

Structure of fuzzy dot BF-subalgebras

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Abstract—The concept of fuzzy dot BF-subalgebra has been introduced. Fuzzy dot product of BF-subalgebras and strong fuzzy product in a fuzzy dot BF-subalgebra has been discussed.

Different theorems, Lemmas and propositions has been proved. We have also investigated different characterizations.

Index Terms—BF-subalgebra, fuzzy BF-subalgebras, Ideal in BF-subalgebras, Fuzzy dot BF-subalgebras.

I. INTRODUCTION

In [8] the idea of fuzzy set was first introduced by Zadeh and the concept of B-algebras was introduced by Neggers and Kim. They defined a B-algebra as an algebra $(X, *, 0)$ of type (2,0) satisfying the following axioms:

- 1) $x * x = 0$.
- 2) $x * 0 = x$.
- 3) $(x * y) * z = x * (z * (0 * y))$.

Andrzej Walendziak initiated the idea of a BF-algebra and characterized different structures in [1] and in [3] Borumand Saeid and Rezrani investigated Fuzzy BF-Algebras.

In this paper, we introduced the concept of fuzzy dot BF-algebras and study its structures. We state and prove some theorem discussed in fuzzy dot BF-subalgebras and level subalgebras. Finally some of results on homomorphic images and inverse images in fuzzy dot BF-subalgebras are investigated.

II. PRELIMINARIES

A BF-algebra is a nonempty set X with a constant '0' and a binary operation '*' satisfying the following conditions:

- 1) $x * x = 0$.
- 2) $x * 0 = x$.
- 3) $0 * (x * y) = (y * x)$ For all $x, y \in X$.

Example 2.1: Let $X = [0, \infty)$. Define the binary operation '*' on X as follows: $x * y = |x - y|$, for all $x, y \in X$, Then $(X, *, 0)$ is a BF-algebra.

Let X be a BF-algebra. Then for any x and y in X , the following hold:

- 1) $0 * (0 * x) = x$.
- 2) if $0 * x = 0 * y$, then $x = y$.
- 3) if $x * y = 0$, then $y * x = 0$.

Any BF-algebra $(X, *, 0)$ that satisfies the identity $(x * z) * (y * z) = x * y$ is a B-algebra. A subset I of X is called an ideal of X if it satisfies:

- 1) $0 \in I$.
- 2) $x * y$ and $y \in I$ imply $x \in I$ for any $x, y \in X$.

A nonempty subset S of X is called a sub algebra of A if $x * y \in S$ for any $x, y \in X$.

Let $(X, *, 0_X)$ and $(Y, *, 0_Y)$ be BF-algebras. A mapping $\phi : X \rightarrow Y$ is called a homomorphism from X into Y if $\phi(x * y) = \phi(x) * \phi(y)$ for any $x, y \in X$.

Let X be a nonempty set. A fuzzy (sub) set μ of the set X is a mapping $\mu : X \rightarrow [0, 1]$.

and μ is the fuzzy set of a set X . For a fixed $s \in [0, 1]$, the set $\mu_s = \{x \in X : \mu(x) \geq s\}$ is called an upper level of μ or level subset of μ . Furthermore let X be a set. A fuzzy set A in X is characterized by a membership function $\mu_A : X \rightarrow [0, 1]$. Let f be a mapping from the set X to the set Y and let B be a fuzzy set in Y with membership function μ_B .

The inverse image of B , denoted $f^{-1}(B)$, is the fuzzy set in X with membership function $\mu_{f^{-1}B}$ defined by $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ for all $x \in X$. Conversely, let A be a fuzzy set in X with membership function μ_A . Then the image of A , denoted by $f(A)$, is the fuzzy set in Y such that $\mu_{f(A)}(y) = \begin{cases} \sup \mu_A(x)_{x \in f^{-1}(y)} & \text{if } f^{-1}(y) = \{x : f(x) = y\} \\ 0 & \text{if otherwise} \end{cases}$.

A fuzzy set A in the BF-algebra X with the membership function μ_A is said to have the sup property if for any subset $T \subseteq X$ there exists $x_0 \in T$ such that $\mu_A(x_0) = \sup \mu_A(t)_{t \in T}$; and let μ be a fuzzy set in a BF-algebra. Then μ is called a fuzzy BF-subalgebra (algebra) of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

A fuzzy set μ of a BF-algebra X is called a fuzzy ideal of X if it satisfies the following conditions.

- 1) $\mu(0) \geq \mu(x)$
- 2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$

III. RESULTS

A. Fuzzy dot BF-Sub algebra

Definition 3.1: Let A be a fuzzy set in BF-Subalgebra X and μ be a fuzzy subset a BF-Subalgebra of X . Then the fuzzy dot BF-Subalgebra of X is $\mu_A(x * y) \geq \mu_A(x) \cdot \mu_A(y)$, for all $x, y \in X$.

Example 3.2: Let $X = \{0, a, b\}$ be a set with the table given below

*	0	a	b
0	0	a	b
a	a	0	0
b	b	0	0

. Then $(X, *, 0)$ is a BF- algebra. Define $\mu : X \rightarrow [0, 1]$ by $\mu(0) = 0.8, \mu(a) = 0.2$, and $\mu(b) = 0.4$. Then μ is a fuzzy dot BF- Subalgebra of X . Since $\mu(a * b) = \mu(0) = 0.8 \geq (0.2)(0.4) = 0.08 = \mu(a) \cdot \mu(b)$. Hence $\mu(a * b) \geq \mu(a) \cdot \mu(b)$, for all $a, b \in X$.

Lemma 3.3: Let A be a fuzzy set in BF- algebra. If μ_A is a fuzzy dot BF-Subalgebra of X , then for all $x \in X, \mu(0) \geq (\mu(x))^2$, for all $x \in X$.

Corollary 3.1: If A is a fuzzy subset of a BF-algebra X and μ_A is a a fuzzy dot subalgebra of X , then $\mu_A(0^n * x) \geq (\mu_A(x))^{2n+1}$, for all $x \in X$ and $n \in N$, where $0^n * x = 0 * (0 * (0 * \dots(0 * x) \dots))$ in which 0 occurs n -times.

Proof. By Lemma 3.3. we have $\mu_A(0) \geq (\mu_A(x))^2$, for all $x \in X$. Put $n = 1$ in $0^n * a$ we have $0 * x$.
 $\mu_A(0 * x) \geq \mu_A(0) \cdot \mu_A(x) \geq (\mu_A(x))^2 \cdot \mu_A(x) = (\mu_A(x))^3$.
 Hence $\mu_A(0 * x) \geq (\mu_A(x))^3$, for all $x \in X$.
 Assume the result holds for $n = k, \mu_A(0^k * x) \geq (\mu_A(x))^{2k+1}$, for all $x \in X$.

$$\begin{aligned} \text{Now } \mu_A(0^{k+1} * x) &= \mu_A(0 * (0^k * x)) \\ &\geq \mu(0) \cdot \mu(0^k * x) \\ &\geq (\mu_A(x))^2 \cdot (\mu_A(x))^{2k+1} \\ &= (\mu_A(x))^{2(k+1)+1} \end{aligned}$$

Hence $\mu_A(0^n * x) \geq (\mu_A(x))^{2n+1}$, for all $x \in X$ and $n \in N$. \square

Theorem 3.2: Let A be a fuzzy subset of BF-Subalgebra and let μ_A be a fuzzy dot BF-Subalgebra of X . If there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} \mu_A(x_n) = 1$. then $\mu_A(0) = 1$.

Theorem 3.3: Let A_1 and A_2 be fuzzy subsets of BF-Subalgebras and let μ_{A_1} and μ_{A_2} be fuzzy dot BF- Subalgebras of X . Then $\mu_{A_1 \cap A_2}$ is a fuzzy dot subalgebras of X .

Proof. Let $x, y \in A_1 \cap A_2$. Then $x, y \in A_1$ and $x, y \in A_2$, since A_1 and A_2 are subsets of fuzzy BF-Subalgebra of X .

$$\begin{aligned} \mu_{A_1 \cap A_2}(x * y) &= \min\{\mu_{A_1}(x * y), \mu_{A_2}(x * y)\} \\ &\geq \min\{\mu_{A_1}(x) \cdot \mu_{A_1}(y), \mu_{A_2}(x) \cdot \mu_{A_2}(y)\} \\ &\geq \mu_{A_1 \cap A_2}(x) \cdot \mu_{A_1 \cap A_2}(y). \end{aligned}$$

Hence $\mu_{A_1 \cap A_2}(x * y) \geq \min\{\min\{\mu_{A_1}(x), \mu_{A_2}(x)\} \cdot \min\{\mu_{A_1}(y), \mu_{A_2}(y)\}\} \geq \mu_{A_1 \cap A_2}(x) \cdot \mu_{A_1 \cap A_2}(y)$. \square

Corollary 3.4: Let $\{A_i | i \in I\}$ be a family of fuzzy dot BF-Subalgebras of X . Then $\cap_{i \in I} A_i$ is also a fuzzy dot BF-Subalgebras of X .

Proof. Let $\{A_i | i \in I\}$ be a family of fuzzy dot BF-Subalgebra. Then μ_{A_i} is a fuzzy dot BF-Subalgebra of X for each $i \in I$.

We have to show $\mu_{\cap_{i \in I} A_i}$ is a fuzzy dot BF-Subalgebra of X .

$$\begin{aligned} \text{Consider } \mu_{\cap_{i \in I} A_i}(a * b) &= \min\{\mu_{A_i}(a * b), \mu_{A_j}(a * b)\} \\ &\geq \min\{\mu_{A_i}(a) \cdot \mu_{A_i}(b), \mu_{A_j}(a) \cdot \mu_{A_j}(b)\} \\ &\geq \min\{\min\{\mu_{A_i}(a), \mu_{A_j}(a)\} \cdot \min\{\mu_{A_i}(b), \mu_{A_j}(b)\}\} \\ &\geq \mu_{\cap_{i \in I} A_i}(a) \cdot \mu_{\cap_{i \in I} A_i}(b). \end{aligned}$$

Hence $\mu_{\cap_{i \in I} A_i}(a * b) \geq \mu_{\cap_{i \in I} A_i}(a) \cdot \mu_{\cap_{i \in I} A_i}(b)$. \square

Definition 3.5: Let A be a fuzzy set in X and $\theta \in [0, 1]$. Then the level BF- Subalgebra of $U(A, \theta)$ of A and strong level BF-Subalgebra $U(A, >, \theta)$ of X are defined as follows:

$$\begin{aligned} U(A, \theta) &= \{x \in X | \mu_A(x) \geq \theta\}. \\ U(A, >, \theta) &= \{x \in X | \mu_A(x) > \theta\}. \end{aligned}$$

Theorem 3.4:

Let A be a non-empty subset of X . Then A is sub algebra of X if and only if X_A is fuzzy dot sub algebra of X .

Proof. Let A be sub algebra of X and $x, y \in A$. Then $x, y \in A$, Then we have

$$X_A(x * y) \geq X_A(x) \cdot X_A(y)$$

since $1 \geq (1) \cdot (1) = 1 \geq 1$, because $x, y \in A$ then $x * y \in A$

If $x \in A$ and $y \notin A$ then we get $X_A(x) = 1$ or $X_A(y) = 0$

$$\begin{aligned} X_A(x * y) &\geq \mu_A(x) \cdot \mu_A(y) \\ 1 \geq (1) \cdot (0) &= 1 \geq 0, \text{ because } A \text{ is subset of } X \text{ and } x, y \in X. \end{aligned}$$

If $x \notin A$ and $y \in A$ then we get $X_A(x) = 0$ or $X_A(y) = 1$

$$\begin{aligned} X_A(x * y) &\geq X_A(x) \cdot X_A(y) \\ 1 \geq (0) \cdot (1) &\geq 1, \text{ because } A \text{ is subset of } X \text{ and } x \in X. \end{aligned}$$

Conversely that X_A is a fuzzy dot sub algebra of X and let $x, y \in A$. Then

$$X_A(x * y) \geq X_A(x) \cdot X_A(y)$$

$1 \geq X_A(x * y) \geq 1$
 $X_A(x * y) = 1$. Then $x * y \in A$. Hence A is a sub algebra of X . \square

Theorem 3.5: Let A be a fuzzy set in a fuzzy BF -Sub algebra of X and μ_A be a fuzzy dot BF -sub algebra of X with least upper bound $\lambda_0 \in [0, 1]$. Then the following are equivalent.

- 1) μ_A is a fuzzy dot BF -subalgebra of X .
- 2) For all $\lambda \in Im(\mu_A)$, the nonempty level subset $U(A, \lambda)$ of A is a BF -subalgebra of X .
- 3) For all $\lambda \in Im(\mu_A)/\lambda_0$, the non empty strong level subset $U(A, >, \lambda)$ of A is a BF -subalgebra of X .
- 4) For all $\lambda \in [0, 1]$, the nonempty strong level subset $U(A, \geq, \lambda)$ of A is a BF -subalgebra of X .
- 5) For all $\lambda \in [0, 1]$, the nonempty level subset $U(A, \lambda)$ of A is a BF -sub algebra of X .

Proof.

- 1) $1 \Rightarrow 4$, let A be a fuzzy set in a BF -sub algebra and μ_A be a fuzzy dot BF -Sub algebra of $X, A \in [0, 1]$ and let $x, y \in U(A, >, \lambda)$. Then $\mu_A(x * y) \geq \mu_A(x) \cdot \mu_A(y) > \lambda$. Imply that $x * y \in U(A, >, \lambda)$. Hence $U(A, >, \lambda)$ is a BF -sub algebra of X .
- 2) $4 \Rightarrow 3$ for each $\lambda \in [0, 1], U(A, >, \lambda)$ be a BF -sub algebra of X . Let $\lambda \in Im \mu_A / \lambda_0$ and $x, y \in U(S, >, \lambda)$. Then $\mu_A(a * b) \geq \mu_A(x) \cdot \mu_A(y) > \lambda$ by (4). We have $x * y \in U(A, >, \lambda)$. Hence $U(A, >, \lambda)$ is a BF -sub algebra of X .
- 3) $3 \Rightarrow 2$ let $\lambda_0 \in Im(\mu_A)$. Then $U(A, \lambda)$ is non-empty since $U(A, \lambda) = \cap_{\lambda > \beta} U(A, >, \lambda)$, where $\beta \in Im(\mu_A) / \lambda_0$. Then by (3) $U(A, \lambda)$ is a BF -sub algebra of X .
- 4) $2 \Rightarrow 5$ Let $\lambda \in [0, 1]$ and $U(A, \lambda)$ be nonempty. Suppose $x, y \in U(A, \lambda)$ and let $\alpha = \min\{\mu_A(x), \mu_A(y)\}$, since $\mu_A(x) \geq \lambda$ and $\mu_A > \lambda$. Imply that $\alpha \geq \lambda$, where $\lambda = \mu_A(x) \cdot \mu_A(y)$. Thus $x, y \in U(A, \alpha)$ and $\alpha \in Im \mu_A$ by 2 $U(A, \alpha)$ is a BF -sub algebra of X . Hence $\mu_A(x * y) \geq \mu_A(x) \cdot \mu_A(y) = \lambda$. Thus $x * y \in U(A, \alpha)$. Then we have $x * y \in U(A, \lambda)$. Hence $U(A, \lambda)$ is a BF -sub algebra of X .
- 5) Assume that the non-empty set $U(A, \lambda)$ is a BF -sub algebra of X for every $\lambda \in [0, 1]$. Suppose $x_0, y_0 \in X$ such that $\mu_A(x_0 * y_0) < \mu_A(x_0) \cdot \mu_A(y_0)$. Put $\mu_A(x_0) = \beta, \mu_A(y_0) = \delta$ and $\mu_A(x_0 * y_0) = \theta$. Then $\theta < \min\{\beta, \delta\}$. Consider $\theta_1 = \frac{1}{2}(\mu_A(x_0 * y_0) + \mu_A(x_0) \cdot \mu_A(y_0))$, we get $\theta_1 = \frac{1}{2}(\theta + \beta \cdot \delta)$. Therefore $\beta > \theta_1 = \frac{1}{2}(\theta + \beta \cdot \delta) > \theta$.

$\delta > \theta_1 = \frac{1}{2}(\theta + \beta \cdot \delta) > \theta$.
 Hence $\beta \cdot \delta > \theta_1 > \theta = \mu_A(x_0 * y_0)$.
 Hence $x_0 * y_0 \notin U(A, \theta)$ which is a contradiction since $\mu_A(x_0) = \beta \geq \beta \cdot \delta > \theta_1$.
 $\mu_A(b_0) = \delta \geq \beta \cdot \delta > \theta_1$. Imply that $x_0, y_0 \in U(A, \theta)$.
 Thus $\mu_A(x * y) \geq \mu_A(x) \cdot \mu_A(y)$, for all $x, y \in X$ which completes the proof.

Theorem 3.6: Each BF -Sub algebra of X is a level BF -sub algebra of a fuzzy dot BF -sub algebra of X .

Proof Let B be a BF -sub algebra of X and A be any fuzzy set in X defined by $\mu_A(x) = \begin{cases} \gamma & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$, where $\gamma \in [0, 1]$.

Let $y \in U(A, \gamma)$. Then $\mu_A(y) \geq \gamma$, for all $y \in A$. So that $y \in B$.

Hence $U(A, \gamma) \subseteq B$.

Let $z \in B$. Then $\mu_A(z) = \gamma$. Imply that $\mu_A(z) \geq \gamma$. Hence $z \in U(A, \gamma)$.

Thus $B \subseteq U(A, \gamma)$. It follows that $U(A, \gamma) = B$.

We consider the following cases:

- 1) If $x, y \in B$, then $\mu_A(x * y) = \gamma$
 $\gamma \cdot \gamma = \gamma^2 = \mu_A(x) \cdot \mu_A(y)$. Hence $x * y \in B$.
- 2) If $x, y \notin B$, then $\mu_A(x) = 0 = \mu_A(y)$. So that $\mu_A(x * y) \geq 0 = \min\{0, 0\} = \min\{\mu_A(x), \mu_A(y)\} \geq \mu_A(x) \cdot \mu_A(y)$. Hence $\mu_A(x * y) \geq \mu_A(x) \cdot \mu_A(y)$.
- 3) If $x \in B$ and $y \notin B$, then $\mu_A(x) = \gamma$ and $\mu_A(y) = 0$. Thus $\mu_A(x * y) \geq 0 = \min\{\gamma, 0\} = \min\{\mu_A(x), \mu_A(y)\} \geq \mu_A(x) \cdot \mu_A(y)$. Hence $\mu_A(x * y) \geq \mu_A(x) \cdot \mu_A(y)$.
- 4) If $y \in B$ and $x \notin B$, then by the same argument as in case 3, we can conclude that $\mu_A(x * y) \geq 0 = \min\{0, \gamma\} = \min\{\mu_A(x), \mu_A(y)\} \geq \mu_A(x) \cdot \mu_A(y)$. Hence μ_A is a fuzzy dot BF -subalgebra of X .

\square

Theorem 3.7: Let B be a subset of X and A be a fuzzy set on X which is given by the proof of theorem 5.6 If μ_A is a fuzzy dot BF -sub algebra of X , then B is a BF -sub algebra of X .

Theorem 3.8: Let A be a fuzzy set in BF -sub algebra X . If μ_A is a fuzzy dot BF -sub algebra of X , then the set $X_{\mu_A} = \{x \in X / \mu_A(x) = \mu_A(0)\}$ is a BF -sub algebra of X .

Proof. Let $x, y \in X_{\mu_A}$. Then $\mu_A(x) = \mu_A(0) = \mu_A(y)$ and so $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\} = \min\{\mu_A(0), \mu_A(0)\} \geq \mu_A(0) \cdot \mu_A(0)$. But $\mu_A(0) \geq \mu_A(0) \cdot \mu_A(0)$. Hence $\mu_A(x * y) \geq \mu_A(0)$ and $\mu_A(0) \geq$

$$\mu_A(x * y).$$

Thus $\mu_A(x * y) = \mu_A(0)$ which means that $x * y \in X_{\mu_A}$.

Definition 3.6: Let $A = \{ \langle x, \mu_A(x) \rangle : x \in X \}$ and $B = \{ \langle x, \mu_B(x) \rangle : x \in X \}$ be two fuzzy sets in X . The Cartesian product $A \times B : X \times X \rightarrow [0, 1]$ is defined by

$$(\mu_A \times \mu_B)(x, y) = \mu_A(x) \cdot \mu_B(y) \text{ for all } x, y \in X.$$

Proposition 3.7: Let A and B be two fuzzy dot sub algebras of X , then $A \times B$ is a fuzzy dot sub algebra of $X \times X$.

Proof. Let (x_1, y_1) and $(x_2, y_2) \in X \times X$ then

$$(\mu_A \times \mu_B)((x_1, y_1) * (x_2, y_2)) = (\mu_A \times \mu_B)((x_1 * x_2), (y_1 * y_2))$$

$$(\mu_A \times \mu_B)((x_1 * x_2), (y_1 * y_2)) = \mu_A(x_1 * x_2) \cdot \mu_B(y_1 * y_2)$$

$$\mu_A(x_1 * x_2) \cdot \mu_B(y_1 * y_2) \geq \mu_A(x_1) \cdot \mu_A(x_2) \cdot \mu_B(y_1) \cdot \mu_B(y_2)$$

$$(\mu_A(x_1) \cdot \mu_A(x_2)) \cdot (\mu_B(y_1) \cdot \mu_B(y_2)) = (\mu_A \times \mu_B)(x_1, y_1) \cdot (\mu_A \times \mu_B)(x_2, y_2)$$

Hence, $A \times B$ is fuzzy dot sub algebra of $X \times X$. \square

Proposition 3.8: Let A and B be two fuzzy dot ideals of X , then $A \times B$ is a fuzzy dot ideals of $X \times X$.

Proof. Let (x_1, y_1) and (x_2, y_2) , then

$$(\mu_A \times \mu_B)(x_1, y_1) = \mu_A(x_1) \cdot \mu_B(y_1)$$

$$\mu_A(x_1) \cdot \mu_B(y_1) \geq (\mu_A(x_1 * x_2) \cdot \mu_A(x_2)) \cdot (\mu_B(y_1 * y_2) \cdot \mu_B(y_2))$$

$$= (\mu_A(x_1 * x_2) \cdot \mu_B(y_1 * y_2)) \cdot (\mu_A(x_2) \cdot \mu_B(y_2))$$

$$= (\mu_A \times \mu_B)(x_1 * x_2, y_1 * y_2) \cdot (\mu_A \times \mu_B)(x_2, y_2)$$

$$= (\mu_A \times \mu_B)((x_1, y_1) * (x_2, y_2)) \cdot (\mu_A \times \mu_B)(x_2, y_2)$$

Hence $A \times B$ is a fuzzy dot ideal of $X \times X$. \square

B. Fuzzy ρ -dot product Relation of BF-Algebra

In this section, strongest fuzzy ρ -relation and fuzzy ρ -product relation of BF-algebras are defined and presented some of its properties.

Definition 3.9: Let ρ be a fuzzy sub set of X . The strongest fuzzy ρ -relation on X is the fuzzy subset μ_ρ of $X \times X$ given by $\mu_\rho(x, y) = \rho(x) \cdot \rho(y)$ for $x, y \in X$

Theorem 3.9: Let μ_ρ be the strongest fuzzy ρ -relation on X , where ρ is a sub set of X , If ρ is a fuzzy dot sub algebra of X , then μ_ρ is a fuzzy dot sub algebra of $X \times X$.

Proof. Suppose that ρ is fuzzy dot sub algebra of X .

For any (x_1, y_1) and $(x_2, y_2) \in X \times X$

$$\text{We have } (\mu_\rho(x_1, y_1) * (x_2, y_2)) = \mu_\rho(x_1 * x_2) \cdot \mu_\rho(y_1 * y_2)$$

By the definition of strongest fuzzy ρ -relation of BF-algebras we get

$$\mu_\rho(x_1 * x_2) \cdot \mu_\rho(y_1 * y_2) = \rho(x_1 * x_2) \cdot \rho(y_1 * y_2)$$

$$\rho(x_1 * x_2) \cdot \rho(y_1 * y_2) \geq \rho(x_1) \cdot \rho(x_2) \cdot \rho(y_1) \cdot \rho(y_2) \text{ (since a fuzzy dot sub algebra of)}$$

$$\rho(x_1) \cdot \rho(x_2) \cdot \rho(y_1) \cdot \rho(y_2) =$$

$$(\rho(x_1) \cdot \rho(y_1)) \cdot (\rho(x_2) \cdot \rho(y_2))$$

$$(\rho(x_1) \cdot \rho(y_1)) \cdot (\rho(x_2) \cdot \rho(y_2)) = \mu_\rho(x_1, y_1) \cdot \mu_\rho(x_2, y_2)$$

Hence, μ_ρ is fuzzy dot sub algebra of $X \times X$. \square

Definition 3.10: Let ρ be a fuzzy subset of X . A fuzzy relation μ on X is called a fuzzy ρ -product relation $\mu_\rho(x, y) \geq \rho(x) \cdot \rho(y)$ for $x, y \in X$

Let ρ be a fuzzy subset of X , A fuzzy relation μ on X is called a left fuzzy relation $\mu_\rho(x, y) = \rho(x)$ for $x, y \in X$

Similarly we can define a right fuzzy relation on $\rho, \mu_\rho(x, y) = \rho(y)$ for $x, y \in X$.

Note that a left (respectively, right) fuzzy relation on is a ρ -product relation.

Theorem 3.10: Let μ be a left fuzzy relation on a fuzzy subset μ of X . If μ is fuzzy dot sub algebra of $X \times X$, then ρ is fuzzy dot sub algebra of X .

Theorem 3.11:

Let μ be a fuzzy relation on X satisfying the inequality $\mu(x, y) \leq \mu(x, 0)$ for all $x, y \in X$, Given $s \in S$, let ρ_s be a fuzzy subset of X , defined by $\rho_s(x) = \mu(x, s)$, for all $x \in X$. If μ is a fuzzy dot sub algebra of $X \times X$, then ρ_s is a fuzzy dot sub algebra of X , for all $s \in X$.

Proof. Let $x, y, s \in X$, Then

$$\rho_s(x * y) = \mu(x * y, s)$$

$$\mu(x * y, s) = \mu(x * y, s * 0)$$

$$\mu(x * y, s * 0) = \mu(x, s) * (y, 0)$$

$$\mu(x, s) * (y, 0) \geq \mu(x, s) \cdot \mu(y, 0)$$

$$\mu(x, s) \cdot \mu(y, 0) \geq \mu(x, s) \cdot \mu(y, s)$$

$$\mu(x, s) \cdot \mu(y, s) = \rho_s(x) \cdot \rho_s(y)$$

Therefore, ρ_s is fuzzy dot sub algebra of X . \square

Theorem 3.12:

Let μ be a fuzzy relation on X and let ρ_μ be a fuzzy sub set of X given by $\rho_\mu(x) = \inf_{y \in X} \{ \mu(x, y), \mu(y, x) \}$ for all $x \in X$. If μ is a fuzzy dot sub algebra $X \times X$ satisfying the equality $\mu(x, 0) = 1 = \mu(0, x)$ for all $x \in X$, then ρ_μ is a fuzzy dot sub algebra of X .

Proof. Let $x, y, z \in X$, we have

$$\mu(x * y, z) = \mu(x * y, z * 0)$$

$$\mu(x * y, z * 0) = \mu((x, z) * (y, 0))$$

$$\mu((x, z) * (y, 0)) \geq \mu(x, z) \cdot \mu(y, 0)$$

$$\mu(x, z) \cdot \mu(y, 0) = \mu(x, z),$$

$$\mu(z, x * y) = \mu(z * 0, x * y)$$

$$\mu(z * 0, x * y) = \mu((z, x) * (0, y))$$

$$\mu((z, x) * (0, y)) \geq \mu(z, x) \cdot \mu(0, y)$$

$$\mu(z, x) \cdot \mu(0, y) = \mu(z, x),$$

It follows that $\mu(x * y, z) \cdot \mu(z, x * y) \geq \mu(x, z) \cdot \mu(z, x)$

$$\mu(x, z) \cdot \mu(z, x) \geq (\mu(x, z) \cdot \mu(z, x)) \cdot (\mu(y, z) \cdot \mu(z, y))$$

$$\text{So that } \rho_\mu(x * y) = \inf_{z \in X} \mu(x * y, z) \cdot \mu(z, x * y)$$

$$\inf_{z \in X} \mu(x * y, z) \cdot \mu(z, x * y) =$$

$$(\inf_{z \in X} \mu(x, z) \cdot \mu(z, x)) \cdot (\inf_{z \in X} \mu(y, z) \cdot \mu(z, y))$$

$$(\inf_{z \in X} \mu(x, z) \cdot \mu(z, x)) \cdot (\inf_{z \in X} \mu(y, z) \cdot \mu(z, y)) =$$

$$\rho_\mu(x) \rho_\mu(y)$$

Therefore ρ_μ is fuzzy dot sub algebra of X . \square

Proposition 3.11: Let f be a BF -homomorphism from X into Y and G be a fuzzy dot BF -subalgebra of Y with the membership function μ_G . Then the inverse image $f^{-1}(G)$ of G is a fuzzy dot BF -subalgebra of X .

Proof. Let $f : X \rightarrow Y$ and G be a fuzzy dot BF -subalgebra of Y and let $x, y \in X$. Then

$$\begin{aligned} \mu_{f^{-1}(G)}(x \star y) &= \mu_G(f(x \star y)) \\ &= \mu_G(f(x) \star f(y)) \\ &\geq \min\{\mu_G(f(x)), \mu_G(f(y))\} \\ &\geq \mu_G(f(x)) \cdot \mu_G(f(y)) \\ &= \mu_{f^{-1}(G)}(x) \cdot \mu_{f^{-1}(G)}(y). \end{aligned}$$

Hence $f^{-1}(G)$ is a fuzzy dot BF -subalgebra of X .

□

Proposition 3.12: Let f be a BF -homomorphism from X onto Y and D be a fuzzy BF -subalgebra of X with the supproperty. Then the image $f(D)$ of D is a fuzzy dot BF -subalgebra of Y .

Proof. Let f be a BF -homomorphism from X into Y and let D be a fuzzy dot BF -subalgebra of Y with supproperty and let $a, b \in Y$, let $x_0 \in f^{-1}(a), y_0 \in f^{-1}(b)$ such that $\mu_D(x_0) = \sup\mu_D(t)_{t \in f^{-1}(a)}, \mu_D(y_0) = \sup\mu_D(t)_{t \in f^{-1}(b)}$. Then by the definition $\mu_{f(D)}$, we have

$$\begin{aligned} \mu_{f(D)}(x \star y) &= \sup\mu_D(t)_{t \in f^{-1}(a \star b)} \\ &\geq \mu_D(x_0 \star y_0) \\ &\geq \min\{\mu_D(x_0), \mu_D(y_0)\} \\ &\geq \mu_D(x_0) \cdot \mu_D(y_0). \end{aligned}$$

$$\begin{aligned} \text{Consequently } \mu_{f(D)}(x \star y) &\geq \mu_D(x_0) \cdot \mu_D(y_0) \\ &= \sup\mu_D(t)_{t \in f^{-1}(a)} \cdot \sup\mu_D(t)_{t \in f^{-1}(b)} = \mu_{f(D)}(a) \cdot \mu_{f(D)}(b). \end{aligned}$$

$$\text{Hence } \mu_{f(D)}(x \star y) \geq \mu_{f(D)}(a) \cdot \mu_{f(D)}(b).$$

Thus $f(D)$ is a fuzzy dot BF -subalgebra of X . □

IV. CONCLUSION

The concepts of structure of fuzzy dot BF -Sub algebras has been introduced. The fuzzy dot product of BF -sub algebra, Fuzzy dot ideal of BF -sub algebra, ρ -product of Fuzzy BF -subalgebras have been discussed. Finally different characterization of Fuzzy dot BF -subalgebras have been investigated.

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