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Structure of fuzzy dot BF-subalgebras

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***Abstract*—The concept of fuzzy dot $BF-$subalgebra has been introduced. Fuzzy dot product of$BF-$ subalgebras and strong fuzy product in a fuzzy dot $BF-$subalgebra has been discused.**
Different theorems,Lemmas and propositions has been proved. We have also investigated different characterizations.

***Index Terms*—BF- subalgebra,fuzzy BF-subalgebras, Ideal in BF- subalgebras, Fuzzy dot BF-subalgebras.**

I. Introduction

In [8] the idea of fuzzy set was first introduced by Zadeh and the concept of B-algebras was introduced by Neggers and Kim. They defined a B-algebra as an algebra $(X, *, 0)$ of type (2,0) satisfying the following axioms:

1) $x * x = 0$.
2) $x * 0 = x$.
3) $(x * y) * z = x * (z * (0 * y))$.

Andrzej Walendziak initaited the idea of a BF-algebra and characterized different structures in [1] and in [3] Borumand Saeid and Rezrani investigated Fuzzy BF-Algebras.
In this paper, we introduced the concept of fuzzy dot BF-algebras and study its structures. We state and prove some theorem discussed in fuzzy dot BF-subalgebras and level subalgebras. Finally some of results on homomorphic images and inverse images in fuzzy dot BF-subalgebras are investigated.

II. Preliminaries

A BF-algebra is a nonempty set X with a constant $'0'$ and a binary operation '$*'$ satisfying the following conditions:

1) $x * x = 0$.
2) $x * 0 = x$.
3) $0 * (x * y) = (y * x)$ For all $x, y \in X$.

Example 2.1: Let $X = [0, \infty)$. Define the binary operation $'*'$ on X as follows: $x * y = |x - y|$, for all $x, y \in X$,Then $(X, *, 0)$ is a BF-algebra.

Let X be a BF-algebra. Then for any x and y in X, the following hold:

1) $0 * (0 * x) = x$.
2) if $0 * x = 0 * y$,then $x = y$.
3) if $x * y = 0$,then $y * x = 0$.

Any BF-algebra $(X, *, 0)$ that satisfies the identity $(x * z) * (y * z) = x * y$ is a B-algebra. A subset I of X is called an ideal of X if it satisfies:

1) $0 \in I$.
2) $x * y$ and $y \in I$ imply $x \in I$ for any $x, y \in X$.

A nonempty subset S of X is called a sub algebra of A if $x * y \in S$ for any $x, y \in X$.

Let $(X, *, 0_X)$ and $(Y, *, 0_Y)$ be BF-algebras. A mapping $\phi : X \longrightarrow Y$ is called a homomorphism from X into Y if $\phi(x * y) = \phi(x) * \phi(y)$ for any $x, y \in X$.

Let X be a nonempty set. A fuzzy (sub) set μ of the set X is a mapping $\mu : X \longrightarrow [0, 1]$.

and μ is the fuzzy set of a set X. For a fixed $s \in [0,1]$, the set $\mu_s = \{x \in X : \mu(x) \geq s\}$ is called an upper level of μ or level subset of μ. Furthermore let X be a set. A fuzzy set A in X is characterized by a membership function $\mu_A : X \longrightarrow [0, 1]$. Let f be a mapping from the set X to the set Y and let B be a fuzzy set in Y with membership function μ_B.
The inverse image of B, denoted $f^{-1}(B)$, is the fuzzy set in X with membership function $\mu_{f^{-1}B}$ defined by $\mu_{f^{-1}(B)(x)} = \mu_B(f(x))$ for all $x \in X$. Conversely, let A be a fuzzy set in X with membership function μ_A Then the image of A, denoted by $f(A)$, is the fuzzy set in Y such that $\mu_{f(A)}(y) = \begin{cases} sup\mu_A(x)_{x \in f^{-1}(y)} & \text{if } f^{-1}(y) = \{x : f(x(= y\} \\ 0 & \text{if otherwise} \end{cases}$.
A fuzzy set A in the BF -algebra X with the membership function μ_A is said to be have the sup property if for any subset $T \subseteq X$ there exists $x_0 \in T$such that $\mu_A(x_0) = sup\mu_A(t)_{t \in T}$;and let μ be a fuzzy set in a BF -algebra. Then μ is called a fuzzy BF -subalgebra (algebra) of X if $\mu(x * y) \geq min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

A fuzzy set μ of a BF-algebra X is called a fuzzy ideal of X if it satisfies the following conditions.

1) $\mu(0) \geq \mu(x)$
2) $\mu(x) \geq min\{\mu(x * y), \mu(y)\}$

III. Results

A. Fuzzy dot BF-Sub algebra

Definition 3.1: Let A be a fuzzy set in BF-Subalgebra X and μ be a fuzzy subset a BF-Subalgebra of X. Then the fuzzy dot BF-Subalgebra of X is $\mu_A(x \star y) \geq \mu_A(x).\mu_A(y)$, for all $x, y \in X$.

Example 3.2: Let $X = \{0, a, b\}$ be a set with the table given below

$\star$	0	a	b
0	0	a	b
a	a	0	0
b	b	0	0

. Then $(X, \star, 0)$ is a $BF-$ algebra.
Define $\mu : X \to [0, 1]$ by $\mu(0) = 0.8, \mu(a) = 0.2$, and $\mu(b) = 0.4$. Then μ is a fuzzy dot $BF-$ Subalgebra of X. Since $\mu(a \star b) = \mu(0) = 0.8 \geq (0.2)(0.4) = 0.08 = \mu(a).\mu(b)$.
Hence $\mu(a \star b) \geq \mu(a).\mu(b)$, for all $a, b \in X$.

Lemma 3.3: Let A be a fuzzy set in $BF-$ algebra. If μ_A is a fuzzy dot BF-Subalgebra of X, then for all $x \in X$, $\mu(0) \geq (\mu(x))^2$, for all $x \in X$.

Corollary 3.1: If A is a fuzzy subset of a BF-algebra X and μ_A is a a fuzzy dot subalgebra of X, then $\mu_A(0^n \star x) \geq (\mu_A(x))^{2n+1}$, for all $x \in X$ and $n \in N$, where $0^n \star x = 0 \star (0 \star (0 \star ...(0 \star x)...)$ in which 0 occurs $n-$times.

Proof. By Lemma 3.3. we have $\mu_A(0) \geq (\mu_A(x))^2$, for all $x \in X$. Put n= 1 in $0^n \star a$ we have $0 \star x$.
$\mu_A(0 \star x) \geq \mu_A(0).\mu_A(x) \geq (\mu_A(x))^2.\mu_A(x) = (\mu_A(x))^3$.
Hence $\mu_A(0 \star x) \geq (\mu_A(x))^3$, for all $x \in X$.
Assume the result holds for n= k,$\mu_A(0^k \star x) \geq (\mu_A(x))^{2k+1}$, for all $x \in X$.

$$\begin{aligned} Now\ \mu_A(0^{k+1} \star x) &= \mu_A(0 \star (0^k \star x)) \\ &\geq \mu(0).\mu(o^k \star x) \\ &\geq (\mu_A(x))^2.(\mu_A(x))^{2k+1} \\ &= (\mu_A(x))^{2(k+1)+1} \end{aligned}$$

Hence $\mu_A(0^n \star x) \geq (\mu_A(x))^{2n+1}$, for all $x \in X$ and $n \in N$.□

Theorem 3.2: Let A be a fuzzy subset of $BF-$Subalgebra and let μ_A be a fuzzy dot BF-Subalgebra of X. If there exists a sequence $\{x_n\}$ in X such that $liim_{n\to\infty}\mu_A(x_n) = 1$. then $\mu_A(0) = 1$.

Theorem 3.3: Let A_1 and A_1 be fuzzy subsets of $BF-$Subalgebras and let μ_{A_1} and μ_{A_2} be fuzzy dot $BF-$ Subalgebras of X. Then $\mu_{A_1 \cap A_1}$ is a fuzzy dot subalgebras of X.

Proof. Let $x, y \in A_1 \cap A_2$. Then $x, y \in A_1$ and $x, y \in A_2$, since A_1 and A_2 are subsets of fuzzy $BF-$Subalgebra of X.
$\mu_{A_1 \cap A_2}(x \star y) = min\{\mu_{A_1}(x \star y), \mu_{A_2}(x \star y)\}$
$\geq min\{\mu_{A_1}(x).\mu_{A_1}(x), \mu_{A_2}(x).\mu_{A_2}(y)\}$
$\geq \mu_{A_1 \cap A_2}(x).\mu_{A_1 \cap A_2}(y)$.
Hence $\mu_{A_1 \cap A_2}(x \star y)$
$\geq min\{min\{\mu_{A_1}(x), \mu_{A_2}(x)\}.min\{\mu_{A_1}(y), \mu_{A_2}(y)\}\} \geq \mu_{A_1 \cap A_2}(x), \mu_{A_1 \cap A_2}(y)$.□

Corollary 3.4: Let $\{A_i | i \in I\}$ be a family of fuzzy dot $BF-$Subalgebras of X. Then $\cap_{i \in I} A_i$ is also a fuzzy dot $BF-$Subalgebras of X.

Proof. Let $\{A_i | i \in I\}$ be a family of fuzzy dot $BF-$Subalgebra . Then μ_{A_i} is a fuzzy dot $BF-$Subalgebra of X for each $i \in I$.
We have to show $\mu_{\cap_{i \in I} A_i}$ is a fuzzy dot $BF-$Subalgebra of X.
Consider $\mu_{\cap_{i \in I} A_i}(a \star b) = min\{\mu_{A_i}(x \star y), \mu_{A_j}(x \star y)\}$
$\geq min\{\mu_{A_i}(x).\mu_{A_i}(y), \mu_{A_j}(x).\mu_{A_j}(y)\}$
$\geq min\{min\{\mu_{A_i}(x), \mu_{A_j}(x)\}.min\{\mu_{A_i}(y), \mu_{A_j}(y)\}\}$
$\geq \mu_{\cap_{i \in I} A_i}(x).\mu_{\cap_{i \in I} A_i}(y)$.
Hence $\mu_{\cap_{i \in I} A_i}(x \star y) \geq \mu_{\cap_{i \in I} A_i}(x).\mu_{\cap_{i \in I} A_i}(y)$.□

Definition 3.5: Let A be a fuzzy set in X and $\theta \in [0, 1]$. Then the level $BF-$ Subalgebra of $U(A, \theta)$ of A and strong level $BF-$Subalgebra $U(A, >, \theta)$ of X are defined as follows:
$U(A, \theta) = \{x \in X | \mu_A(x) \geq \theta\}$.
$U(A, >, \theta) = \{x \in X | \mu_A(x) > \theta\}$.

Theorem 3.4:

Let A be a non-empty subset of X.Then A is sub algebra of X if and only if X_A is fuzzy dot sub algebra of X.

Proof. Let A be sub algebra of X and $x, y \in A$.Then $x, y \in A$, Then we have
$X_A(x * y) \geq X_A(x).X_A(y)$
since $1 \geq (1).(1) = 1 \geq 1$, because $x, y \in A$ then $x * y \in A$
If $x \in A$ and $y \notin A$ then we get $X_A(x) = 1$ or $X_A(y) = 0$
$X_A(x * y) \geq \mu_A(x).\mu_A(y)$
$1 \geq (1).(0) = 1 \geq 0$, because A is subset of X and $x, y \in X$.
If $x \notin A$ and $y \in A$ then we get $X_A(x) = 0$ or $X_A(y) = 1$
$X_A(x * y) \geq X_A(x).X_A(y)$
$1 \geq (0).(1) \geq 1$, because A is subset of X and $x \in X$.
Conversely that X_A is a fuzzy dot sub algebra of X and let $x, y \in A$. Then
$X_A(x * y) \geq X_A(x).X_A(y)$

$1 \geq X_A(x * y) \geq 1$
$X_A(x * y) = 1$. Then ,$x * y \in A$. Hence A is a sub algebra of X.□

Theorem 3.5: Let A be a fuzzy set in a fuzzy $BF-$Sub algebra of X and μ_A be a fuzzy dot $BF-$sub algebra of X with least upper bound $\lambda_0 \in [0, 1]$. Then the following are equivalent.

1) μ_A is a fuzzy dot $BF-$subalgebra of X.
2) For all $\lambda \in Im(\mu_A)$, the nonempty level subset $U(A, \lambda)$ of A is a $BF-$subalgebra of X.
3) For all $\lambda \in Im(\mu_A)/\lambda_0$, the non empty strong level subset $U(A, >, \lambda)$ of A is a $BF-$ subalgebra of X.
4) For all $\lambda \in [0, 1]$, the nonempty strong level subset $U(A, \geq, \lambda)$ of A is a $BF-$subalgebra of X.
5) For all $\lambda \in [0, 1]$, the nonempty level subset $U(A, \lambda)$ of A is a $BF-$sub algebra of X.

Proof.

1) $1 \Rightarrow 4$, let A be a fuzzy set in a $BF-$sub algebra and μ_A be a fuzzy dot $BF-$Sub algebra of X,$A \in [0, 1]$ and let $x, y \in U(A, >, \lambda)$. Then $\mu_A(x \star y) \geq \mu_A(x).\mu_A(y) > \lambda$. Imply that $x \star y \in U(A, >, \lambda)$.
Hence $U(A, >, \lambda)$ ie a $BF-$ sub algebra of X.
2) $4 \Rightarrow 3$ for each $\lambda \in [0, 1]$,$U(A, >, \lambda)$ be a $BF-$sub algebra of X.
Let$\lambda \in Im\mu_A/\lambda_0$ and $x, y \in U(S, >, \lambda)$. Then $\mu_A(a \star b) \geq \mu_A(x).\mu_A(y) > \lambda$ by (4).
We have $x \star y \in U(A, >, \lambda)$. Hence $U(A, >, \lambda)$ is a $BF-$sub algebra of X.
3) $3 \Rightarrow 2$ let $\lambda_0 \in Im(\mu_A)$. Then $U(A, \lambda)$ is nonempty since $U(A, \lambda) = \cap_{\lambda > \beta} U(A, >, \lambda)$, where $\beta \in Im(\mu_A)/\lambda_0$. Then by (3) $U(A, \lambda)$ is a $BF-$ sub algebra of X.
4) $2 \Rightarrow 5$ Let $\lambda \in [0, 1]$ and $U(A, \lambda)$ be nonempty. Suppose $x, y \in U(A, \lambda)$ and let $\alpha = min\{\mu_A(x), \mu_A(y)\}$, since $\mu_A(x) \geq \lambda$ and $\mu_A > \lambda$. Imply that $\alpha \geq \lambda$, where $\lambda = \mu_A(x).\mu_A(y)$.
Thus $x, y \in U(A, \alpha)$ and $\alpha \in Im\mu_A$ by 2 $U(A, \alpha)$ is a $BF-$sub algebra of X. Hence $\mu_A(x \star y) \geq \mu_A(x).\mu_A(y) = \lambda$.
Thus $x \star y \in U(A, \alpha)$. Then we have $x \star y \in U(A, \lambda)$. Hence $U(A, \lambda)$ is a $BF-$sub algebra of X.
5) Assume that the non-empty set $U(A, \lambda)$ is a $BF-$sub algebra of X for every $\lambda \in [0, 1]$.
Suppose $x_0, y_0 \in X$such that $\mu_A(x_0 \star y_0 < \mu_A(x_0).\mu_A(y_0)$. Put $\mu_A(x_0) = \beta, \mu_A(y_0) = \delta$ and $\mu_A(x_0 \star y_0) = \theta$. Then $\theta < min\{\beta, \delta\}$.
Consider $\theta_1 = \frac{1}{2}(\mu_A(x_0 \star y_0) + \mu_A(x_0).\mu_A(y_0)$, we get $\theta_1 = \frac{1}{2}(\theta + \beta.\delta)$.
Therefore $\beta > \theta_1 = \frac{1}{2}(\theta + \beta.\delta) > \theta$.
$\delta > \theta_1 = \frac{1}{2}(\theta + \beta.\delta) > \theta$.
Hence $\beta.\delta > \theta_1 > \theta = \mu_A(x_0 \star y_0)$.
Hence $x_0 \star y_0 \notin U(A, \theta)$ which is a contradiction since $\mu_A(x_0) = \beta \geq \beta.\delta > \theta_1$.
$\mu_A(b_0) = \delta \geq \beta.\delta > \theta_1$. Imply that $x_0, y_0 \in U(A, \theta)$.
Thus $\mu_A(x \star y) \geq \mu_A(x).\mu_A(y)$, for all $x, y \in X$ which completes the proof.

Theorem 3.6: Each BF- Sub algebra of X is a level $BF-$ sub algebra of a fuzzy dot $BF-$ sub algebra of X.

Proof Let B be a $BF-$sub algebra of X and A be any fuzzy set in X defined by$\mu_A(x) = \begin{cases} \gamma & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$, where $\gamma \in [0, 1]$.
Let $y \in U(A, \gamma)$. Then $\mu_A(y) \geq \gamma$, for all $y \in A$. So that $y \in B$.
Hence $U(A, \gamma) \subseteq B$.
Let $z \in B$. Then $\mu_A(z) = \gamma$. Imply that $\mu_A(z) \geq \gamma$. Hence $z \in U(A, \gamma)$.
Thus $B \subseteq U(A, \gamma)$. It follows that $U(A, \gamma) = B$.
We consider the following cases:

1) If $x, y \in B$, then $\mu_A(x \star y) = \gamma$
$\gamma.\gamma = \gamma^2 = \mu_A(x).\mu_A(y)$. Hence $x \star y \in B$.
2) If $x, y \notin B$, then $\mu_A(x) = 0 = \mu_A(y)$. So that $\mu_A(x \star y) \geq 0 = min\{0, 0\} = min\{\mu_A(x), \mu_A(y)\} \geq \mu_A(x).\mu_A(y)$.
Hence $\mu_A(x \star y) \geq \mu_A(x).\mu_A(y)$.
3) If $x \in B$and $y \notin B$, then $\mu_A(x) = \gamma$ and $\mu_A(y) = 0$.
Thus $\mu_A(x \star y) \geq 0 = min\{\gamma, 0\} = min\{\mu_A(x), \mu_A(y)\}$
$\geq \mu_A(x).\mu_A(y)$.
Hence $\mu_A(x \star y) \geq \mu_A(x).\mu_A(y)$.
4) If $y \in B$ and $x \notin B$, then by the same argument as in case 3, we can conclude that $\mu_A(x \star y) \geq 0 = min\{0, \gamma\} = min\{\mu_A(x), \mu_A(y)\}$
$\geq \mu_A(x).\mu_A(y)$. Hence μ_A is a fuzzy dot $BF-$ subalgebra of X.

□

Theorem 3.7: Let B be a subset of X and A be a fuzzy set on X which is given by the proof of theorem 5.6 If μ_A is a fuzzy dot $BF-$sub algebra of X, then B is a $BF-$sub algebra of X.

Theorem 3.8: Let A be a fuzzy set in $BF-$ sub algebra X. If μ_A is a fuzzy dot $BF-$sub algebra of X, then the set $X_{\mu_A} = \{x \in X/\mu_A(x) = \mu_A(0)\}$ is a $BF-$sub algebra of X.

Proof. Let $x, y \in X_{\mu_A}$. Then $\mu_A(x) = \mu_A(0) = \mu_A(y)$ and so $\mu_A(x \star y) \geq min\{\mu_A(x), \mu_A(y)\}$
$= min\{\mu_A(0), \mu_A(0)\} \geq \mu_A(0).\mu_A(0)$.But $\mu_A(0) \geq \mu_A(0).\mu_A(0)$. Hence $\mu_A(x \star y) \geq \mu_A(0)$ and $\mu_A(0) \geq$

$\mu_A(x \star y)$.
Thus $\mu_A(x \star y) = \mu_A(0)$ which means that $x \star y \in X_{\mu_A}$.

Definition 3.6: Let $A = \{< x, \mu_A(x) >: x \in X\}$ and $B = \{< x, \mu_B(x) >: x \in X\}$ be two fuzzy sets in X.The Cartesian product $A \times B : X \times X \longrightarrow [0, 1]$ is defined by
$(\mu_A \times \mu_B)(x, y) = \mu_A(x).\mu_B(y)$ for all $x, y \in X$.

Proposition 3.7: Let A and B be two fuzzy dot sub algebras of X,then $A \times B$ is a fuzzy dot sub algebra of $X \times X$.

Proof. Let (x_1, y_1) and $(x_2, y_2) \in X \times X$ then
$(\mu_A \times \mu_B)((x_1, y_1) * (x_2, y_2)) = (\mu_A \times \mu_B)((x_1 * x_2), (y_1 * y_2)$
$(\mu_A \times \mu_B)((x_1 * x_2), (y_1 * y_2) = \mu_A(x_1 * x_2).\mu_B(y_1 * y_2)$
$\mu_A(x_1 * x_2).\mu_B(y_1 * y_2) \geq (\mu_A(x_1).\mu_A(x_2)).(\mu_B(y_1).\mu_B(y_2))$
$(\mu_A(x_1).\mu_A(x_2)).(\mu_B(y_1).\mu_B(y_2)) = (\mu_A \times \mu_B)(x_1, y_1).(\mu_A \times \mu_B)(x_2, y_2)$
Hence,$A \times B$ is fuzzy dot sub algebra of $X \times X$. □

Proposition 3.8: Let A and B be two fuzzy dot ideals of X, then $A \times B$ is a fuzzy dot ideals of $X \times X$.

Proof. Let (x_1, y_1) and (X_2, y_2), then
$(\mu_A \times \mu_B)(x_1, y_1) = \mu_A(x_1).\mu_B(y_1)$
$\mu_A(x_1).\mu_B(y_1) \geq (\mu_A(x_1 * x_2).\mu_A(x_2)).(\mu_B(y_1 * y_2).\mu_B(y_2))$
$= (\mu_A(x_1 * x_2).\mu_B(y_1 * y_2)).(\mu_A(x_2).\mu_B(y_2))$
$= (\mu_A \times \mu_B)(x_1 * x_2, y_1 * y_2).(\mu_A \times \mu_B)(x_2, y_2)$
$= (\mu_A \times \mu_B)((x_1, y_1) * (x_2, y_2)).(\mu_A \times \mu_B)(x_2, y_2)$
Hence $A \times B$ is a fuzzy dot ideal of $X \times X$.□

B. *Fuzzy ρ-dot product Relation of BF-Algebra*

In this section, strongest fuzzy ρ-relation and fuzzy ρ-product relation of BF-algebras are defined and presented some of its properties.

Definition 3.9: Let ρ be a fuzzy sub set of X. The strongest fuzzy ρ-relation on X is the fuzzy subset μ_ρ of $X \times X$ given by $\mu_\rho(x, y) = \rho(x).\rho(y)$ for $x, y \in X$

Theorem 3.9: Let μ_ρ be the strongest fuzzy ρ-relation on X, where ρ is a sub set of X, If ρ is a fuzzy dot sub algebra of X, then μ_ρ is a fuzzy dot sub algebra of $X \times X$.

Proof. Suppose that ρ is fuzzy dot sub algebra of X.
For any (x_1, y_1) and $(x_2, y_2) \in X \times X$
We have $(\mu_\rho(x_1, y_1) * (x_2, y_2)) = \mu_\rho(x_1 * x_2).\mu_\rho(y_1 * y_2)$
By the definition of strongest fuzzy ρ-relation of BF-algebras we get
$mu_\rho(x_1 * x_2).\mu_\rho(y_1 * y_2) = \rho(x_1 * x_2).\rho(y_1 * y_2)$
$\rho(x_1 * x_2).\rho(y_1 * y_2) \geq \rho(x_1).\rho(x_2).\rho(y_1).\rho(y_2)$ (since a fuzzy dot sub algebra of)
$\rho(x_1).\rho(x_2).\rho(y_1).\rho(y_2) = (\rho(x_1).\rho(y_1)).(\rho(x_2).\rho(y_2))$
$(\rho(x_1).\rho(y_1)).(\rho(x_2).\rho(y_2)) = \mu_\rho(x_1, y_1).\mu_\rho(x_2, y_2)$
Hence,μ_ρ is fuzzy dot sub algebra of $X * X$.□

Definition 3.10: Let ρ be a fuzzy subset of X. A fuzzy relation μ on X is called a fuzzy ρ-product relation $\mu_\rho(x, y) \geq \rho(x).\rho(y)$ for $x, y \in X$
Let ρ be a fuzzy subset of X,A fuzzy relation μ on X is called a left fuzzy relation$\mu_\rho(x, y) = \rho(x)$ for $x, y \in X$
Similarly we can define a right fuzzy relation on ρ,$\mu_\rho(x, y) = \rho(y)$ for $x, y \in X$.
Note that a left (respectively, right) fuzzy relation on is a ρ-product relation.

Theorem 3.10: Let μ be a left fuzzy relation on a fuzzy subset μ of X.If μ is fuzzy dot sub algebra of $X \times X$,then rho is fuzzy dot sub algebra of X.

Theorem 3.11:

Let μ be a fuzzy relation on X satisfying the inequality $\mu(x, y) \leq \mu(x, 0)$ for all $x, y \in X$, Given $s \in S$, let ρ_s be a fuzzy subset of X,defined by $\rho_s(x) = \mu(x, s)$, for all $x \in X$. If μ is a fuzzy dot sub algebra of $X \times X$, then ρ_s is a fuzzy dot sub algebra of X,for all $s \in X$.

Proof. Let $x, y, s \in X$, Then
$\rho_s(x * y) = \mu(x * y, s)$
$\mu(x * y, s) = \mu(x * y, s * 0)$
$\mu(x * y, s * 0) = \mu(x, s) * (y, 0)$
$\mu(x, s) * (y, 0) \geq \mu(x, s).\mu(y, 0)$
$\mu(x, s).\mu(y, 0) \geq \mu(x, s).\mu(y, s)$
$\mu(x, s).\mu(y, s) = \rho_s(x).\rho_s(y)$
Therefore,ρ_s is fuzzy dot sub algebra of X.□

Theorem 3.12:

Let μ be a fuzzy relation on X and let ρ_μ be a fuzzy sub set of X given by $\rho_\mu(x) = inf_{y \in X}\{\mu(x, y), \mu(y, x)\}$ for all $x \in X$. If μ is a fuzzy dot sub algebra $X \times X$ satisfying the equality $\mu(x, 0) = 1 = \mu(0, x)$ for all $x \in$, then ρ_μ is a fuzzy dot sub algebra of X.

Proof. Let $x, y, z \in X$, we have
$\mu(x * y, z) = \mu(x * y, z * 0)$
$\mu(x * y, z * 0) = \mu((x, z) * (y, 0))$
$\mu((x, z) * (y, 0)) \geq \mu(x, z).\mu(y, 0)$
$\mu(x, z).\mu(y, 0) = \mu(x, z)$,
$\mu(z, x * y) = \mu(z * 0, x * y)$
$\mu(z * 0, x * y) = \mu((z, x) * (0, y))$
$\mu((z, x) * (0, y)) \geq \mu(z, x).\mu(0, y)$
$\mu(z, x).\mu(0, y) = \mu(z, x)$, It follows that $\mu(x * y, z).\mu(z, x * y) \geq \mu(x, z).\mu(z, x)$
$\mu(x, z).\mu(z, x) \geq (\mu(x, z).\mu(z, x)).(\mu(y, z).\mu(z, y))$
So that $\rho_\mu(x * y) = inf_{z \in X}\mu(x * y, z).\mu(z, x * y)$
$inf_{z \in X}\mu(x * y, z).\mu(z, x * y) = (inf_{z \in X}\mu(x, z).\mu(z, x)).(inf_{z \in X}\mu(y, z).\mu(z, y))$
$(inf_{z \in X}\mu(x, z).\mu(z, x)).(inf_{z \in X}\mu(y, z).\mu(z, y)) = \rho_\mu(x)\rho_\mu(y)$
Therefore ρ_μ is fuzzy dot sub algebra of X.□

Proposition 3.11: Let f be a $BF-$ homomorphism from X into Y and G be a fuzzy dot $BF-$subalgebra of Y with the membership function μ_G. Then the inverse image $f^{-1}(G)$ of G is a fuzzy dot $BF-$ subalgebra of X.

Proof. Let $f : X \to Y$ and G be a fuzzy dot $BF-$subalgebra of Y and let $x, y \in X$. Then

$$\begin{aligned}\mu_{f^{-1}(G)}(x \star y) &= \mu_G(f(x \star y))\\ &= \mu_G(f(x) \star f(y))\\ &\geq min\{\mu_G(f(x)), \mu_G(f(y))\}\\ &\geq \mu_G(f(x)).\mu_G(f(y))\\ &= \mu_{f^{-1}(G)}(x).\mu_{f^{-1}(G)}(y).\end{aligned}$$

$$Hence \quad f^{-1}(G) is a fuzzy dot BF-subalgebra of X.$$

□

Proposition 3.12: Let f be a $BF-$homomorphism from X onto Y and D be a fuzzy $BF-$subalgebra of X with the suppropery. Then the image $f(D)$ of D is a fuzzy dot $BF-$subalgebra of Y .

Proof. Let f be a $BF-$homomorphism fron X into Y and let D be a fuzzy dot $BF-$subalgebra of Y with suppropperty and let $a, b \in Y$, let $x_0 \in f^{-1}(a), y_0 \in f^{-1}(b)$ such that $\mu_D(x_0) = sup \mu_D(t)_{t\in f^{-1}(a)}$,$\mu_D(y_0) = sup\mu_D(t)_{t\in f^{-1}(b)}$. Then by the definition $\mu_{f(D)}$, we have

$$\begin{aligned}\mu_{f(D)}(x \star y) &= sup\mu_D(t)_{t\in f^{-1}(a\star b)}\\ &\geq \mu_D(x_0 \star y_0)\\ &\geq min\{\mu_D(x_0), \mu_D(y_0)\}\\ &\geq \mu_D(x_0).\mu_D(y_0).\end{aligned}$$

$$Consequently \mu_{f(D)}(x \star y) \geq \mu_D(x_0).\mu_D(y_0)$$

$$= sup\mu_D(t)_{t\in f^{-1}(a)}.sup\mu_D(t)_{t\in f^{-1}(b)} = \mu_{f(D)}(a).\mu_{f(D)}(b).$$

$$Hence \quad \mu_{f(D)}(x \star y) \geq \mu_{f(D)}(a).\mu_{f(D)}(b).$$

Thus f(D) is a fuzzy dot BF-subalgebra of X. □

IV. CONCLUSION

The concepts of structure of fuzzy dot BF-Sub algebras has been introduced. The fuzzy dot product of BF-sub algebra,Fuzzy dot ideal of BF-sub algebra , ρ-product of Fuzzy BF-subalgebras have been discussed. Finally different characterization of Fuzzy dot BF-subalgebras have been investigated.

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