Resolution of time-dependent Navier-Stokes Equations with a new Boundary Condition

Abdeslam El Akkad, Najib Guessous, and Ahmed Elkhalfi,

Abstract—In this work, a numerical solution of the unsteady incompressible Navier- Stokes equations with a new boundary condition is proposed. The method suggested is based on an algorithm of discretization by finite element method in space and the Euler full-implicit scheme in time. The matrix system is solved at each iteration with a preconditioned GMRES method. Also, we proposed two types of a posteriori error indicator, with one being for the time discretization and the other for the space discretization. We prove the equivalence between the sum of the two types of error indicators and the full error.
In order to evaluate the performance of the method, the numerical results of two-dimensional backward-facing step flow are compared with some previously published works or with others coming from commercial code like ADINA (Automatic Dynamic Incremental Nonlinear Analysis) system.

Keywords—Unsteady Navier-Stokes equations, Finite element method, Projection operator, Error estimates, Iterative solvers, Adina system.

I. Introduction

It is well known that the nonstationary incompressible Navier-Stokes equations are one of the main equations studied in mathematical physics and fluid mechanics fields. Numerous works have been devoted to numerical solutions of the above equations using finite element methods (FEMs). For example, Bernardi, and Raugel [7] for the conforming FEM, He [8] for the fully discrete penalty FEM, John and Kaya [14] for the variational multiscale method, and we quote Refs. [15, 19] for the stabilized FEMs. The finite element method, which is one of the well-known methods in the theory of partial differential equations, has been used to prove existence properties and to study the finite element approximation for the solutions of the equations [20, 21].

This paper presents numerical studies for the Navier-Stokes equations in the case of two-dimensional laminar time-dependent flows where the numerical problem is well posed with boundary conditions and other aspects of the problem. For the incompressible Navier-Stokes equations, we use the approximation with Euler fully-implicit scheme, and a finite element discretizations on a quadrilateral element mesh, whereas the discrete Navier-Stokes equations require a method such as the generalized minimum residual method (GMRES), which is designed for non symmetric systems [3, 10]. The key for fast solution lies in the choice of effective preconditioning strategies. The package offers a range of options, including algebraic methods such as incomplete LU factorizations, as well as more sophisticated and state-of-the-art multigrid methods designed to take advantage of the structure of the discrete linearized Navier-Stokes equations. In addition, there is a choice of iterative strategies, Picard iteration or Newton's method, for solving the nonlinear algebraic systems arising from the latter problem.

There are several ways to define error estimators by using the residual equation. In particular, for the Navier-Stokes problem, M. Ainsworth and J. Oden [16] and R. Verfurth [17] introduced several error estimators and provided that they are equivalent to the energy norm of the errors.

The paper is organised as follows. Section 2 presents the model problem used in this paper. The discretization by mixed finite elements is described in section 3. Section 4 shows the methods of a posteriori error bounds of the computed solution. Numerical experiments carried out within the framework of this publication and their comparisons with other results are shown in section 5.

Abdeslam El Akkad is with the Département de mathématiques, Centre Régional des Métiers d'Education et de Formation de Fès Meknès, Rue de Koweit, Ville Nouvelle, BP : 49 - Fès. Ville : Fès, Maroc e-mail: elakkadabdeslam1@gmail.com

Najib Guessous is with Département de mathématiques et informatique, Ecole normale Supérieure de Fès, B.P. 5206, Bensouda, Fès, Maroc.

Ahmed Elkhalfi is with Laboratoire Génie Mécanique, Faculté des Sciences et Techniques B.P. 2202 - Route d'Imouzzer - Fès, Maroc.

Manuscript received Juin 29, 2020; revised Juillet 30, 2020.

II. Time-dependent Navier-Stokes Equations

Let Ω be a bounded simply-connected open domain in $\mathbb{R}^d$, d = 2, 3, with a Lipschitz continuous connected boundary $\partial\Omega$. We consider the unsteady Navier-Stokes equations for the flow of a Newtonian incompressible viscous fluid with constant viscosity

$$\begin{cases} \frac{\partial \overrightarrow{u}}{\partial t} - \nu\nabla^2\overrightarrow{u} + \overrightarrow{u}.\nabla\overrightarrow{u} + \nabla p = \overrightarrow{f} \ in\ \Omega\times(0,T], \\ \nabla.\overrightarrow{u} = 0 \ in\ \Omega\times(0,T], \\ \overrightarrow{u}(x,0) = \overrightarrow{u}_0(x) \ in\ \Omega, \end{cases} \tag{1}$$

Where $\nu > 0$ is a given constant called the kinematic viscosity, $T > 0$ is some final time, $\overrightarrow{u}$ is the fluid velocity, $\overrightarrow{u}_0(x)$ is the initial velocity, p is the pressure field, ∇ is the gradient and $\nabla.$ is the divergence operator.

The boundary conditions on $\partial\Omega$ given by:

$$C_\beta : \overrightarrow{u} + \beta\ (\nu\nabla\overrightarrow{u} - pI)\overrightarrow{n} = \overrightarrow{g} \ in\ \Gamma =: \partial\Omega, \tag{2}$$

where $\overrightarrow{n}$ denote the outward unit normal vector, $\overrightarrow{g} \in H^{\frac{1}{2}}(\Gamma)$ and β nonzero defined on $\partial\Omega$ verify:

There are two strictly positive constants a_1 and b_1 such that:

$$a_1 \le \frac{1}{\beta(x)} \le b_1 \ for\ all\ x \in \Gamma. \tag{3}$$

DOI: 10.46300/9102.2020.14.22 Volume 14, 2020

Remark : If β is strictly positive constant such that $\beta \prec\prec 1$ then C_β , is the Dirichlet boundary condition and if $\beta \succ\succ 1$ then the C_β, is the Neumann boundary condition. For this, β is called the Neumann coefficient.

We set

$$X = \{\overrightarrow{v} \in L^2(\Omega)^2 : div\overrightarrow{v} = 0 \ , \ \overrightarrow{v}.\overrightarrow{n}|_{\partial\Omega} = 0\}, \quad (4)$$

$$Y = \{\overrightarrow{v} \in H_0^1(\Omega)^2 : div\overrightarrow{v} = 0\}, \quad (5)$$

$$V = H_0^1(\Omega) \times H_0^1(\Omega), \ W = L^2(\Omega)^2, \quad (6)$$

and

$$Q = \{q \in L^2(\Omega) : \int_\Omega q(x)dx = 0\}. \quad (7)$$

Let the Stokes operator $A = -P\ \Delta$, where P is the L^2-orthogonal projection of W onto X, and $D(A) = H^2(\Omega)^2 \cap Y$.
Let the assumption (B_1) on Ω [13]:
(B_1): We suppose that Ω is smooth so that the unique solution $(\overrightarrow{v}, q) \in (V, Q)$ of the Stokes problem

$$-\nu\Delta\overrightarrow{v} + \nabla q = \overrightarrow{g}_1, \quad div\overrightarrow{v} = 0 \ \ in\ \Omega \ , \ \overrightarrow{v}|_{\partial\Omega} = 0, \quad (8)$$

exists and satisfies $\|\overrightarrow{v}\|_2 + \|q\|_1 \le C\|\overrightarrow{g}_1\|_0$, for all $\overrightarrow{g}_1 \in W$, where $C > 0$ is a constant depending on Ω and ν.

Let the bilinear forms $a : V \times V \longrightarrow \mathbb{R}$, $b : V \times Q \longrightarrow \mathbb{R}$, $d_1 : Q \times Q \longrightarrow \mathbb{R}$, and the trilinear form $d : V \times V \times V \longrightarrow \mathbb{R}$

$$a(\overrightarrow{u}, \overrightarrow{v}) = \nu \int_\Omega \nabla\overrightarrow{u} : \nabla\overrightarrow{v} dx + \int_\Gamma \frac{1}{\beta}\ \overrightarrow{u}.\overrightarrow{v},$$

$$b(\overrightarrow{v}, q) = \int_\Omega (q\nabla.\overrightarrow{v})dx, \quad (9)$$

Let

$$d_1(p, q) = \int_\Omega p\ q\ dx,$$

$$D(\overrightarrow{u}, \overrightarrow{v}) = (\overrightarrow{u}.\nabla)\overrightarrow{v} + \frac{1}{2}(\nabla.\overrightarrow{u})\overrightarrow{v}, \ \overrightarrow{u}, \overrightarrow{v} \in V, \quad (10)$$

$$\begin{aligned} d(\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}) &= \langle D(\overrightarrow{u}, \overrightarrow{v}), \overrightarrow{w}\rangle_{V',V} \\ &= ((\overrightarrow{u}.\nabla)\overrightarrow{v}, \overrightarrow{w}) + \frac{1}{2}((\nabla.\overrightarrow{u})\overrightarrow{v}, \overrightarrow{w}) \quad (11) \\ &= \frac{1}{2}((\overrightarrow{u}.\nabla)\overrightarrow{v}, \overrightarrow{w}) - \frac{1}{2}((\overrightarrow{u}.\nabla)\overrightarrow{w}, \overrightarrow{v}) \end{aligned}$$

for all $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w} \in V$.
These inner products induce norms on V and Q denoted by $\|.\|_V$ and $\|.\|_Q$ respectively.

$$\|\overrightarrow{v}\|_V = a(\overrightarrow{v}, \overrightarrow{v})^{\frac{1}{2}} \quad \forall\overrightarrow{u} \in V, \quad (12)$$

$$\|q\|_Q = d_1(q, q)^{\frac{1}{2}} \quad \forall q \in Q. \quad (13)$$

Let the norm [9, 22]

$$[\overrightarrow{v}](t) = (\|\overrightarrow{v}(., t)\|^2_{L^2(\Omega)^2} + \nu \int_0^t \|\nabla\overrightarrow{v}(., s)\|^2_{L^2(\Omega)} ds)^{\frac{1}{2}}. \quad (14)$$

Let (B_2) the assumption:
(B_2) : $\overrightarrow{f}(x, t) \in C^0(0, T, W) \cap L^2(0, T, H^1(\Omega))$, $\overrightarrow{f}_t(x, t) \in L^2(0, T, L^2(\Omega))$ and $\overrightarrow{u}_0(x) \in D(A)$.

Given the continuous functional $l : V \longrightarrow \mathbb{R}$

$$l(\overrightarrow{v}) = \int_\Omega \overrightarrow{f}.\overrightarrow{v}\ dx + \int_\Gamma \frac{1}{\beta}\overrightarrow{g}.\overrightarrow{v}\ dx. \quad (15)$$

Then the standard weak formulation of the unsteady Navier-Stokes problem (1)-(2) is the following:
Find $(\overrightarrow{u}, p) \in V \times Q$ such that

$$\overrightarrow{u}(., 0) = \overrightarrow{u}_0\ in\ \Omega, \quad (16)$$

$$(\frac{\partial\overrightarrow{u}}{\partial t}, \overrightarrow{v}) + a(\overrightarrow{u}, \overrightarrow{v}) - b(\overrightarrow{v}, p) + d(\overrightarrow{u}, \overrightarrow{u}, \overrightarrow{v}) = l(\overrightarrow{v}), \quad (17)$$

$$-b(\overrightarrow{u}, q) = 0, \quad (18)$$

for all $(\overrightarrow{v}, q) \in V \times Q$ and $t \in (0, T)$.

III. Finite element approximation

Our goal here is to consider the unsteady Navier-Stokes equations with a new boundary conditions in a two dimensional domain and to approximate them by a finite element method in space and the Euler full-implicit scheme in time.
Let τ_h, $h > 0$, be a family of triangulations of Ω. We denote by h_K the diameter of a simplex K, by h_E the diameter of a face E of K, and we set $h = \max_{K\in\tau_h}\{h_K\}$.
For any $K \in \tau_h$, we denote by $\varepsilon(K)$ and N(K) the set of its edges and vertices, respectively.
We let $\varepsilon_h = \bigcup_{K\in\tau_h} \varepsilon(K)$ denotes the set of all edges split into interior and boundary edges.
$\varepsilon_h = \varepsilon_{h,\Omega} \bigcup \varepsilon_{h,\Gamma}$, where $\varepsilon_{h,\Omega} = \{E \in \varepsilon_h : E \subset \Omega\}$ and $\varepsilon_{h,\Gamma} = \{E \in \varepsilon_h : E \subset \partial\Omega\}$.

Let $0 = t_0 < t_1 < ... < t_N = T$, $\tau_n = \Delta t_n = t_n - t_{n-1}$, by τ the N-tuple $(\tau_1, ..., \tau_N)$ and $\delta_\tau = \max_{2\le n\le N}\ \frac{\Delta t_n}{\Delta t_{n-1}}$ the regularity parameter.
We define the function $\overrightarrow{v}_\tau$ on [0, T] which is affine on each interval $[t_{n-1}; t_n]$, $1 \le n \le N$ by

$$\overrightarrow{v}_\tau = \frac{t - t_{n-1}}{\Delta t_n}\overrightarrow{v}^n + \frac{t_n - t}{\Delta t_n}\overrightarrow{v}^{n-1}, \quad (19)$$

for all $t \in [t_{n-1}, t_n]$, $1 \le n \le N$.
For any Banach space F, and each family $(\overrightarrow{v}^n)_{0\le n\le N} \in F^{N+1}$, we denote by $W_\tau(F)$ the space of such functions.
Let the discrete norm on space $W_\tau(H_0^1(\Omega))$

$$[[\overrightarrow{v}_\tau]](t_n) = (\|\overrightarrow{v}^n\|^2_{L^2(\Omega)^2} + \nu\sum_{m=1}^n \Delta t_m\|\nabla\overrightarrow{v}^m\|^2_{L^2(\Omega)})^{\frac{1}{2}}, \quad (20)$$

for all n, $1 \le n \le N$.
The finite element approximation to (1)-(2) by the Euler's scheme is then
Find $(\overrightarrow{u}^n)_{0\le n\le N} \in W \times V^N$ and $(p^n)_{1\le n\le N} \in Q^N$, such that

$$\overrightarrow{u}^0 = \overrightarrow{u}_0\ in\ \Omega, \quad (21)$$

$$\frac{1}{\Delta t_n}(\overrightarrow{u}^n - \overrightarrow{u}^{n-1}, \overrightarrow{v}) + a(\overrightarrow{u}^n, \overrightarrow{v}) - b(\overrightarrow{v}, p^n) + d(\overrightarrow{u}^n, \overrightarrow{u}^n, \overrightarrow{v}) = (\overrightarrow{f}^n, \overrightarrow{v}) + \frac{1}{\beta}(\overrightarrow{g}^n.\overrightarrow{v})_\Gamma, \tag{22}$$

$$-b(\overrightarrow{u}^n, q) = 0, \tag{23}$$

for all $(\overrightarrow{v}, q) \in V \times Q$.

Let V_h and Q_h the approximation spaces for $Q_1 - P_0$ approximation. Using the stabilized $Q_1 - P_0$ method and a Trapezoid Rule time stepping [27], we find the pair $(\overrightarrow{d}_h^{n+1}, p_h^{n+1}) \in V_h \times Q_h$, such that

$$2(\overrightarrow{d}_h^{n+1}, \overrightarrow{v}_h) + \nu k_{n+1}(\nabla \overrightarrow{d}_h^{n+1}, \nabla \overrightarrow{v}_h) + k_{n+1}(\overrightarrow{w}_h^{n+1}.\nabla \overrightarrow{d}_h^{n+1}, \overrightarrow{v}_h) - (p_h^{n+1}, \nabla.\overrightarrow{v}_h) \tag{24}$$
$$= (\frac{\partial \overrightarrow{u}_h^n}{\partial t}, \overrightarrow{v}_h) - \nu(\nabla \overrightarrow{u}_h^n, \nabla \overrightarrow{v}_h) - (\overrightarrow{w}_h^{n+1}.\nabla \overrightarrow{u}_h^n, \overrightarrow{v}_h),$$

$$-(\nabla.\overrightarrow{d}_h^{n+1}, q_h) - \alpha\ \gamma(p_h^{n+1}, q_h) = 0, \tag{25}$$

for all $(\overrightarrow{v}_h, q_h) \in V_h \times Q_h$, where $\overrightarrow{w}_h^{n+1} = (1 + \frac{k_{n+1}}{k_n})\overrightarrow{u}_h^n - \frac{k_{n+1}}{k_n}\overrightarrow{u}_h^{n-1}$ and $k_{n+1} := t_{n+1} - t_n$ is the current time step.
The velocity and acceleration at t_{n+1} are defined by $\overrightarrow{u}_h^{n+1} = \overrightarrow{u}_h^n + k_{n+1}\overrightarrow{d}_h^n$, $\frac{\partial \overrightarrow{u}_h^{n+1}}{\partial t} = 2\overrightarrow{d}_h^n - \frac{\partial \overrightarrow{u}_h^n}{\partial t}$.

α is the stabilization parameter, and the stabilization term $\gamma(p_h, q_h)$ is defined by [3]

$$\gamma_K(p_h, q_h) := \frac{|K|}{4}\sum_{E\in\Gamma_K}\frac{1}{h_E}\int_E [p_h]_E[q_h]_E, \tag{26}$$

$$\gamma(p_h, q_h) := \sum_{K\in T_K}\gamma_K(p_h, q_h), \tag{27}$$

where Γ_K is the set consisting of the four interior element edges in the macroelement K, T_K is a macroelement partitioning of the domain Ω, $|K|$ is the mean element area within the macroelement, $[\ .\ ]_E$ is the jump across edge E and h_E is the length of E.

Let $V_{n,h} \subset V$ and $Q_{n,h} \subset Q$. We assume that:
(B_3): (1) $X_{n,h}^1 \subset V_{n,h}$ such that:

$$X_{n,h}^1 = \{\overrightarrow{v}_n \in V : \forall K \in \tau_{n,h}, \overrightarrow{v}_h|_K \in P_2(K)\}, \tag{28}$$

where $P_2(K)$ is the space of polynomials of degree ≤ 2, for $K \in \tau_{n,h}$.

(2) for $1 \leq n \leq N$, there exists a constant $\gamma_{n,h} > 0$ such that

$$\sup_{\overrightarrow{v}_h\in V_{n,h}}\frac{(\nabla.\overrightarrow{v}_h, q_h)}{\|\nabla\overrightarrow{v}_h\|_{L^2(\Omega)}} \geq\ \gamma_{n,h}\|q_h\|_{L^2(\Omega)}, \tag{29}$$

for all $q_h \in Q_{n,h}$.

Let the space

$$Y_{n,h} = \{\overrightarrow{v}_n \in V_{n,h}; (\nabla.\overrightarrow{v}_h, q_h) = 0, \forall q_h \in Q_{n,h}\}. \tag{30}$$

Let π_h the projection operator from $L^2(\Omega)$ onto $V_{0,h}$.
Let $\overrightarrow{u}_h^0 \in V_{0,h}$ and $p_h^0 = 0$. We find $(\overrightarrow{u}_h^n)_{0\leq n\leq N} \in \prod_{n=0}^N V_{n,h}$ and $(p_h^n)_{1\leq n\leq N} \in \prod_{n=0}^N Q_{n,h}$ such that

$$\overrightarrow{u}_h^0 = \pi_h\overrightarrow{u}_0\ in\ \Omega, \tag{31}$$

$$\frac{1}{\Delta t_n}(\overrightarrow{u}_h^n - \overrightarrow{u}_h^{n-1}, \overrightarrow{v}_h) + d(\overrightarrow{u}_h^n, \overrightarrow{u}_h^n, \overrightarrow{v}_h) - b(\overrightarrow{v}_h, p_h^n) + a(\overrightarrow{u}_h^n, \overrightarrow{v}_h) = (\overrightarrow{f}^n, \overrightarrow{v}_h) + \frac{1}{\beta}(\overrightarrow{g}^n.\overrightarrow{v}_h)_\Gamma, \tag{32}$$

$$-b(\overrightarrow{u}_h^n, q_h) = 0, \tag{33}$$

for all $(\overrightarrow{v}_h, q_h) \in V_{n,h} \times Q_{n,h}$ and $1 \leq n \leq N$.

We use a set of vector-valued basis functions $\{\overrightarrow{\varphi_i}\}_{i=1,\ldots,n_u}$ so that

$$\overrightarrow{u}_h = \sum_{i=1}^{n_u} u_i\overrightarrow{\varphi_i}. \tag{34}$$

We introduce a set of pressure basis functions $\{\psi_k\}_{k=1,\ldots,n_p}$ and set

$$p_h = \sum_{k=1}^{n_p} p_k\psi_k, \tag{35}$$

Where n_u and n_p are the numbers of velocity and pressure basis functions, respectively.
We obtain a non linear system of algebraic equations:

$$D\frac{dU}{dt}(t) + [N(U(t)) + M]U(t) + BP(t) = L(t), \tag{36}$$

$$B^TU(t) = 0. \tag{37}$$

Where

$$U(t) = (u_1(t), u_2(t), ..., u_{n_u}(t))^T, \tag{38}$$

$$P(t) = (p_1(t), p_2(t), ..., p_{n_p}(t))^T. \tag{39}$$

The matrix B is the divergence matrix

$$B = [b_{k,j}]; b_{k,j} = -\int_\Omega \psi_k\nabla.\overrightarrow{\varphi}_j, \tag{40}$$

and

$$D = [d_{ij}],\ d_{ij} = \int_\Omega \overrightarrow{\varphi}_i.\overrightarrow{\varphi}_j, \tag{41}$$

$$N = [n_{ij}],\ n_{ij} = \sum_{k=1}^{n_u} u_k(t)\int_\Omega(\overrightarrow{\varphi}_j.\nabla\overrightarrow{\varphi}_k).\overrightarrow{\varphi}_i, \tag{42}$$

$$M = [m_{ij}],\ m_{ij} = \nu\int_\Omega \nabla\overrightarrow{\varphi}_i : \nabla\overrightarrow{\varphi}_j + \int_{\partial\Omega}\frac{1}{\beta}\overrightarrow{\varphi}_i.\overrightarrow{\varphi}_j, \tag{43}$$

$$L = [l_i];\ l_i = \int_\Omega \overrightarrow{f}.\overrightarrow{\varphi}_i + \int_{\partial\Omega}\frac{1}{\beta}\overrightarrow{g}.\overrightarrow{\varphi}_i, \tag{44}$$

for $i, j = 1, ..., n_u$, $k = 1, ..., n_p$.
Solution of the nonlinear system of equations (36)-(37), can be carried out efficiently using Picards method. The linear system we need to solve within each iteration of Picards method has the following generic form:

$$\begin{pmatrix} A_0 + N & B_0^T \\ B_0 & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} L \\ 0 \end{pmatrix}. \quad (45)$$

We use the generalized minimum residual method (GMRES) for solving the nonsymmetric systems [3, 31].
Preconditioning is a technique used to enhance the convergence of an iterative method to solve a large linear system iteratively. Instead of solving a system $\Lambda x = b$, one solves a system , $P^{-1}\Lambda x = P^{-1}b$ where P is the preconditioned. A good preconditioned should lead to fast convergence of the Krylov method. Furthermore, systems of the form Pz = r should be easy to solve. For the Navier-Stokes equations, the objective is to design a preconditioned that increases the convergence of an iterative method independent of the Reynolds number and number of grid points. We use a least-squares commutator preconditioning [3, 23, 30].

IV. Error estimates

In this section we consider a posteriori error estimator for the unsteady incompressible Navier-Stokes equation. We propose two types of error indicators: the time error indicators and the space error indicators, and we derive the upper bounds for the error estimators. We prove the equivalence between the sum of the two types of error indicators and the full error. For simplicity, we suppose that $\beta = 0$ and $\overrightarrow{g} = \overrightarrow{0}$.

Let $\overrightarrow{f}_h^n$ the approximation of $\overrightarrow{f}^n$ which is polynomial of degree $\leq l$ on all elements of $\tau_{n;h}$, and $[.]_E$ the jump of across E in the direction $\overrightarrow{n}_E$, for each $E \in \varepsilon(K)$.
Let the time error indicators

$$\eta^n = \sqrt{\frac{\Delta t_n}{3}\nu} \quad \|\nabla(\overrightarrow{u}_h^n - \overrightarrow{u}_h^{n-1})\|_{L^2(\Omega)}, \ 1 \leq n \leq N, \quad (46)$$

and let the space error indicators

$$\begin{aligned} \eta_K^n \quad = \quad & h_K \|\overrightarrow{f}_h^n - \frac{\overrightarrow{u}_h^n - \overrightarrow{u}_h^{n-1}}{\Delta t_n} + \nu \Delta \overrightarrow{u}_h^n - \nabla p_h^n - \\ & (\overrightarrow{u}_h^n.\nabla)\overrightarrow{u}_h^n\|_{L^2(K)} + \sum_{E \in \varepsilon(K)} h_E^{\frac{1}{2}} \|[\nu \partial_{n_E} \overrightarrow{u}_h^n - \\ & p_h^n \overrightarrow{n}_E]_E\|_{L^2(E)} + \nu \|div \overrightarrow{u}_h^n\|_{L^2(K)}. \quad (47) \end{aligned}$$

The time error indicators η^n is local in time and global in space, and the space error indicators η_K^n is local both in time and in space.

Theorem 1: We suppose that the assumptions (B_1) and (B_2) holds, then, the problem (17)-(18) has a unique solution $\overrightarrow{u} \in L^{\propto}(0, t; X) \cap L^2(0, t; Y)$ such that

$$\|\overrightarrow{u}(t)\|_0^2 + \|\nabla \overrightarrow{u}(t)\|_0^2 + \|A\overrightarrow{u}(t)\|_0^2 + \|\nabla p(t)\|_0^2 + \|\overrightarrow{u}_t(t)\|_0^2$$

$$\leq K_1, \quad (48)$$

$$\int_0^t \{\|\nabla \overrightarrow{u}\|_0^2 + \|\overrightarrow{u}_t\|_0^2 + \|A\overrightarrow{u}\|_0^2 + \|\nabla p\|_0^2 + \|\nabla \overrightarrow{u}_t\|_0^2\} ds$$

$$\leq K_1, \quad (49)$$

where K_1 is a positive constant.
We have

$$[\overrightarrow{u}](t) \leq (\frac{1}{\nu}\|\overrightarrow{f}\|^2_{L^2(0,t;H^{-1}(\Omega)^2)} + \|\overrightarrow{u}_0\|^2_{L^2(\Omega)})^{\frac{1}{2}}, \quad (50)$$

and

$$\begin{aligned} & \|\frac{\partial \overrightarrow{u}}{\partial t} + (\overrightarrow{u}.\nabla)\overrightarrow{u} + \nabla p\|_{L^2(0,t;H^{-1}(\Omega)^2)} \\ & \leq 2(\|\overrightarrow{f}\|^2_{L^2(0,t;H^{-1}(\Omega)^2)} + \frac{\nu}{2}\|\overrightarrow{u}_0\|^2_{L^2(\Omega)})^{\frac{1}{2}}. \quad (51) \end{aligned}$$

Proof: See [11] .

Let

$$\overrightarrow{v}_{h\tau} = \frac{t - t_{n-1}}{\Delta t_n}\overrightarrow{v}_h^n + \frac{t_n - t}{\Delta t_n}\overrightarrow{v}_h^{n-1}, \quad (52)$$

for all $t \in [t_{n-1}, t_n]$, $1 \leq n \leq N$.

Theorem 2: We suppose that the assumptions $(B_1) - (B_2)$ holds, $\overrightarrow{u}$ be a solution of the problem (17)-(18), and $\overrightarrow{u}_\tau$ is the solution of (22)-(23), we have

$$\begin{aligned} [\overrightarrow{u} - \overrightarrow{u}_\tau](t_n) \quad \leq \quad & \beta_1(\sum_{m=1}^{n} \frac{1}{\nu}(\eta^m)^2 + \|\overrightarrow{u}_\tau - \overrightarrow{u}_{h\tau}\|^2_{L^2(0,t_n;H^1)} \\ & + \frac{1}{\nu}\|\overrightarrow{f} - \Pi_\tau \overrightarrow{f}\|^2_{L^2(0,t_n;H^{-1})})^{\frac{1}{2}}, \quad (53) \end{aligned}$$

for all n, $1 \leq n \leq N$,
Here β_1 is a positive constant depends on ν and $\overrightarrow{f}$, and $\Pi_\tau \overrightarrow{f}$ is the step function which is constant and equal to $\overrightarrow{f}(t_n)$ on each interval $(t_{n-1}; t_n)$; $1 \leq n \leq N$.

Proof: Using (17)-(18) and (22)-(23), the pair $(\overrightarrow{u} - \overrightarrow{u}_\tau, p - \Pi_\tau p_\tau)$ satisfies

$$(\overrightarrow{u} - \overrightarrow{u}_\tau)(.,0) = 0 \quad in \ \Omega, \quad (54)$$

and

$$\begin{aligned} & (\frac{\partial}{\partial t}(\overrightarrow{u} - \overrightarrow{u}_\tau), \overrightarrow{v}) + a(\overrightarrow{u} - \overrightarrow{u}_\tau, \overrightarrow{v}) - \\ & b(\overrightarrow{v}, p - \Pi_\tau p_\tau) + d(\overrightarrow{u}, \overrightarrow{u}, \overrightarrow{v}) - d(\overrightarrow{u}_\tau, \overrightarrow{u}_\tau, \overrightarrow{v}) \\ = \quad & (\overrightarrow{f} - \Pi_\tau \overrightarrow{f}, \overrightarrow{v}) + a(\overrightarrow{u}^n - \overrightarrow{u}_\tau, \overrightarrow{v}) + \\ & d(\overrightarrow{u}^n, \overrightarrow{u}^n, \overrightarrow{v}) - d(\overrightarrow{u}_\tau, \overrightarrow{u}_\tau, \overrightarrow{v}), \quad (55) \end{aligned}$$

$$-b(\overrightarrow{u} - \overrightarrow{u}_\tau, q) = 0, \quad (56)$$

for all $(\overrightarrow{v}, q) \in V \times Q$.
Setting $(\overrightarrow{v}, q) = (\overrightarrow{u} - \overrightarrow{u}_\tau, p - \Pi_\tau p_\tau)$ in (55)-(56), we obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|\overrightarrow{u} - \overrightarrow{u}_\tau\|^2_{L^2(\Omega)} + \nu\|\nabla(\overrightarrow{u} - \overrightarrow{u}_\tau)\|^2_{L^2(\Omega)} + \\ & d(\overrightarrow{u}, \overrightarrow{u}, \overrightarrow{u} - \overrightarrow{u}_\tau) - d(\overrightarrow{u}_\tau, \overrightarrow{u}_\tau, \overrightarrow{u} - \overrightarrow{u}_\tau) \\ = & (\overrightarrow{f} - \Pi_\tau \overrightarrow{f}, \overrightarrow{u} - \overrightarrow{u}_\tau) + a(\overrightarrow{u}^n - \overrightarrow{u}_\tau, \overrightarrow{u} - \overrightarrow{u}_\tau) + \\ & d(\overrightarrow{u}^n, \overrightarrow{u}^n, \overrightarrow{u} - \overrightarrow{u}_\tau) - d(\overrightarrow{u}_\tau, \overrightarrow{u}_\tau, \overrightarrow{u} - \overrightarrow{u}_\tau). \end{aligned}$$

Using the bound of $d(\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w})$ and (48), (49), (51), we have

$$\begin{aligned} & d(\overrightarrow{u}^n, \overrightarrow{u}^n, \overrightarrow{u} - \overrightarrow{u}_\tau) - d(\overrightarrow{u}_\tau, \overrightarrow{u}_\tau, \overrightarrow{u} - \overrightarrow{u}_\tau) \\ & \leq \beta_2|\overrightarrow{u}^n - \overrightarrow{u}_\tau|_1|\overrightarrow{u} - \overrightarrow{u}_\tau|_1, \quad (57) \end{aligned}$$

$$\begin{aligned} & d(\overrightarrow{u}, \overrightarrow{u}, \overrightarrow{u} - \overrightarrow{u}_\tau) - d(\overrightarrow{u}_\tau, \overrightarrow{u}_\tau, \overrightarrow{u} - \overrightarrow{u}_\tau) \\ & \leq \beta_3|\overrightarrow{u} - \overrightarrow{u}_\tau|_1\|\overrightarrow{u} - \overrightarrow{u}_\tau\|_{0,\Omega}. \quad (58) \end{aligned}$$

Let $\beta_4 = max\{\beta_2^2,\ \beta_3^2\}$. We have

$$\frac{1}{2}\frac{d}{dt}\|\overrightarrow{u} - \overrightarrow{u}_\tau\|^2_{L^2(\Omega)} + \nu\|\nabla(\overrightarrow{u} - \overrightarrow{u}_\tau)\|^2_{L^2(\Omega)} \leq \frac{1}{\nu}\|\overrightarrow{f} - \Pi_\tau\overrightarrow{f}\|^2_{H^{-1}} + \frac{\nu}{4}|\overrightarrow{u} - \overrightarrow{u}_\tau|^2_1 + \frac{3\nu}{16}|\overrightarrow{u} - \overrightarrow{u}_\tau|^2_{1,\Omega} + 4\nu|\overrightarrow{u}^n - \overrightarrow{u}_\tau|^2_{1,\Omega} + \frac{4\beta_4}{\nu}|\overrightarrow{u}^n - \overrightarrow{u}_\tau|^2_{1,\Omega} + \frac{4\beta_4}{\nu}\|\overrightarrow{u} - \overrightarrow{u}_\tau\|^2_{0,\Omega} + \beta_4|\overrightarrow{u}^n - \overrightarrow{u}_\tau|^2_{1,\Omega} + \beta_4\|\overrightarrow{u} - \overrightarrow{u}_\tau\|^2_{0,\Omega}. \quad (59)$$

We have the following inequality [9]

$$\int_{t_{m-1}}^{t_m}\|\nabla(\overrightarrow{u}^m - \overrightarrow{u}_\tau)(.,x)\|^2_{0,\Omega}dx \leq \frac{3}{\nu}(\eta^m)^2 + 6\int_{t_{m-1}}^{t_m}\|\nabla(\overrightarrow{u}_\tau - \overrightarrow{u}_{h\tau})(.,x)\|^2_{0,\Omega}dx. \quad (60)$$

Then

$$\|(\overrightarrow{u} - \overrightarrow{u}_\tau)(t_m)\|^2_{L^2(\Omega)} + \nu\int_{t_{m-1}}^{t_m}\|\nabla(\overrightarrow{u} - \overrightarrow{u}_\tau)\|^2_{L^2(\Omega)}dt \leq \beta_5(\eta^m)^2 + \|(\overrightarrow{u} - \overrightarrow{u}_\tau)(t_{m-1})\|^2_{0,\Omega} + \beta_6\int_{t_{m-1}}^{t_m}\|\overrightarrow{u} - \overrightarrow{u}_\tau\|^2_{0,\Omega}dt + 6\beta_5\nu\int_{t_{m-1}}^{t_m}\|\nabla(\overrightarrow{u}_\tau - \overrightarrow{u}_{h\tau})(.,x)\|^2_{L^2(\Omega)}dx + \frac{2}{\nu}\|\overrightarrow{f} - \Pi_\tau\overrightarrow{f}\|^2_{L^2(t_{m-1},t_m;H^{-1}(\Omega))}. \quad (61)$$

Using (61) and (20), we obtain (53).

Theorem 3: We suppose that the assumptions $(B_1) - (B_2)$ holds. $(\overrightarrow{u}, p)$ be a solution of the problem (17)-(18) and $(\overrightarrow{u}_\tau, \Pi_\tau p_\tau)$ the solution of (22)-(23), we have

$$\sum_{m=1}^{n}\int_{t_{m-1}}^{t_m}\|\frac{\partial}{\partial t}(\overrightarrow{u} - \overrightarrow{u}_\tau) + (\overrightarrow{u}.\nabla)\overrightarrow{u} - (\overrightarrow{u}^m.\nabla)\overrightarrow{u}^m + \nabla(p - \Pi_\tau\ p_\tau)\|^2_{H^{-1}(\Omega)}dt \leq C_1(\sum_{m=1}^{n}\nu(\eta^m)^2 + \sum_{m=1}^{n}\int_{t_{m-1}}^{t_m}(\nu^2\|\overrightarrow{u}_\tau - \overrightarrow{u}_{h\tau}\|^2_1 + \|\overrightarrow{f} - \Pi_\tau\overrightarrow{f}\|^2_{H^{-1}(\Omega)})dt), \quad (62)$$

for all n, $1 \leq n \leq N$,
Here C_1 is a positive constant depends on ν, $\overrightarrow{f}$ and Ω.

Proof: Using (55)-(56) gives

$$\|\frac{\partial}{\partial t}(\overrightarrow{u} - \overrightarrow{u}_\tau) + (\overrightarrow{u}.\nabla)\overrightarrow{u} - (\overrightarrow{u}^m.\nabla)\overrightarrow{u}^m + \nabla(p - \Pi_\tau p_\tau)\|_{-1} = \sup_{\overrightarrow{v}\in H^1_0(\Omega)}\frac{(\overrightarrow{f} - \Pi_\tau\overrightarrow{f}, \overrightarrow{v}) - a(\overrightarrow{u} - \overrightarrow{u}^m, \overrightarrow{v})}{\|\nabla\overrightarrow{v}\|_{L^2(\Omega)}} \leq \|\overrightarrow{f} - \Pi_\tau\overrightarrow{f}\|_{H^{-1}(\Omega)} + \nu|\overrightarrow{u} - \overrightarrow{u}_\tau|_1 + \nu|\overrightarrow{u}_\tau - \overrightarrow{u}^m|_1. \quad (63)$$

Using (53), (60), we obtain (62).

Let the assumption (B_4)
(B_4): $Q^0_{n,h} \subset Q_{n,h}$ or $Q^1_{n,h} \subset Q_{n,h}$ such that

$$Q^0_{n,h} = \{q_h \in L^2_0(\Omega); q_h|_K \in P_0(K),\ \forall\ K \in \tau_{n,h}\}, \quad (64)$$

$$Q^1_{n,h} = \{q_h \in H^1(\Omega) \cap L^2_0(\Omega); q_h|_K \in P_1(K),\ \forall\ K \in \tau_{n,h}\}. \quad (65)$$

Let the assumption (B_5)
(B_5): For all $1 \leq p \leq N$, there exists a conforming triangulation $\widetilde{\tau}_{p,h}$, such that each element K of $\tau_{p-1,h}$ or of $\tau_{p,h}$ is the union of elements $\widetilde{K}$ of $\widetilde{\tau}_{p,h}$ such that $h_K \sim h_{\widetilde{K}}$.

Lemma 4: Let π: $V \mapsto V$ the operator $\pi\overrightarrow{v} = \overrightarrow{w}$, $\forall\overrightarrow{v} \in V$, where $(\overrightarrow{w}, r) \in V \times Q$ is the unique solution of the Stokes problem

$$\left\{\begin{array}{l} -\triangle\overrightarrow{w} + \bigtriangledown r = 0\ in\ \Omega, \\ \nabla.\overrightarrow{w} = \nabla.\overrightarrow{v}\ in\ \Omega, \\ \overrightarrow{w} = \overrightarrow{0}\ in\ \partial\Omega. \end{array}\right. \quad (66)$$

Then, we have
(i) $\pi\overrightarrow{v} = \overrightarrow{0}\quad \forall\overrightarrow{v} \in Y$.
(ii) We have

$$|\overrightarrow{v} - \pi\overrightarrow{v}|_1 \leq |\overrightarrow{v}|_1,\ |\pi\overrightarrow{v}|_1 \leq \frac{1}{\lambda}|div\,\overrightarrow{v}|_{L^2(\Omega)}, \quad \forall\overrightarrow{v} \in V. \quad (67)$$

where

$$\lambda = \inf_{q\in Q}\sup_{\overrightarrow{v}\in V}\frac{b(\overrightarrow{v}, q)}{|\overrightarrow{v}|_1|q|_0}.$$

(iii) We suppose that the assumption (B_4) hold, then

$$\|\pi\overrightarrow{v}_h\|_{L^2(\Omega)} \leq Ch^\theta_n\|div\,\overrightarrow{v}_h\|_{L^2(\Omega)}, \quad (68)$$

for all $\overrightarrow{v}_h \in Y_{n,h}$, $1 \leq n \leq N$,
where

$$\left\{\begin{array}{ll} \theta = 1 & if\ \Omega\ is\ convex, \\ \theta = \frac{1}{2} & otherwise. \end{array}\right. \quad (69)$$

Proof: See [9].

Theorem 5: We suppose that the assumptions $(B_2) - (B_4)$ holds. Let $\overrightarrow{u}_\tau$ the solution of (22)-(23) and $\overrightarrow{u}_{h\tau}$ associated with the solution $(\overrightarrow{u}^n_h)_{0\leq n\leq N}$ of (32)-(33), we have

$$[[\overrightarrow{u}_\tau - \overrightarrow{u}_{h\tau}]](t_n) \leq C_2(\sum_{m=1}^{n}\Delta t_m\sum_{K\in\tau_{mh}}((1 + \xi_{h\tau})(\eta^m_K)^2 + \frac{h^2_K}{\nu}\|\overrightarrow{f}^m - \overrightarrow{f}^m_h\|^2_{0,K}))^{\frac{1}{2}} + C_3\|\overrightarrow{u}_0 - \pi_h\overrightarrow{u}_0\|_{0,\Omega}, \quad (70)$$

for all n, $1 \leq n \leq N$,
Here C_2 and C_3 are two positive constants depending on ν and $\overrightarrow{f}$, $\xi_{h\tau}$ is defined by

$$\xi_{h\tau} = \sup_{1\leq n\leq N}\frac{\sup_{K\in\tau_{n,h}}h^{2\theta_K}_K}{\nu\Delta t_n},$$

and

$$\left\{\begin{array}{ll} \theta_K = 1 & if\ K \cap \partial\Omega \neq \emptyset, \\ \theta_K = \frac{1}{2} & otherwise. \end{array}\right. \quad (71)$$

Proof: Combining (32)-(33) and(17)-(18), we obtain

$$(\frac{(\overrightarrow{u}^n - \overrightarrow{u}^n_h) - (\overrightarrow{u}^{n-1} - \overrightarrow{u}^{n-1}_h)}{\Delta t_n}, \overrightarrow{v}) + a(\overrightarrow{u}^n - \overrightarrow{u}^n_h, \overrightarrow{v}) - b(\overrightarrow{v}, p^n - p^n_h) + d(\overrightarrow{u}^n - \overrightarrow{u}^n_h, \overrightarrow{u}^n - \overrightarrow{u}^n_h, \overrightarrow{v})$$
$$= [(\overrightarrow{f}^n_h - \frac{\overrightarrow{u}^n_h - \overrightarrow{u}^{n-1}_h}{\Delta t_n} - (\overrightarrow{u}^n_h.\nabla)\overrightarrow{u}^n_h, \overrightarrow{v} - \overrightarrow{v}_h) - a(\overrightarrow{u}^n_h, \overrightarrow{v} - \overrightarrow{v}_h) + b(\overrightarrow{v} - \overrightarrow{v}_h, p^n_h)] + [(\overrightarrow{f}^n - \overrightarrow{f}^n_h, \overrightarrow{v} - \overrightarrow{v}_h)] + [-d(\overrightarrow{u}^n - \overrightarrow{u}^n_h, \overrightarrow{u}^n_h, \overrightarrow{v}) + d(\overrightarrow{u}^n_h, \overrightarrow{u}^n - \overrightarrow{u}^n_h, \overrightarrow{v})]$$
$$= F_1 + F_2 + F_3. \quad (72)$$

Let $\overrightarrow{e}^n = \overrightarrow{u}^n - \overrightarrow{u}^n_h$, $\overrightarrow{v} = \overrightarrow{e}^n - \pi\overrightarrow{e}^n$, and $\overrightarrow{v}_n = R_{nh}(\overrightarrow{e}^n - \pi\overrightarrow{e}^n)$. We have $div(\overrightarrow{e}^n - \pi\overrightarrow{e}^n) = 0$,
where R_{nh} is a Clement type regularization operator [24].
We obtain

$$\begin{aligned}(\overrightarrow{e}^n - \overrightarrow{e}^{n-1}, \overrightarrow{e}^n) + \Delta t_n\nu(\nabla\overrightarrow{e}^n, \nabla\overrightarrow{e}^n) + \\ \Delta t_n d(\overrightarrow{e}^n, \overrightarrow{e}^n, \overrightarrow{e}^n) \\ = (\overrightarrow{e}^n - \overrightarrow{e}^{n-1}, \pi\overrightarrow{e}^n) + \Delta t_n(\nu(\nabla\overrightarrow{e}^n, \nabla\pi\overrightarrow{e}^n) + \\ d(\overrightarrow{e}^n, \overrightarrow{e}^n, \pi\overrightarrow{e}^n) + \sum_{i=1}^{3} F_i). \quad (73)\end{aligned}$$

From (73), Lemma 4 and using $\pi\overrightarrow{e}^n = -\pi\overrightarrow{u}^n_h$, we obtain

$$\begin{aligned}(\overrightarrow{e}^n - \overrightarrow{e}^{n-1}, \pi\overrightarrow{e}^n) \quad \leq \quad & \frac{1}{2}\|\overrightarrow{e}^n - \overrightarrow{e}^{n-1}\|^2_{L^2(\Omega)} \\ & C\xi_{h\tau}\ \nu\Delta t_h\|div\overrightarrow{u}^n_h\|^2_{L^2(\Omega)},\end{aligned}$$

$$\begin{aligned}\nu\Delta t_n(\nabla\overrightarrow{e}^n, \nabla\pi\overrightarrow{e}^n) \quad \leq \quad & \frac{\nu\Delta t_n}{4}\|\nabla\overrightarrow{e}^n\|^2_{L^2(\Omega)} + \\ & \frac{\nu\Delta t_n}{\lambda^2}\|div\overrightarrow{u}^n_h\|^2_{L^2(\Omega)},\end{aligned}$$

$$\begin{aligned}d(\overrightarrow{e}^n, \overrightarrow{e}^n, \pi\overrightarrow{e}^n) + F_3 \quad \leq \quad & \frac{\nu}{8}|\overrightarrow{e}^n|^2_1 + C_4\|div\overrightarrow{u}^n_h\|^2_{0,\Omega} + \\ & C_5|\overrightarrow{e}^n|^2_{0,\Omega}.\end{aligned}$$

We have

$$\begin{aligned}F_1 \leq C\Delta t_n(\sum_{K\in\tau_{n,h}}(h_K\|\overrightarrow{f}^n_h - \frac{\overrightarrow{u}^n_h - \overrightarrow{u}^{n-1}_h}{\Delta t_n} - \\ (\overrightarrow{u}^n_h.\nabla)\overrightarrow{u}^n_h + \nu\Delta\overrightarrow{u}^n_h - \nabla p^n_h\|_{L^2(K)}) + \\ \sum_{E\in\varepsilon(K)} h_E^{\frac{1}{2}}\|[[\nu\partial_{n_E}\overrightarrow{u}^n_h - p^n_h\overrightarrow{n}_E]]_E\|_{L^2(E)})|\overrightarrow{v}|_1. \quad (74)\end{aligned}$$

$$\begin{aligned}F_2 \quad &= \quad (\overrightarrow{f}^n - \overrightarrow{f}^n_h, \overrightarrow{v} - \overrightarrow{v}_h) \\ &\leq \quad C\sum_{K\in\tau_{n,h}} h_K\|\overrightarrow{f}^n - \overrightarrow{f}^n_h\|_{L^2(K)}|\overrightarrow{v}|_1. \quad (75)\end{aligned}$$

Using (73), we obtain

$$\begin{aligned}\frac{1}{2}\|\overrightarrow{e}^n\|^2_0 - \frac{1}{2}\|\overrightarrow{e}^{n-1}\|^2_0 + \frac{1}{2}\nu\Delta t_n|\overrightarrow{e}^n|^2_1 \\ \leq C_4\{\sum_{K\in\tau_{n,h}}((\eta^n_K)^2 + \frac{h^2_K}{\nu}\|\overrightarrow{f}^n - \overrightarrow{f}^n_h\|^2_{0,K}) + \\ \sum_{K\in\tau_{n,h}}(\xi_{h\tau}\nu\Delta t_n + \frac{\nu\Delta t_n}{\lambda^2} + (\Delta t_n)^2\xi_{h\tau})\|div\overrightarrow{u}^n_h\|^2_{0,K} + \\ \Delta t_n\|\overrightarrow{e}^n\|^2_0\}. \ (76)\end{aligned}$$

Using (76) and the discrete Gronwall Lemma [4], we obtain (70).

Corollary: We suppose that the conditions of Theorem 4 holds, we have following results

$$\begin{aligned}(\sum_{m=1}^{n}\int_{t_{m-1}}^{t_m}\|\frac{\partial}{\partial_t}(\overrightarrow{u}_\tau - \overrightarrow{u}_{h\tau}) + (\overrightarrow{u}^m.\nabla)\overrightarrow{u}^m - \\ (\overrightarrow{u}^m_h.\nabla)\overrightarrow{u}^m_h + \nabla\ \Pi_\tau(p_\tau - p_{h\tau})\|^2_{H^{-1}(\Omega)}dx)^{\frac{1}{2}} \\ \leq C_5(\sum_{m=1}^{n}\Delta t_m\sum_{K\in\tau_{m,h}}(\nu(1+\xi_{h\tau})(\eta^m_K)^2 + \\ h^2_K\|\overrightarrow{f}^m - \overrightarrow{f}^m_h\|^2_{0,K}))^{\frac{1}{2}} + C_6\nu^{\frac{1}{2}}\|\overrightarrow{u}_0 - \pi_h\overrightarrow{u}_0\|^2_{L^2(\Omega)}, \quad (77)\end{aligned}$$

for all n, $1 \leq n \leq N$.

Using the results and the standard results of [17], we have following results.

Theorem 6: We suppose that the assumption (B_5) hold, and $\exists k$, $\forall n \in [1, N]$, $\forall K \in \tau_{n,h}$, $\forall H \in V_{n,h} \cup Q_{n;h}$, $H_{|K} \in P_k$. We have

$$\begin{aligned}\eta^n_K \quad \leq \quad & C_7(\sqrt{\nu}\ \|\nabla(\overrightarrow{u}^n - \overrightarrow{u}^n_h)\|_{0,\overline{\omega}_K} + \\ & \nu^{-\frac{1}{2}}\|\frac{(\overrightarrow{u}^n - \overrightarrow{u}^n_h) - (\overrightarrow{u}^{n-1} - \overrightarrow{u}^{n-1}_h)}{\Delta t_n} + \\ & \nabla(p^n - p^n_h) + (\overrightarrow{u}^n.\nabla)\overrightarrow{u}^n - \\ & (\overrightarrow{u}^n_h.\nabla)\overrightarrow{u}^n_h\|_{H^{-1}(\overline{\omega}_K)} + \\ & \sqrt{\nu}\ h_K\|\overrightarrow{f}^n - \overrightarrow{f}^n_h\|_{0,\overline{\omega}_K}), \quad (78)\end{aligned}$$

for all $n \in [1, N]$,
where $\overline{\omega}_K$ denote the union of elements of $\tau_{n;h}$ that share at least a vertex with K.
Moreover, we have

$$\begin{aligned}\eta^n \quad \leq \quad & \sqrt{\nu}\|\nabla(\overrightarrow{u} - \overrightarrow{u}_\tau)\|_{L^2(t_{n-1},t_n;L^2(\Omega))} + \\ & \nu^{-\frac{1}{2}}\|\frac{\partial}{\partial t}(\overrightarrow{u} - \overrightarrow{u}_\tau) + (\overrightarrow{u}.\nabla)\overrightarrow{u} - (\overrightarrow{u}^n.\nabla)\overrightarrow{u}^n + \\ & \nabla(p - \Pi_\tau p_\tau)\|_{L^2(t_{n-1},t_n;H^{-1}(\Omega))} + \\ & \nu^{-\frac{1}{2}}\|\overrightarrow{f} - \Pi_\tau\overrightarrow{f}\|_{L^2(t_{n-1},t_n,H^{-1}(\Omega))} + \\ & \sqrt{\frac{\Delta t_n}{3}\nu}\ \ (\|\nabla(\overrightarrow{u}^n - \overrightarrow{u}^n_h)\|_{L^2(\Omega)} + \\ & \|\nabla(\overrightarrow{u}^{n-1} - \overrightarrow{u}^{n-1}_h)\|_{L^2(\Omega)}). \quad (79)\end{aligned}$$

We define the error $\varepsilon(t_n)$ by:

$$\begin{aligned}\varepsilon^2(t_n) \quad = \quad & [\overrightarrow{u} - \overrightarrow{u}_\tau]^2(t_n) + [\overrightarrow{u}_\tau - \overrightarrow{u}_{h\tau}]^2(t_n) + \\ & \frac{1}{\nu}\sum_{m=1}^{n}\int_{t_{m-1}}^{t_m}\|\frac{\partial}{\partial t}(\overrightarrow{u} - \overrightarrow{u}_\tau) + \nabla(p - \Pi_\tau\ p_\tau) + \\ & (\overrightarrow{u}.\nabla)\overrightarrow{u} - (\overrightarrow{u}^m.\nabla)\overrightarrow{u}^m\|^2_{H^{-1}(\Omega)}dx + \\ & \frac{1}{\nu}\sum_{m=1}^{n}\int_{t_{m-1}}^{t_m}\|\frac{(\overrightarrow{u}^m - \overrightarrow{u}^m_h) - (\overrightarrow{u}^{m-1} - \overrightarrow{u}^{m-1}_h)}{\triangle t_m} \\ & +\nabla\ \Pi_\tau(p_\tau - p_{h\tau}) + (\overrightarrow{u}^m.\nabla)\overrightarrow{u}^m - \\ & (\overrightarrow{u}^m_h.\nabla)\overrightarrow{u}^m_h\|^2_{H^{-1}(\Omega)}dx, \quad (80)\end{aligned}$$

for all $n = 1, ..., N$.
Let

$$\eta_S = (\sum_{m=1}^{n}((\eta^m)^2 + \triangle t_m\sum_{K\in\tau_{m,h}}(\eta^m_K)^2))^{\frac{1}{2}}. \quad (81)$$

Summarizing and incorporating the the previous results, we have

Theorem 7: We suppose that the assumptions $(B_1) - (B_5)$ holds, the full error $\varepsilon(t_n)$ is equivalent to the error η_S : there exist positive constants m_1 and M_2 such that

$$m_1\ \eta_S \leq \varepsilon(t_n) \leq M_2\ \eta_S. \quad (82)$$

V. Numerical simulation

Example. L-shaped domain Ω, parabolic inflow boundary condition, natural outflow boundary condition.
This example represents flow in a rectangular duct with a sudden expansion; a Poiseuille flow profile is imposed on the inflow boundary (x=-1; $0 \leq y \leq 1$), and a no-flow (zero velocity) condition is imposed on the walls.
The Neumann condition (83) is applied at the outflow boundary (x=5; $-1 < y < 1$) and automatically sets the mean outflow pressure to zero.

$$\begin{cases} \nu \frac{\partial u_x}{\partial x} - p = 0, \\ \frac{\partial u_y}{\partial x} = 0. \end{cases} \quad (83)$$

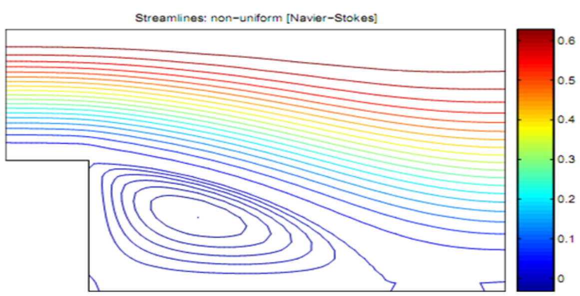


Fig.1: Equally spaced streamline plot at t = 100, with a 32×96 square grid, $Q_1 - Q_0$ approximation and $\nu = 1/600$.

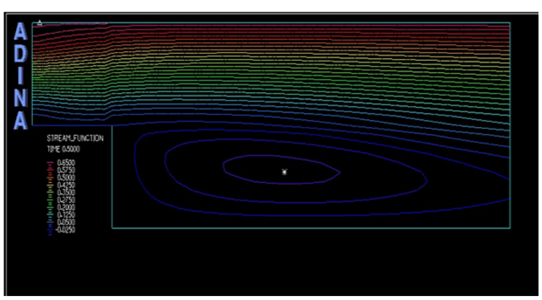


Fig.2: The solution computed with ADINA system. The plots show the Stream function at t = 100, with a 32×96 square grid and $\nu = 1/600$.

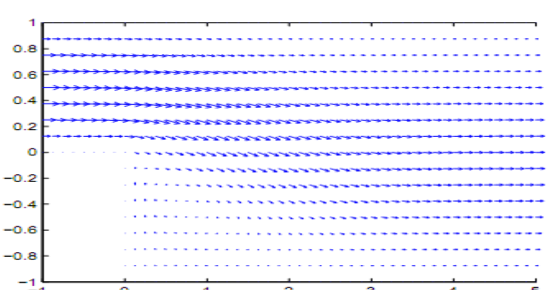


Fig.3: Quiver plot of flux solution at t = 100, with a 32×96 square grid and $\nu = 1/600$.

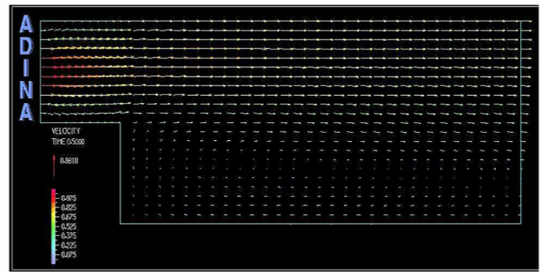
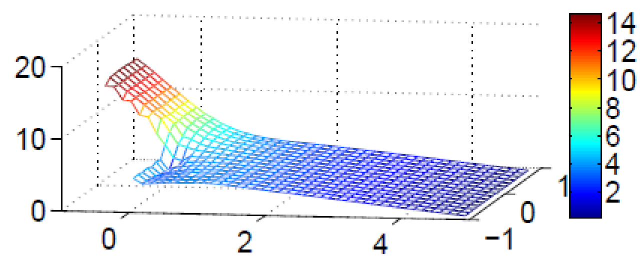
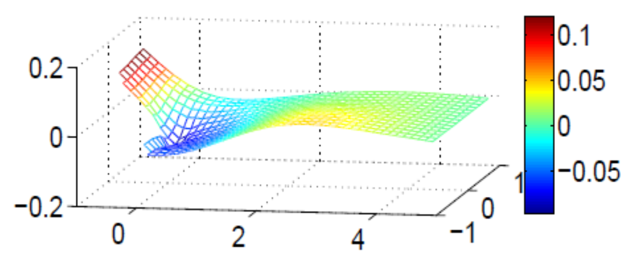


Fig.4: The solution computed with ADINA system. The plots show the Velocity vectors solution at t = 100, with a 32×96 square grid and $\nu = 1/600$.

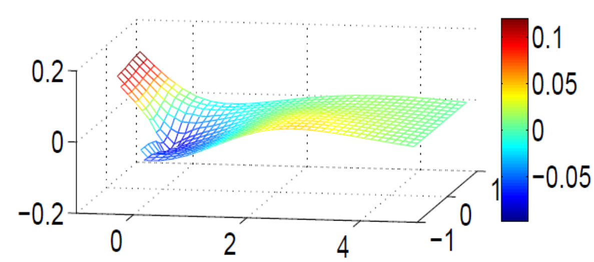
The two solutions are therefore essentially identical.This is very good indication that my solver is implemented correctly.



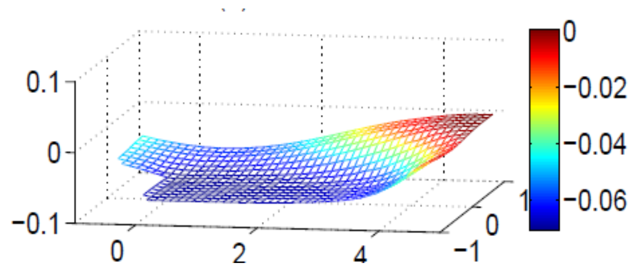
(a) Pressure at $\nu = 1$.



(b) Pressure at $\nu = \frac{1}{40}$.

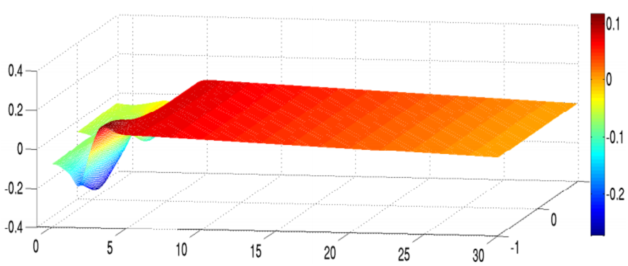


(c) Pressure at $\nu = \frac{1}{100}$.

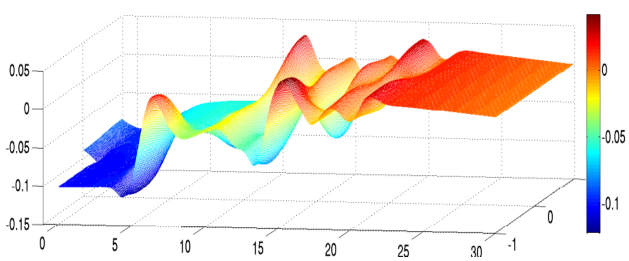


(d) Pressure at $\nu = \frac{1}{500}$.

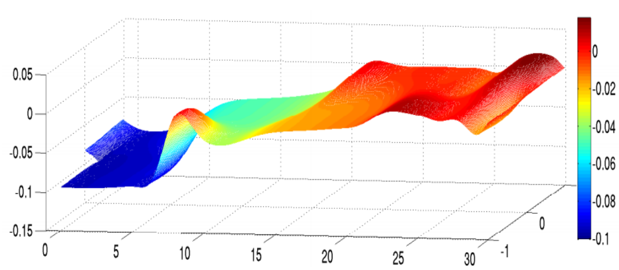
Fig.5: Pressure solutions at t = 120, for: $\nu = 1$, $\nu = \frac{1}{40}$, $\nu = \frac{1}{100}$, $\nu = \frac{1}{500}$, with a 32×96 square grid and $Q_1 - P_0$ approximation.



(a) Pressure at t = 10.25.



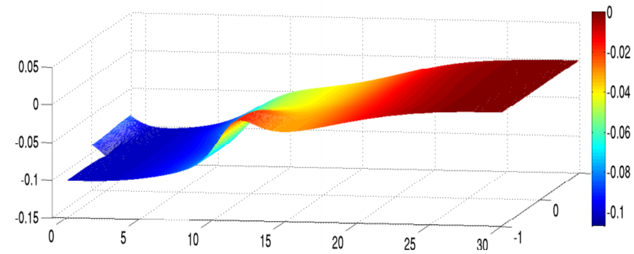
(b) Pressure at t = 50.15.



(c) Pressure at t = 100.33.

TABLE I
COMPARISON OF FINE MESH EDDY STRUCTURE AT TIME T = 450

Method	Lower length	Upper start	Upper end	Upper length
Gartling [26]	12.20	9.70	20.96	11.26
$Q_2 - P_1$	11.4369	9.2809	20.4372	11.1560
$Q_1 - P_0$ with $\alpha = 0$	11.4059	9.2501	20.4372	11.1873
$Q_1 - P_0$ with $\alpha = \frac{1}{4}\nu$	11.4059	9.2501	20.4372	11.1873
$Q_1 - P_0$ with $\alpha = \frac{1}{4}$	11.4059	9.1561	20.3123	11.1564



(d) Pressure at t = 450.02.

Fig.6: Pressure generalized by stabilized $Q_1 - P_0$ and $\nu = 1/600$.

Table I shows the comparison of fine mesh eddy structure at t = 450. The results with $\alpha = 0$ and $\alpha = \frac{1}{4}\nu$ are indistinguishable. On the coarse mesh, the $Q_1 - P_0$ approximations with $\alpha = 0$ or $\alpha = \frac{1}{4}\nu$ are much closer to the reference $Q_2 - P_1$ results than the results with the results with $\alpha = \frac{1}{4}$. In adition, all four results are in close agreement when computed using the finer mesh. The reference values provided by Gartling [26] are presented in Table I. It can be seen that our fine mesh results at the final time are slightly smaller than the reference values. Since as discussed in [32], the blunt inlet channel in [26] is known to give longer separation eddy lengths when the viscosity is small.

Table II and Table III show the number of preconditioned GMRES iterations for coarse mesh and fine mesh respectively, at the time t = 190 . The optimally stabiliized system with $\alpha = \frac{1}{4}\nu$ is significantly better conditioned than the over-stabilized system with $\alpha = \frac{1}{4}$.

Looking at Fig.5, the spurious pressure oscillations of unstabilized $Q_1 - P_0$ can be seen to diminish inmagnitude as the viscosity parameter is reduced. This suggests that the stabilization parameter should be scaled in proportion to the viscosity in order to avoid over-stabilizing the pressure approximation. The pressure solution evolution is shown in Fig.6. These pictures show that the pressure changes rapidly at the beginning and goes to a steady-state at the end of the time interval.

TABLE II
NUMBER OF PRECONDITIONED GMRES ITERATIONS FOR COARSE MESH AT TIME T = 190

Method	Standard	Rescaled
$Q_2 - P_1$	14	12
$Q_1 - P_0$ with $\alpha = 0$	7	7
$Q_1 - P_0$ with $\alpha = \frac{1}{4}\nu$	7	10
$Q_1 - P_0$ with $\alpha = \frac{1}{4}$	34	82

TABLE III
NUMBER OF PRECONDITIONED GMRES ITERATIONS FOR FINE MESH AT TIME T = 190

Method	Standard	Rescaled
$Q_2 - P_1$	10	9
$Q_1 - P_0$ with $\alpha = 0$	7	7
$Q_1 - P_0$ with $\alpha = \frac{1}{4}\nu$	8	9
$Q_1 - P_0$ with $\alpha = \frac{1}{4}$	32	68

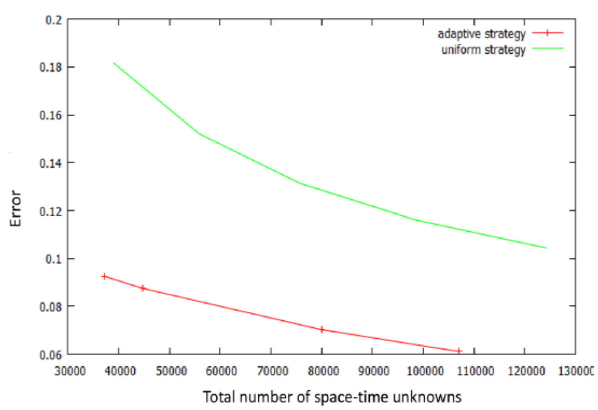


Fig.7: Comparison between uniform and adaptive method.

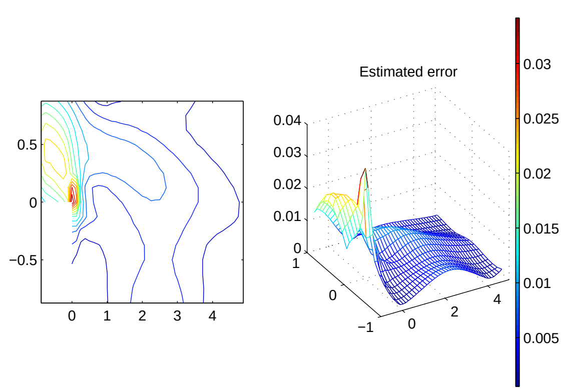


Fig.8: Estimated error η_T associated with 32×96, square grid of a $Q_1 - Q_0$ solution for the flow at t = 100, with $\nu = 1/100$.

VI. CONCLUSION

In this paper, we were interested in the numerical solution of the partial differential equations by simulating the flow of an incompressible fluid. We applied the finite element method to the resolution of the unsteady Navier-Stokes equations. The matrix system is solved at each iteration with a preconditioned GMRES method. We obtain a faster convergence. We also study a posteriori error estimates for the finite element approximation of the unsteady Navier-Stokes problem and we proposed two types of a posteriori error indicator, with one being for the time discretization and the other for the space discretization. We prove the equivalence between the sum of the two types of error indicators and the full error.

Numerical experiments were carried out and compared with satisfaction with other numerical results, either resulting from the literature, or resulting from calculation with commercial software like Adina system.
The comparisons show good agreement.

ACKNOWLEDGMENT

The authors would like to express their sincere thanks for the referee for his/her helpful suggestions.

REFERENCES

[1] J. Roberts and J.M. Thomas, *Mixed and Hybrid methods, Handbook of Numerical Analysis II, Finite Element Methods 1*, P. Ciarlet and J. Lions, Amsterdam, 1989.
[2] P.A. Raviart and J.M. Thomas, *A mixed finite element method for second order elliptic problems, in: Mathematical Aspects of Finite Element Method, Lecture Notes in Mathematics*, Springer, New York, 606, pp. 292-315, 1977.
[3] H.C. Elman, D.J. Silvester and A.J. Wathen, *Finite Elements and Fast Iterative Solvers with Applications in Incompressible Fluid Dynamics*, Oxford University Press, 2005.
[4] V. Giraul and P.A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, Springer-Verlag, Berlin, 1986.
[5] S. Turek, *A comparative study of some time-stepping techniques for the incompressible Navier-Stokes equations: From fully implicit nonlinear schemes to semi-implicit projection methods*, Int. J. Numer. Meth. Fluids., vol. 22, pp. 987-1011, 1996.
[6] S. Garcia, *The lid-driven square cavity flow: from stationary to time periodic and chaotic*, Commun. Comput. Phys., vol. 2, pp. 900-932, 2007.
[7] C. Bernardi and G. Raugel, *A conforming finite element method for the time-dependent Navier- Stokes equations*, SIAM J. Numer. Anal., vol. 22, pp. 455-473, 1985.
[8] Y. He, *Optimal error estimate of the penalty finite element method for the time- dependent Navier-Stokes equations*, Math. Comput., vol. 74, pp. 1201-1216, 2005.
[9] C. Bernardi and R. Verfurth, *A posteriori error analysis of the fully discretized time- dependent Stokes equations*, ESAIM: M2AN., vol. 38, pp. 437-455, 2004.
[10] A. Nejat and C. Ollivier-Gooch, *Effect of discretization order on preconditioning and convergence of a high-order unstructured Newton-GMRES solver for the Euler equations*, J. Comput. Phys., vol. 227, pp. 2366-2386, 2008.
[11] Y. He, *The Euler implicit/explicit scheme for the 2D time-dependent Navier-Stokes equations with smooth or non-smooth initial data*, Math. Comput., vol. 77, pp. 2097-2124, 2008.
[12] J.G. Heywood and R. Rannacher, *Finite-element approximation of the nonstationary Navier-Stokes problem. Part IV: Error analysis for second-order time discretization*, SIAM J. Numer. Anal., vol. 27, pp. 353-384, 1990.
[13] R. Temam, *Theory and Numerical Analysis of the Navier-Stokes Equations*, North- Holland, 1977.

[14] V. John and S. Kaya, *A finite element variational multiscale method for the Navier-Stokes equations*, SIAM J. Sci. Comput., vol. 26, pp. 1485-1503, 2005.

[15] J. Li, Y.N. He and Z.X. Chen, *A new stabilized finite element method for the transient Navier-Stokes equations*, Comput. Methods. Appl. Mech. Eng ., vol. 197, pp. 22-35, 2007.

[16] M. Ainsworth and J. Oden, *A Posteriori Error Estimation in Finite Element Analysis*, Wiley, New York, [264, 266, 330, 334, 335], 2000.

[17] R. Verfurth, *A Review of a posteriori Error Estimation and Adaptive Mesh-Refinement Techniques*, Wiley-Teubner, Chichester, 1996.

[18] G. Barrenecheaa and J. Blasco, *Pressure stabilization of finite element approximations of time-dependent incompressible flow problems*, Comput. Methods Appl. Mech. Engrg., vol. 197, pp. 219-231, 2007.

[19] L. Shan and Y.R. Hou, *A fully discrete stabilized finite element method for the time-dependent Navier-Stokes equations*, Appl. Math. Comput., vol. 215, pp. 85-99, 2009.

[20] Y. He and K. Li, *Two-level stabilized finite element methods for the steady Navier-Stokes problem*, Computing, vol. 74, pp. 337-351, 2005.

[21] Y. Yan, *Galerkin finite element methods for stochastic parabolic partial differential equations*, SIAM J. Numer. Anal., vol. 43, pp. 1363-1384, 2005.

[22] S. Nicaise and N. Soualem, *A posteriori error estimates for a nonconforming finite element discretization of the time-dependent Stokes problem*, J. Numer. Math., vol. 15, pp. 137-162, 2007.

[23] H. Elman, , V.E. Howle, J. Shadid, D. Silvester and R. Tuminaro, *Least squares preconditioners for stabilized discretizations of the Navier- Stokes equations*, SIAM J. Sci. Comput., vol. 30, pp. 290-311, 2007.

[24] P. Clement, *Approximation by finite element functions using local regularization*, RAIRO. Anal. Numer., vol. 9, pp. 77-84, 1975.

[25] D. Kay, P. Gresho, D. Griffiths and D. Silvester, *Adaptive time-stepping for incompressible flow part ii: Navier-Stokes equations*, SIAM J. Sci. Comput., vol. 32, pp. 111-128, 2010.

[26] D. Gartling, *A test problem for outflow boundary conditions - flow over a backward-facing step*, Int. J. Numer. Meth. Fluids., vol. 11, pp. 953-967, 1990.

[27] Q. Liao and D. Silvester, *Fast Implicit Solvers using Stabilized Mixed Approximation*, Int. J. Numer. Meth. Fluids., vol. 36, pp. 1-26, 2011.

[28] A. El Akkad and A. Elkhalfi, *A posteriori error estimation for incompressible viscous fluid with a new boundary condition*, Bol. Soc. Paran. Mat., vol. 36, pp. 53-74, 2018.

[29] A. Elakkad, A. Elkhalfi and N. Guessous, *An a posteriori error estimate for mixed finite element approximations of the Navier-Stokes equations*, J. Korean Math. Soc., vol. 48, pp. 529-550, 2011.

[30] A. Gauthier, F. Saleri and A. Veneziani, *A fast preconditioner for the incompressible Navier-Stokes equations*, Comput. Visual. Sci., vol. 6, pp. 105-112, 2004.

[31] Y. Saad and M.H. Schultz, *GMRES: a generalizedminimal residual algorithm for solving non- symmetric linear systems*, SIAM J. Sci. Stat. Comput., vol. 7, pp. 856-869, 1986.

[32] I. Barton, *The entrance effect of laminar flow over a backward-facing step geometry*, Int. J. Numer. Meth. Fluids., vol. 25, pp. 633-644, 1997.

[33] J. Frutos, B.G. Archilla, J. Novo, *Fully Discrete Approximations to the Time-Dependent Navier-Stokes Equations with a Projection Method in Time and Grad-Div Stabilization*, Journal of Scientific Computing, vol. 80, pp. 1330-1368, 2019.