the numerical example of section 5 and considering varying intervals $[a, b]$ all centered at 0.8, the results are shown in Table 2 below.

The results suggest that the approximation is very accurate when the interval is small; that is, when the standard deviation is small. However, the error increases as the interval becomes larger though the approximation remains accurate. This is due to the fact that the mathematical model was developed base on the percentage of perfect quality items which is expected to be closer to 1 than 0. If the formulation of the model was based on the percentages of imperfect quality items, which is expected to be closer to 0 than 1, the approximation would become sensitive as this percentages becomes closer to 0.

Table 2: Comparing the Exact and Approximate Optimal Solutions

a	*b*	Q* (Exact)	Q* (Approx.)	Percent Error
0.79	0.81	1006.87	1006.85	0.00%
0.78	0.82	1006.73	1006.64	0.01%
0.77	0.83	1006.5	1006.29	0.02%
0.76	0.84	1006.17	1005.80	0.04%
0.75	0.85	1005.76	1005.18	0.06%
0.74	0.86	1005.24	1004.41	0.08%
0.73	0.87	1004.64	1003.51	0.11%
0.72	0.88	1003.94	1002.47	0.15%
0.71	0.89	1003.15	1001.30	0.18%
0.7	0.9	1002.27	1000.00	0.23%
0.69	0.91	1001.3	998.57	0.27%
0.68	0.92	1000.23	997.00	0.32%
0.67	0.93	999.079	995.31	0.38%
0.66	0.94	997.832	993.49	0.44%
0.65	0.95	996.496	991.55	0.50%
0.64	0.96	995.069	989.48	0.56%
0.63	0.97	993.553	987.30	0.63%
0.62	0.98	991.948	985.00	0.70%
0.61	0.99	990.254	982.59	0.77%

VIII. Conclusion

Application of the renewal reward theorem to an economic order quantity (EOQ) model with quality was examined by developing and comparing the exact and an approximate optimal solution. The model assumes that items received from a supplier contain both perfect and imperfect quality items. The percentage of perfect quality is a random variable having a known probability distribution. The imperfect quality items are sold at a discounted price and the demands for both perfect and imperfect quality items occur continuously during the inventory cycle. The inventory model accounted for the case when the supplier offers a delay in payment for items received at the beginning of the cycle. A mathematical model is developed and a closed form formula for the optimal order quantity. The formula is based on maximizing the total profit per unit time.

A numerical example was presented to illustrate the model. Comparison between the exact and approximate solutions showed that the approximation is generally accurate. However, in certain situations the approximation may become inaccurate.

For future research, we suggest analyzing the effects of the various parameters on the approximate optimal solution.

REFERENCES

[1] M. K. Salameh, and M. Y. Jaber, “Economic production quantity model for items with imperfect quality,” *International journal of production economics*, vol. 64, no. 1, pp. 59-64, 2000.

[2] A. Eroglu, and G, Ozdemir, “An economic order quantity model with defective items and shortages,” *International journal of production economics*, vol. 106, no. 2, pp. 544-549, 2007.

[3] M. Khan, M. Y. Jaber, A. L. Guiffrida, and S. Zolfaghari, “A review of the extensions of a modified EOQ model for imperfect quality items,“ *International Journal of Production Economics*, vol. 132, no. 1, pp. 1-12, 2011.

[4] N. C. Simpson, “Questioning the relative virtues of dynamic lot sizing rules,” *Computers & Operations Research*, vol. 28, no. 9, pp. 899-914, 2001.

[5] T. C. E. Cheng, “An economic order quantity model with demand-dependent unit production cost and imperfect production processes,” *IIE transactions*, vol. 23, no. 1, pp. 23-28, 1991.

[6] E. L. Porteus, “Optimal lot sizing, process quality improvement and setup cost reduction,” *Operations research*, vol. 34, no. 1, pp. 137-144, 1986.

[7] R. L. Schwaller, “EOQ under inspection costs,” *Production and Inventory Management*, vol. 29, no. 3, pp. 22-24, 1988.

[8] Rosenblatt, M. J., and Lee, H. L. “Economic production cycles with imperfect production processes,” *IIE transactions*, vol. 18, no. 1, pp. 48-55, 1986.

[9] P. A. Hayek, and M. K. Salameh, “Production lot sizing with the reworking of imperfect quality items produced,” *Production Planning & Control*, vol. 12, no. 6, pp. 584-590, 2001.

[10] P. Y. Chiu, “Determining the optimal lot size for the finite production model with random defective rate, the rework process, and backlogging, “ *Engineering Optimization*, vol. 35, no. 4, pp. 427-437, 2003.

[11] Y. S. P. Chiu, S. W. Chiu, and H. C. Chao, “Numerical method for determination of reworking or scraping the defective items in a finite production rate model,” *Communications in numerical methods in engineering*, vol. 22, no. 5, pp. 377-386, 2006.

[12] A. N. El-Kassar, A. Dah, and M. K. Salameh, “Optimal Lot Size for EPQ Inventory Model for Items of Different Qualities,” *Journal of the Academy of Business & Economics*, vol. 8, no. 4, 2008.

[13] M. Salameh, and A. N. El-Kassar, “Optimality of the single period inventory model with credit facility,” *International Journal of Applied Mathematics*, vol. 1, no. 7, pp. 817-828, 1999.

[14] M. K. Salameh, N. E. Abboud, A. N. El-Kassar, and R. E. Ghattas, “Continuous review inventory model with delay in payments,” *International Journal of Production Economics*, vol. 85, no. 1, pp. 91-95, 2003.

[15] M. K. Salameh, M. A. Abdul-Malak, and A. N. El-Kassar, “The exact time-discounting of the instantaneous replenishment model,” *International Journal of Applied Mathematics*, vol. 1, no. 8, pp. 941-948, 1999

[16] M. K. Salameh, and A. N. El-Kassar, “The exact time discounting of the finite production model”, in *Proc. 6th International Conference on Industrial Engineering and Production Management Conference*, Porto, Portugal, vol. 1, pp. 432-441, 2003.

[17] A. Dah, and A. N. El-Kassar, ”The uniform replenishment inventory model with payment credit facilities,“ *International Journal of Applied Mathematics*, vol. 21, no. 4, pp. 637-644, 2008.

[18] M. K. Salameh, and A. N. El-Kassar, *“Accounting for the holding cost of raw material in the finite production model,”* in *Proc. BIMA Inaugural Conference, Sharjah*, pp. 72-81, 2007.

[19] A. N. El-Kassar, and A. Dah,” Effect of time discounting and cost of raw material on the economic production model,” *Global Review of Business and Economic Research Journal*, vol. 5, no. 1, pp. 113-124, 2009.

[20] A. N. El-Kassar, N. Yassin, and K. Maknieh, “Lot sizing a multistage production process,” *The Business Review, Cambridge*, vol. 14, no. 2, pp. 265-273, 2010.

[21] A. N. El-Kassar, and N. Yassin. “The discounted cash flows of multistage and multicycle production models,” in *Proc. Global Conference On Business And Finance*, Las Vegas, vol. 6, no. 1, pp. 717-727, 2011.

[22] A.N. El-Kassar, M. Salameh and M. Bitar, “EPQ model with imperfect quality raw material,” *Math. Balkanica*, vol. 26, pp. 123-132, 2012.

[23] A.N. El-Kassar, M. Salameh and M. Bitar,” Effects of time value of money on the EPQ Model with the imperfect quality items of Raw material”, in *Proc. Academy of Information and Management Sciences*, New Orleans, vol. 16, no. 1, pp. 11-18, 2012.

[24] T. Mikdashi, A. N. El-Kassar, and A. Joudi, “Effects of Probabilistic Imperfect Quality Items of Raw Material on the Optimal Production Lot Size,” *Journal of the Academy of Business & Economics*, vol. 12, no. 3, pp. 75-84. 2012.

[25] N. Yassine, “Joint Probability Distribution and the Minimum of a Set of Normalized Random Variables,” *Procedia - Social and Behavioral Sciences*, vol. 230, no. 12, pp. 235–239, 2016.

[26] N. Yassine, “The Minimum of a Set of Random Variables and the Optimal Lot Size Model with Quality,” *International Journal of Mathematical Models and Methods in Applied Sciences*, to be published.

[27] A. N. M. El-Kassar, ”Optimal order quantity for imperfect quality items,” *Proc. Academy of Information and Management Sciences*, vol. 13, no. 1, pp. 24-30, 2009.

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