Contact of a shell and rigid body though the heat-conducting layer temperature field

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Abstract— A problem of heat conducting and unilateral contact of a shell through the heat-conducting layer is formulated. An approach consists in considering a change of layer thickness in the process of the shell deformation. Three dimensional connected equations of the thermoelasticity and the heat conductivity are created. These equations take into account change of the conditions of heat exchange between the shell and the rigid body during the structure deformation and a possibility of close mechanical contact. Three dimensional equations of thermoelasticity and heat coduction are expanded into a polynomial Legendre series in terms of the thickness. The first-approximations, Timoshenko's and Kirchhoff-Love's equations have been studied. Numerical example of the unilateral thermoelastic contact of the cylindrical shells and rigid body through the heat-conducting layer is considered.

***Keywords*—**Heat-conductivity, cylindrical shell, heat-conducting layer, mechanical contact.

I. Introduction

Many elements of machines and structures during their exploitation are affected by high temperature and mechanical loading. Contact interaction is the most common way to transfer load from one body to another. In the case if contacting bodies have different temperature between them take place heat-contact interactions. Therefore not only condition of the mechanical contact, but also conditions of the thermal contact have to be considered. Usually perfect thermal contact is supposed, i.e. it is supposed that the temperature and the thermal flux of the contacting bodies in the contact area are the same [1], [4]. In [6], [7] it was shown that in many cases these contact conditions are not acceptable because they can not take into account physical processes related to deformation and heat exchange. In these publications it have been considered the problem of thermoelastic contact of plates and shells thought a heat-conduction layer with considering change of the layer thickness during the plates and shells deformation. Numerical examples presented there show that in many important for science and engineering cases the result obtained using a perfect thermoelastic contact conditions and the conditions with considering change of the layer thickness in the process of deformation are very different. In some cases the difference is not only quantitative but also qualitative. Therefore it is very important to consider contact conditions introduced in [6], [7] in the problems were thin-walled structures may have contact though the heat-conducting layer in the intensive temperature field. Such kind of problems takes place in many important structures, equipment, and devices in chemical, airspace, nuclear industries etc.

The approach developed in [6], [7] have been applied to the plates and shells thermoelastic contact problems [11], [12], the laminated composite materials with possibility of delamination and thermoelastic contact in temperature field in [8], [9], and the pencil-thin nuclear fuel rods modeling [10].

In this paper some new results related to unilateral thermoelastic contact of the thin-walled structures through the heat-conducting layer are formulated. The connected equations of thermoelasticity and heat conductivity are created. These equations take into account change of the conditions of heat exchange between the shell-like structures and the rigid body during the structures deformation and possibility of close unilateral mechanical contact. Numerical example of the heat conductivity of the cylindrical shells through the heat-conducting layer is considered. The thermomechanical effects caused by contact interaction and their influence on the thermomechanics parameters were investigated.

II. 3-D Statement of the Problem

Let us consider an elastic homogeneous deformable and rigid bodies in the temperature field situated in an initial, undeformed state in a distance $h_0(\mathbf{x})$ apart. There is a heat-conducting medium in the gap between the bodies. The medium does not resist the body deformation, and heat exchange between the bodies is due to the thermal conductivity of the medium. We assume that gap h_0 is commensurable with the body displacements and we assume those displacements to be small.

The thermodynamic state of the deformable body and the heat-conducting medium, is defined by the following parameters: $\sigma_{ij}(\mathbf{x})$, $\varepsilon_{ij}(\mathbf{x})$ and $u_i(\mathbf{x})$ are the components of the stress and strain tensors and displacement vector, and $\theta(\mathbf{x})$, $\chi(\mathbf{x})$, $\theta^*(\mathbf{x})$, $\chi^*(\mathbf{x})$ are the temperature and specific strength of the internal heat sources at the body and the medium respectively.

We denote by V volume occupied by deformable body and by ∂V its boundary. The body, boundary may be presented in the forms

$$\partial V = \partial V_p \cup \partial V_u \cup \partial V_e \text{ and } \partial V = \partial V_\theta \cup \partial V_q \cup \partial V_e .$$

On the parts ∂V_p and ∂V_u boundary conditions for displacements and traction are prescribed. On parts ∂V_θ and ∂V_q boundary conditions for temperature and heat flux are prescribed. On the part of the boundary ∂V_e contact of bodies takes place. Different parts of the boundary do not intersect:

$\partial V_p \cap \partial V_u \cap \partial V_e = \varnothing$ and $\partial V_\theta \cap \partial V_q \cap \partial V_e = \varnothing$.

We denote by V^* volume occupied by the heat-conducting medium and by ∂V^* its boundary. The boundary of the medium may be presented in the forms

$$\partial V^* = \partial V_\theta^* \cup \partial V_q^* \cup \partial V_e \ .$$

On parts ∂V_θ^* and ∂V_q^* boundary conditions for temperature and heat flux are prescribed.

A. Equations of thermoelasticity

We assume that displacements of the body points and their gradients are small and relations between deformations, stress and temperature are linear. In this case thermodynamic state of the deformable body is defined by linear equations of thermoelasticity. The stress-strain state is described by small strain deformation tensor $\varepsilon_{ij}(\mathbf{x})$. The strain tensor and displacement vector are connected by Cauchy relations

$$\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \tag{1}$$

The components of the strain tensor must also satisfy the Saint-Venant's relations

$$\partial_{kl}^2 \varepsilon_{ij} - \partial_{il}^2 \varepsilon_{kj} = \partial_{kj}^2 \varepsilon_{il} - \partial_{ij}^2 \varepsilon_{kl} \tag{2}$$

From the balance of impulse and moment of impulse lows follow that the stress tensor is symmetric one and satisfy the equations of equilibrium

$$\partial_j \sigma_{ij} + b_i = 0_i, \qquad \forall \mathbf{x} \in V. \tag{3}$$

Here and throughout the article the summation convention applies to repeated indices.

The stress $\sigma_{ij}(\mathbf{x})$ tensor, tensor of deformation $\varepsilon_{ij}(\mathbf{x})$ and temperature are related by Hook's law

$$\sigma_{ij} = c_{ijkl}\sigma_{kl} + \beta_{ij}\theta \ , \ c_{ijkl} = c_{jikl} = c_{klij}\, , \ \beta_{ij} = \beta_{ji} \tag{4}$$

where $\partial_i = \partial/\partial x_i$ are partial derivatives with respect to the space variables x_i, c_{ijkl} and β_{ij} are elastic modulus and the coefficients of linear thermal expansion. In the isotropic case

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}),\ \beta_{ij} = (\mu + 3\lambda)\alpha\delta_{ij} \tag{5}$$

where λ and μ are the Lame constants, α are the coefficients of linear thermal expansion.

The differential equations of equilibrium for the displacement vector components may be presented in the form

$$A_{ij}u_j + A_i\,\theta + b_i = 0 \ , \tag{6}$$

with

$$A_{ij} = c_{ijkl}\partial_k\partial_l \ , \ A_i = \beta_{ij}\partial_j \ , \tag{7}$$

$$A_i = (\mu + 3\lambda)\alpha\partial_i\, , \ \ A_{ij} = \mu^2\delta_{ij}\partial_k\partial_k + (\lambda + \mu)\partial_i\partial_j\, ,$$

in anisotropic and isotropic case respectively.

B. Mechanical boundary and contact conditions

On the parts ∂V_p and ∂V_u boundary conditions for displacements and traction have the form

$$\begin{gathered} u_i = \varphi_i \ \ \forall \mathbf{x} \in \partial V_u \\ p_i = \sigma_{ij}n_j = P_{ij}[u_j(\mathbf{x})] = \psi_i\ , \forall \mathbf{x} \in \partial V_p \end{gathered} \ , \tag{8}$$

where φ_i and ψ_i prescribed displacements and tractions on the boundary respectively.

The differential operator $P_{ij} : u_j \to p_i$ is called stress operator. It transforms the displacements into the tractions. For homogeneous anisotropic and isotropic body they have the forms

$$P_{ij} = c_{ikjl}n_k\partial_l \ \text{and} \ P_{ij} = \lambda n_i\partial_k + \mu\left(\delta_{ij}\partial_n + n_k\partial_i\right) \tag{9}$$

respectively. Here n_i are components of the outward normal vector, $\partial_n = n_i\partial_i$ is a derivative in direction of the vector $\mathbf{n}(\mathbf{x})$ normal to the surface ∂V_p.

In the area ∂V_e unilateral mechanical contact with friction may occur. Therefore boundary conditions have form of inequalities [2]

$$\begin{gathered} u_n = \geq h_0\ , q_n \geq 0\, , \left(u_n - h_0\right)q_n = 0 \\ \left|\mathbf{p}_\tau\right| < k_\tau\, p_n \to \partial_t\mathbf{u}_\tau = 0 ; \left|\mathbf{p}_\tau\right| = k_\tau\, p_n \to \partial_t\mathbf{u}_\tau = -\lambda_\tau\mathbf{p}_\tau \end{gathered} \tag{10}$$

where p_n, u_n, $\mathbf{p}_\tau$ and $\mathbf{u}_\tau$ are the normal and tangential components of the contact force vector and the displacement vector respectively, k_τ and λ_τ are coefficients which depend upon the properties of the contact surfaces.

C. Equations of heat conductivity

We assume that heat distribute in the body and in the media according to Fourier low

$$q_i = \lambda_{ij}\partial_j\theta \tag{11}$$

Here q_i is a vector of thermal flow, λ_{ij} is the tensor of coefficients of thermal conductivity of the body. In the isotropic case

$$\lambda_{ij} = \delta_{ij}\lambda_T\, , \tag{12}$$

where λ_T is the coefficients of thermal conductivity of the body

Then linear equations for heat conductivity for the body have the form

$$\lambda_{ij}\partial_i\partial_j\theta - \chi = 0 \ , \ \forall\, \mathbf{x}\in V \tag{13}$$

The temperature distribution within the heat-conducting medium is described by the equations of heat conductivity

$$\lambda^*_{ij}\partial_i\partial_j\theta_* - \chi_* = 0 \ , \ \forall \mathbf{x}\in V^* \tag{14}$$

Here λ^*_{ij} is the tensor of coefficients of thermal conductivity of the body. In the isotropic case

$$\lambda^*_{ij} = \delta_{ij}\lambda^*_T \, , \tag{15}$$

where λ^*_T is the coefficients of thermal conductivity of the body

D. Thermal boundary and contact conditions

On the parts ∂V_θ and ∂V_q boundary conditions for temperature and heat flux have the form

$$\theta = \theta^b \ , \forall \mathbf{x}\in \partial V_\theta \ , q_i = q_i^b \ , \forall \mathbf{x}\in \partial V_q \ , \tag{16}$$

where θ^b and q_i^b prescribed temperature and thermal flux on the boundary respectively.

Boundary conditions on the lateral sides of the heat-conducting medium will be considered in the form

$$n_i\lambda_{ij}\partial_j\theta + \beta\left(\theta - \theta^b\right) = 0 \tag{17}$$

where coefficient β depends on thermal properties of surroundings.

We assume that on the part of the body boundary that is in thermal contact with media classical thermal contact conditions take place. It means that temperature and thermal flux of the body and media on contact area equals. Therefore conditions of heat conductivity through the heat-conducting medium have the form

$$\theta_* = \theta \, , \lambda^*_{ij}\partial_n\theta_* = \lambda_{ij}\partial_n\theta \ \ , \ \forall \mathbf{x}\in\partial V_e \tag{18}$$

In the area of close mechanical contact the thermal conditions are transformed into the form

$$q_\theta = \alpha_e\left(\theta - \theta^b\right) \ , \ \forall \mathbf{x}\in \partial V_e \tag{19}$$

where q_θ is the heat flux passing across the close mechanical contact area, α_e is the coefficient of the contact surface thermal conductivity.

Now problem consists in join solution of the equations of themoelasticity (6) with boundary conditions (8) and unilateral contact conditions with friction (10), equations of heat conductivity for shell (13) and heat conducting medium (14) and thermal boundary and contact conditions (16)-19). Analysis of the problem encounters mathematical difficulties caused by the dimension of the problem, as well as by its non-linearity. The problem can be partially simplified considering thin-walled bodies. In this case we can reduce the dimension of the problem

III. 2-D STATEMENT OF THE PROBLEM

Let a deformable body be an elastic homogeneous shell of arbitrary geometry with 2*h* thickness. In this case the region V occupied by the body and its boundary ∂V may be represented as

$$V = \Omega\times[-h,h] \text{ and } \partial V = S\cup\Omega^+\cup\Omega^- .$$

Here Ω is the middle surface of the shell, $\partial\Omega$ is its boundary, Ω^+ and Ω^- are the outer sides and $S = \partial\Omega\times[-h,h]$ is a sheer side.

Let it be assumed that the component parameters, which describe the stress-strain state of a deformable body as a three-dimensional body are sufficiently smooth functions of x_3 coordinate and may be expanded into Legandre's polynomial series. Using the approach developed in [3], [5], they can be expressed as

$$u_i(\mathbf{x}) = \sum_{k=0}^{\infty} u_i^k(\mathbf{x}_\alpha)P_k(\omega) \, , \ \sigma_{ij}(\mathbf{x}) = \sum_{k=0}^{\infty}\sigma_{ij}^k(\mathbf{x}_\alpha)P_k(\omega),$$

$$\varepsilon_{ij}(\mathbf{x}) = \sum_{k=0}^{\infty}\varepsilon_{ij}^k(\mathbf{x}_\alpha)P_k(\omega), \ \theta(\mathbf{x}) = \sum_{k=0}^{\infty}\theta^k(\mathbf{x}_\alpha)P_k(\omega)\, , \tag{20}$$

$$\theta_*(\mathbf{x}) = \sum_{n=0}^{\infty}\theta_*^k(\mathbf{x}_\alpha)P_k(\omega),$$

where

$$u_i^k(\mathbf{x}_\alpha) = \frac{2k+1}{2h}\int_{-h}^{h} u_i(\mathbf{x}_\alpha, x_3)P_k(\omega)dx_3 \ ,$$

$$\sigma_{ij}^k(\mathbf{x}_\alpha) = \frac{2k+1}{2h}\int_{-h}^{h}\sigma_{ij}(\mathbf{x}_\alpha, x_3)P_k(\omega)dx_3 \ ,$$

$$\varepsilon_{ij}^k(\mathbf{x}_\alpha) = \frac{2k+1}{2h}\int_{-h}^{h}\varepsilon_{ij}(\mathbf{x}_\alpha, x_3)P_k(\omega)dx_3 \ , \tag{21}$$

$$\theta^k(\mathbf{x}_\alpha) = \frac{2k+1}{2h}\int_{-h}^{h}\theta(\mathbf{x}_\alpha, x_3)P_k(\omega)dx_3 \ ,$$

$$\theta_*^k(\mathbf{x}_\alpha) = \frac{2k+1}{2h}\int_{-h}^{h}\theta_*(\mathbf{x}_\alpha, x_3)P_k(\omega)dx_3 \ ,$$

$\omega = x_3/h$ is a dimensionless coordinate.

Then we will get the equations of the problem in terms of the coefficients of this expansion. As a result, we obtain a 2-D system of equations for coefficients of Legandre's polynomial series.

A. Equations of thermoelsticity

In order to obtain 2-D equations of thermoelasticity we have to substitute expansion (20) into 3-D equations (1)-(6). Expansion of the corresponding derivatives gives us

$$\frac{2k+1}{2h}\int_{-h}^{h}\partial_\alpha \sigma_{ij} P_k(\omega)dx_3 = \partial_\alpha \sigma_{ij}^k \,,\ \frac{2k+1}{2h}\int_{-h}^{h}\partial_\alpha u_i P_k(\omega)dx_3 = \partial_\alpha u_i^k$$

$$\frac{2k+1}{2h}\int_{-h}^{h}\partial_3 \sigma_{i3} P_k(\omega)dx_3 = \frac{2k+1}{h}\left[\sigma_{i3}^+ - (-1)^k \sigma_{i3}^-\right] - \frac{2k+1}{h}\left(\sigma_{i3}^{k-1} + \sigma_{i3}^{k-3} + \dots\right)$$

$$\frac{2k+1}{2h}\int_{-h}^{h}\partial_3 u_i P_k(\omega)dx_3 = \frac{2k+1}{h}\left(u_i^{k+1} + u_i^{k+3} + \dots\right),$$

$$\frac{2k+1}{2h}\int_{-h}^{h}\partial_\alpha \theta P_k(\omega)dx_3 = \partial_\alpha \theta^k \,,\ \frac{2k+1}{2h}\int_{-h}^{h}\partial_3 \theta P_k(\omega)dx_3 = Q_3^k$$

Then the differential equations of thermoelasticity (3) are transformed into its 2-D form

$$\begin{aligned}
&\sum_{l=0}^{\infty}\left(L_{11}^{kl}u_1^l + L_{12}^k u_2^k + L_{13}^{kl}u_3^l\right) - \beta_\tau \partial_1\left(\theta^k - \theta_0^k\right) + p_1^k = 0 \\
&\sum_{l=0}^{\infty}\left(L_{21}^{kl}u_1^l + L_{22}^k u_2^k + L_{23}^{kl}u_3^l\right) - \beta_\tau \partial_2\left(\theta^k - \theta_0^k\right) + p_2^k = 0 \\
&\sum_{l=0}^{\infty}\left(L_{31}^{kl}u_1^l + L_{32}^k u_2^k + L_{33}^{kl}u_3^l\right) + (k_1 + k_2)\beta_\tau\left(\theta^k - \theta_0^k\right) + \\
&+ 2\beta_\tau \frac{2k+1}{2h}\left(\theta^{k-1} - \theta_0^{k-1} + \theta^{k-3} - \theta_0^{k-3} \dots\right) + p_3^k = 0
\end{aligned} \quad (21)$$

where $p_i^k = \dfrac{2k+1}{2h}\left[p_i^+ - (-1)^k p_i^-\right] - b_i^k$.

As the result instead of 3-D system of the differential equations we get infinite system of 2-D differential equations. For some specific types of shells geometry and for plates analytical expressions for differential operators L_{ij}^{kl} may be found in [3], [5], [6].

B. Mechanical boundary and contact conditions

The boundary conditions at the sheer side (4) and (7) easy can be transformed into 2-D form. Applying expansion into Legendre's polynomial series we obtain boundary conditions for coefficients of the expansion in the form

$$p_i^k = \psi_i^k \,,\ \forall\, \mathbf{x} \in \partial\Omega_p \,;\ u_i^k = \varphi_i^k \,,\ \forall\, \mathbf{x} \in \partial\Omega_u \,. \quad (22)$$

Surface forces and displacements on upper and lower sides of the shell are calculated in the form

$$\begin{aligned}
&\sum_{k=0}^{\infty} p_i^k = p_i^+ \,,\ \sum_{k=0}^{\infty}(-1)^k u_i^k = u_i^+ \,,\quad \forall\, \mathbf{x} \in \Omega^+ \\
&\sum_{k=0}^{\infty}(-1)^k p_i^k = p_i^- \,,\ \sum_{k=0}^{\infty} u_i^k = u_i^- \,,\ \forall\, \mathbf{x} \in \Omega^-
\end{aligned} \quad (23)$$

We used here relations for Legendre's polynomial

$$P_k(1) = 1 \,,\ P_k(-1) = (-1)^k$$

The contact conditions (10) can not be formulated for the coefficients of Legendre's polynomial series because of their nonlinearity. They are transformed into 2-D form with considering the representations for surface forces and displacements on contact surface using (23).

C. Equations of heat conductivity

In order to obtain 2-D equations of heat conductivity we have to substitute expansion (20) into 3-D equation of heat conductivity (13). Expansion of the second derivatives of the temperature with respect to x_3 gives us

$$\frac{2k+1}{2h}\int_{-h}^{h}\partial_3^2 \theta P_k(\omega)dx_3 = \frac{2k+1}{h}\left[Q_3^+ - (-1)^k Q_3^-\right] - \frac{2k+1}{h}\left(Q_3^{k-1} + Q_3^{k-3} + \dots\right)$$

Then the 3-D differential equation of heat conductivity (13) is transformed into 2-D form

$$\begin{aligned}
&\Delta_0 \theta^k + \frac{2k+1}{2h}\left[Q_3^+ - (-1)^k Q_3^-\right] - \frac{2k+1}{h}\left[Q_3^{k-1} + Q_3^{k-3} \dots\right] + \\
&+ (k_1 + k_2)Q_3^k + \frac{\chi_k}{\lambda_0} = 0
\end{aligned} \quad (24)$$

The 2-D equations of heat-conductivity for the layer have more complicate form. It is because of the layer thickness is variable, i.e. $h^+(\mathbf{x})$ and $h^-(\mathbf{x})$ are functions of coordinates. Therefore expansion of the corresponding derivative gives us

$$\begin{aligned}
&\frac{2k+1}{2h}\int_{h^-}^{h^+}\partial_1\left(\frac{A_2}{A_1}\partial_1 \theta_*\right)P_k(\omega_*)dx_3 = \partial_1\left(\frac{A_2}{A_1}\partial_1 \theta_*^k\right) + \{\partial_1\left(\frac{1}{h}\partial_1 h\right)T_2^k + \\
&+ \partial_1\left(\frac{1}{h}\partial_1 h_*\right)T_1^k + \frac{1}{h}\partial_1 h \partial_1 T_2^k - \frac{2k+1}{2h}\left[\partial_1\left(\frac{1}{h}\partial_1 h^+\right)\theta_*^+ - (-1)^k \partial_1\left(\frac{1}{h}\partial_1 h^-\right)\theta_*^-\right] + \\
&+ \frac{1}{h}\partial_1 h \underline{Q}_1^k + \frac{1}{h}\partial_1 h \underset{-}{Q}_1^k\}\frac{A_2}{A_1}
\end{aligned}$$

Here $\omega_* = \dfrac{2x_3 - h_*}{h}$, $h = h^+ - h^-$, $h_* = h^+ + h^-$,

$$T_1^k = (2k+1)\left(\theta_*^{k-1} + \theta_*^{k-3} + \dots\right), T_2^k = (k+1)\theta_*^k + (2k+1)\left(\theta_*^{k-2} + \theta_*^{k-4} + \dots\right)$$

$$\begin{aligned}
\underline{Q}^k = (2k+1)\Big\{&\partial_1 \theta_*^{k-1} + \partial_1 \theta_*^{k-3} + \dots + \frac{1}{h}\partial_1 h\Big[k\theta_*^{k-1} + \\
&+ (3k-3)\theta_*^{k-3} + \dots\Big] + \frac{1}{h}\partial_1 h\left[(2k-1)\theta_*^{k-2} + (4k-6)\theta_*^{k-4} + \dots\right] - \\
&- \left(\frac{2k-1}{2} + \frac{2k-5}{2} + \dots\right)\frac{1}{h}\left[\partial_1 h^+ \theta_*^+ - (-1)^k \partial_1 h^- \theta_*^-\right]\Big\}
\end{aligned}$$

$$\begin{aligned}
\underset{-}{Q}^k = (k+1)\Big\{&\partial_1 \theta_*^k + \frac{2k+1}{2h}\left[\partial_1 h^+ \theta_*^+ - (-1)^k \partial_1 h^- \theta_*^-\right] + \frac{1}{h}\partial_1 h T_2^k + \frac{1}{h}\partial_1 h_* T_1^k + \\
&+ (2k+1)\{\partial_1 \theta_*^{k-2} + \partial_1 \theta_*^{k-4} + \dots + \frac{1}{h}\partial_1 h\left[(k-1)\theta_*^{k-2} + (3k-6)\theta_*^{k-4} + \dots\right] + \\
&+ \partial_1 h_*\left[(2k-3)\theta_*^{k-3} + (4k-10)\theta_*^{k-5} + \dots\right] - \\
&- \left(\frac{2k-3}{2} + \frac{2k-7}{2} + \dots\right)\frac{1}{h}\left[\partial_1 h^+ \theta_*^+ - (-1)^k \partial_1 h^- \theta_*^-\right]\Big\}
\end{aligned}$$

Then 2-D equations of heat-conductivity for the layer are transformed into 2-D form

$$\begin{aligned}
&\Delta_0 \theta_*^k + \Delta_* h T_2^k + \Delta_* h_* T_1^k + \left(\nabla_* h \cdot \nabla T_2^k\right) + \left(\nabla_* h_* \cdot \nabla T_1^k\right) - \\
&- \frac{2k+1}{2}\left[\theta_*^+ \Delta_* h^+ + (-1)^k \theta_*^- \Delta_* h^-\right] + \left(\nabla_* h \cdot \mathbf{Q}_2^k\right) + \\
&+ \left(\nabla_* h \cdot \mathbf{Q}_1^k\right) + \frac{2k+1}{2h}\left[Q_3^+ - (-1)^k Q_3^-\right] -
\end{aligned} \quad (25)$$

$$-\frac{2k+1}{2h}\left(Q_3^{k-1}+Q_3^{k-3}+\ldots\right)+\left(k_1+k_2\right)Q_3^k+\frac{\chi^k}{\lambda_*}=0$$

where

$$\Delta_0=\frac{1}{A_1A_2}\left(\partial_1\frac{A_2}{A_1}\partial_1+\partial_2\frac{A_1}{A_2}\partial_2\right) \quad ,$$

$$\Delta_*=\frac{A_2}{A_1}\partial_1\frac{1}{h}\partial_1+\frac{A_1}{A_2}\partial_2\frac{1}{h}\partial_2 \quad , \quad \nabla_*=\frac{1}{h}\left(\frac{A_2}{A_1}\partial_1+\frac{A_1}{A_2}\partial_2\right) \ ,$$

$$\nabla=\partial_1+\partial_2 \ , \ \mathbf{Q}_1^k=\left(\underline{Q}_1^k,\underline{Q}_2^k\right), \mathbf{Q}_2^k=\left(\underline{Q}_{-1}^k,\underline{Q}_{-2}^k\right)$$

As the result instead of 3-D system of the differential equations we get infinite system of 2-D differential equations. The equations of heat conductivity for the layer (25) are complicate because it contains information about deformation of the shell.

The equations thermo-elasticity and heat conductivity are written in coordinates related to the principal curvatures of the shell surfaces. Here A_α are coefficients of the first quadratic form, and k_α are principal curvatures.

D. Thermal boundary and contact conditions

The boundary conditions (4) and (7) can be transformed into 2-D form in the same way like it was done in (22). As result we have

$$\theta^k=\theta_b^k \ , \forall\, \mathbf{x}\in\partial\Omega_\theta \, ; q_i^k=Q_i^k \ , \forall\, \mathbf{x}\in\partial\Omega_q \qquad (26)$$

The thermal contact conditions are transformed into 2-D form considering the surface temperature and heat flux on upper and lower sides of the shell

$$\sum_{k=0}^{\infty}\theta^k=\theta^+ \ , \ \sum_{k=0}^{\infty}\left(-1\right)^k q_i^k=Q_i^+ \ , \quad \forall\, \mathbf{x}\in\Omega^+ \qquad (27)$$

$$\sum_{k=0}^{\infty}\left(-1\right)^k\theta^k=\theta^- \ , \ \sum_{k=0}^{\infty}q_i^k=Q_i^- \ , \ \forall\, \mathbf{x}\in\Omega^-$$

Then contact conditions (18) will be presented in the form

$$\sum_{k=0}^{\infty}\left(-1\right)^k\theta^k=\sum_{k=0}^{\infty}\theta_*^k \, ; \lambda\sum_{k=0}^{\infty}\left(-1\right)^k Q_3^k=\lambda_*\sum_{k=0}^{\infty}Q_{3(*)}^k \qquad (28)$$

$$\sum_{k=0}^{\infty}\theta^k=\sum_{k=0}^{\infty}\left(-1\right)^k\theta_*^k \, ; \lambda\sum_{k=0}^{\infty}Q_3^k=\lambda_*\sum_{k=0}^{\infty}\left(-1\right)^k Q_{3(*)}^k$$

The coefficients of Legandre's polynomial series for temperature and its derivative with respect to x_3 are related by equation

$$\frac{2\theta_i^k}{h}=\frac{Q_i^{k-1}}{2k-1}-\frac{Q_i^{k+1}}{2k+31} \ , \ (k=1,\ldots,n)$$

Now instead of one 3-D boundary value problem for equations of thermoelasticity and heat conductivity we have infinite set of 2-D boundary value problems for coefficients of the Legandre's polynomial series expansion. In order to simplify the problem we have construct approximate theory and keep only finite set of members in (20).

IV. APPOXOMATE EQUATIONS

As it was mentioned earlier, we consider a deformable body be an elastic homogeneous shell of arbitrary geometry with 2*h* thickness. In developed here approach the shell is substituted by its middle surface and it thermodynamical state is described by infinite system of differential equations (21), (24), (25). Using regular approximation theorem, we can use only finite number of members in Legandre's polynomial series (20). Order of the system of equations depends on assumption regarding thickness distribution of the thermodynamical parameters. The thickness is relatively small in comparison with other dimensions of the shell. Therefore following [3], [5] we can use only two members in polynomial expansion (20). In this case we will get first approximation equations of shell and heat conductivity.

We will consider here the first approximation shell equations, which usually refer as Vekua's shell theory, Timosheko's shell equations, Kirchhoff-Love's shell equations, and equations of heat conductivity with linear distribution of temperature along the thickness.

A. Vekua's shell equations

In the first approximation, the shell theory considers only the first two terms of the Legendre polynomials series [3], [5]. In this case the thermodynamic parameters, which describe the state of the shell, can be presented in the form

$$\begin{aligned}\sigma_{ij}\left(\mathbf{x}\right)&=\sigma_{ij}^0\left(\mathbf{x}_\nu\right)P_0\left(\omega\right)+\sigma_{ij}^1\left(\mathbf{x}_\nu\right)P_1\left(\omega\right) \ ,\\ u_i\left(\mathbf{x}\right)&=u_i^0\left(\mathbf{x}_\nu\right)P_0\left(\omega\right)+u_i^1\left(\mathbf{x}_\nu\right)P_1\left(\omega\right) \ ,\\ \varepsilon_{ij}\left(\mathbf{x}\right)&=\varepsilon_{ij}^0\left(\mathbf{x}_\nu\right)P_0\left(\omega\right)+\varepsilon_{ij}^1\left(\mathbf{x}_\nu\right)P_1\left(\omega\right) \ ,\end{aligned} \qquad (29)$$

Then the 2-D equations of thermo-elasticity for the shell can be obtained substituting these parameters into 3-D equations (1)-(6) or directly from (21). They have the form

$$\begin{aligned}L_{ij}^{00}u_j^0+L_{ij}^{01}u_j^1+L_i^0\left(\theta^0-\theta_0^0\right)+b_i^0&=0\\ L_{ij}^{10}u_j^0+L_{ij}^{11}u_j^1+L_i^1\left(\theta^1-\theta_0^1\right)+b_i^1&=0\end{aligned} \qquad (30)$$

We obtain system of six differential equations for unknown coefficients u_j^0 and u_j^1 of the displacements vector. The operators $L_{ij}^{00}, L_{ij}^{00}, L_{ij}^{00}$ and L_{ij}^{00} are second–order and first–order differential operators, the operators L_i^0 and L_i^1 are first–order differential operators. Their expressions are given in [3], [5] for some types of shell geometry.

B. Timoshenko's shell equations

Timoshenko's theory of shells is based on assumptions concerning the nature of the stress-strain state of the shell. Thus, according to those assumptions $\sigma_{33}=0$ and $\varepsilon_{33}=0$. In this theory the thermodynamic state of shells is determined by quantities specified on the middle surface. The stress state is characterized by the normal $n_{\alpha\alpha}$, tangential $n_{\alpha\beta}$ ($\alpha \neq \beta$) and shear $n_{\alpha 3}$ forces, as well as the bending $m_{\alpha\alpha}$ and twisting $m_{\alpha\beta}$ ($\alpha \neq \beta$) moments. The components of the stress tensor are given by the equations

$$\sigma_{\alpha\beta}(\mathbf{x}) = \frac{n_{\alpha\beta}(\mathbf{x}_\nu)}{2h} + \frac{3m_{\alpha\beta}(\mathbf{x}_\nu)x_3}{h^3}, \qquad (31)$$
$$\sigma_{\alpha 3}(\mathbf{x}) = \frac{n_{\alpha 3}(\mathbf{x}_\nu)}{2h}, \sigma_{33}(\mathbf{x}) = 0$$

The components of the stress tensor are

$$\varepsilon_{\alpha\beta}(\mathbf{x}) = e_{\alpha\beta}(\mathbf{x}_\nu) + \kappa_{\alpha\beta}(\mathbf{x}_\nu)x_3, \qquad (32)$$
$$\varepsilon_{\alpha 3}(\mathbf{x}) = e_{\alpha 3}(\mathbf{x}_\nu), \ \varepsilon_{33}(\mathbf{x}) = 0,$$

where $e_{\alpha i}$ characterize the deformation that is uniform throughout the thickness of the shell and is associated with the extension and the compression at the middle surface and the displacement in the perpendicular planes, while $\kappa_{\alpha\beta}$ is associated with bending and twisting at the middle surface [3], [5].

The components of the displacement vector are given by the equations

$$u_\alpha(\mathbf{x}) = v_\alpha(\mathbf{x}_\nu) + \gamma_\alpha(\mathbf{x}_\nu)x_3, \ u_3(\mathbf{x}) = v_3(\mathbf{x}_\nu) \qquad (33)$$

where v_i is the displacement of the points on the middle surface and γ_α is the angle of rotation of the middle surface.

Differential equations of thermo-elasticity for shells according to Timoshenko's theory have the form

$$L_{ij}^{00}v_j + L_{i\beta}^{01}\gamma_\beta + L_i^0\left(\theta^0 - \theta_0^0\right) + \bar{b}_i = 0$$
$$L_{ij}^{10}v_j + L_{\alpha\beta}^{11}\gamma_\beta + L_\alpha^1\left(\theta^1 - \theta_0^1\right) + \bar{m}_\alpha = 0 \qquad (34)$$

where $\bar{b}_i$ and $\bar{m}_\alpha$ are external loads acting on Ω^+ and Ω^- and reduced to the middle surface, L_{ij}^{00}, $L_{i\beta}^{01}$, $L_{\alpha j}^{10}$, and $L_{\alpha\beta}^{11}$ are second–order and first–order differential operators, L_i^0 and L_α^1 are first–order differential operators. Their expressions are given in [3], [5] for some types of shell geometry.

C. Kirchhoff-Love's shell equations

In the classical Kirchhoff-Love's theory of shells in addition to the assumptions of the Timoshenko's theory it is assumed that $\varepsilon_{\alpha 3} = 0$ and that the angles of rotation of the normal to the middle surface vector become dependent and are given by the equations

$$\gamma_\alpha(\mathbf{x}_\nu) = -\frac{1}{A_\alpha(\mathbf{x}_\nu)}\partial_\alpha v_3(\mathbf{x}_\nu) + k_\alpha(\mathbf{x}_\nu)v_\alpha(\mathbf{x}_\nu) \qquad (35)$$

The inconsistencies of the classical Kirchhoff-Love's theory of shells resulting from these hypotheses are well known [3], [5]. Nevertheless differential equations of thermo-elasticity for shells in this case have simple form

$$L_{11}v_1 + L_{12}v_2 + L_{13}v_3 + \sum_{k=0}^{1} L_1^k\left(\theta^k - \theta_0^k\right) + b_1 = 0$$
$$L_{21}v_1 + L_{22}v_2 + L_{23}v_3 + \sum_{k=0}^{1} L_2^k\left(\theta^k - \theta_0^k\right) + b_2 = 0 \qquad (36).$$
$$L_{31}v_1 + L_{32}v_2 + L_{33}v_3 + \sum_{k=0}^{1} L_3^k\left(\theta^k - \theta_0^k\right) + b_3 = 0$$

where b_i are external loads acting on Ω^+ and Ω^- and reduced to the middle surface, L_{ij} are differential operators of the order up to four, L_i^k are differential operators of the order up to two Their expressions are given in [3], [5] for some types of shell geometry.

D. Equations of heat conductivity

In the first approximation approach it is assumed that temperature linearly distributed along the thickness. Therefore we considers only the first two terms of the Legendre polynomials series. Temperature in the shell and heat-conducting layer can be presented in the form

$$\theta^{(q)}(\mathbf{x}) = \theta^0(\mathbf{x}_\nu)P_0(\omega) + \theta^1(\mathbf{x}_\nu)P_1(\omega), \qquad (37)$$
$$\theta_*(\mathbf{x}) = \theta_*^0(\mathbf{x}_\nu)P_0(\omega) + \theta_*^1(\mathbf{x}_\nu)P_1(\omega)$$

Then the 2-D equations of heat-conductivity for the shell have the form

$$\Delta_0\theta^0 + \frac{1}{2h}\left(Q_3^+ - Q_3^-\right) + (k_1 + k_2)Q_3^0 + \frac{\chi^0}{\lambda_0} = 0$$
$$\Delta_0\theta^1 + \frac{3}{2h}\left(Q_3^+ + Q_3^-\right) + (k_1 + k_2)Q_3^1 + \frac{\chi^1}{\lambda_0} = 0 \qquad (38)$$

where

$$Q_3^+ - Q_3^- = \frac{3}{4h}\left(\theta^+ + T_k\right) + \frac{3\theta^0}{2h}, \quad Q_3^0 = \frac{1}{2h}\left(\theta^+ - T_k\right), \qquad (39)$$
$$Q_3^+ + Q_3^- = \frac{3}{2h}\left(\theta^+ - T_k\right) - \frac{5\theta^1}{2h}, \ Q_3^1 = \frac{3}{2h}\left(\theta^+ + T_k\right) - \frac{3\theta^1}{2h}$$
$$T_k = \frac{\lambda_0(h_0 - u_3)\left(3\theta^+ + 6\theta^0 - 10\theta^1\right) + \lambda_* h\theta^-}{9\lambda_0(h_0 - u_3) + \lambda_* h} \qquad (40)$$

We will consider only one term in the Legendre polynomials series for θ_*. Then the 2-D equations of heat conductivity for the layer have the form

$$\Delta_0\theta_*^0 + \Delta h\theta_*^0 + \left(\nabla^* h \cdot \nabla\theta_*^0\right) - \frac{1}{2}\left(\theta_*^+\Delta_* h^+ + \theta_*^-\Delta_* h_*\right) +$$
$$+ \left(\nabla^* h \cdot \mathbf{Q}_2^0\right) + \frac{1}{2h}(Q^{*+}_3 - Q^{*-}_3) + \qquad (41)$$
$$+ (k_1 + k_2)Q^{*0}_3 + \frac{\chi^0}{\lambda_*} = 0$$

In this case the differential equation (41) is not depend on (38) and also differential for shell (30), (34), (36). See [6] for details.

V. NUMERICAL EXAMPLEs

Let us consider axisymmetrical cylindrical shell placed inside of the cylindrical hole in the rigid body with gap $h_0(x)$. The heat is transferred from the body to the shell though heat conducting layer. The possibility for unilateral mechanical contact is also taken into account.

Differential equations of thermo-elasticity and heat-conductivity for the axisymmetrical cylindrical shell in the classic Kirchhoff-Love's theory have the form

$$\frac{d^4 w}{dx^4} + 4\beta^4 w - \beta_0 \theta^0 - \beta_1 \frac{d^2\theta^1}{dx^2} = \frac{1}{D}(p-q) \,, \quad (42)$$

$$\frac{d^2\theta^0}{dx^2} - \varepsilon_0^2\theta^0 + F_0 = 0 \,, \ \frac{d^2\theta^1}{dx^2} - \varepsilon_1^2\theta^1 + F_1 = 0 \,,$$

where

$$\varepsilon_0 = \frac{3}{h^2}, \varepsilon_1 = \frac{15}{h^2} \,, \beta^4 = \frac{3(1-\nu^2)}{4h^2r^2} \,, \ \beta_0 = \frac{3(1-\nu)\alpha_\tau}{h^2 r} \,,$$

$$\beta_1 = \frac{(1+\nu)\alpha_\tau}{h} \,, \ D = \frac{2Eh^3}{3(1-\nu^2)}$$

and

$$F_0 = 0.5\varepsilon_0(\theta^- + T_k) + \frac{1}{2hr}(T_k - \theta^-) \,,$$

$$F_1 = 0.5\varepsilon_1(T_k - \theta^-) + \frac{3}{2hr}(T_k + \theta^-) - \frac{3}{hr}\theta^1$$

In [8], [9], [11] it was shown that the differential equations (42) can be transformed into the integral equations of Hammerstein's type

$$\int_l G_\alpha(x,y)F_\alpha(y)dy = \theta^\alpha \,, \quad (43)$$

$$\int_l W(x,y)\left\{\frac{1}{D}[p(y)-q(y)] - \beta_0 F_3(y)\right\}dy = w \,,$$

where the kernels in these integral equations are fundamental solutions for corresponding differential operators of the form

$$G_i(x,y) = \exp(-\varepsilon_i|x-y|)/2\varepsilon_i \,, \ i = 0,1 \,, \quad (44)$$

$$W(x,y) = \frac{1}{8\beta^3 D}\exp(-\beta|x-y|)[\cos(\beta|x-y|) + \sin(\beta|x-y|)]$$

and $F_3 = \beta_1(F_1 + \varepsilon_1^2\theta^1) - \beta_0\theta^0$.

Stresses in the axisymmetrical cylindrical shell are calculated by formulas

$$\sigma_x = \frac{E}{1-\nu^2}\left[\frac{d^2w}{dx^2}z - (1+\nu)\alpha_t\theta^1\frac{z}{2h}\right] \,, \quad (45)$$

$$\sigma_\theta = \frac{Ew}{r} - \alpha_t\theta^0 E + \frac{E}{1-\nu^2}\left[\nu\frac{d^2w}{dx^2}z - (1+\nu)\alpha_t\theta^1\frac{z}{2h}\right]$$

Substituting expression for the second derivative of the displacements into (45) we obtain

$$\sigma_x(x) = b_0\int_0^l \theta^0(\xi)\frac{d^2G(\xi,x)}{dx^2}d\xi + \quad (46)$$

$$b_1\int_0^l G(\xi,x)\theta^1(\xi)d\xi - b_2\theta^1(x) + b_3\int_0^l [p(\xi)-q(\xi)]\frac{d^2G(\xi,x)}{dx^2}d\xi$$

$$\sigma_\theta(x) = \nu\sigma_x(x) + \frac{Ew(x)}{r} - \alpha_t E[\theta^0(x) + \theta^1(x)z]$$

where

$$b_0 = \frac{3E\alpha_t zr^5}{(1+\nu)h^2h_0} \,, \ b_1 = \frac{3(1+\nu)E\alpha_\tau r^6 z}{4h^3h^2{}_0} \,, \ b_2 = \frac{E\alpha_\tau r^4 z}{(1-\nu)hh_0} \,,$$

$$b_3 = \frac{3r^6 z}{2h^3h_0}, \beta_0 = \frac{3\alpha_t Erz}{h^2} \,, \ \beta_1 = \frac{3(1-\nu)\alpha_t Er^2 z}{h^2} \,, \ \beta_3 = \frac{r^2 z}{8h^3} .$$

Mizes stresses are calculated by the equations

$$\sigma_i^- = \sqrt{(\sigma_x^-)^2 + (\sigma_x^-)(\sigma_0^-) + (\sigma_0^-)^2} \,, \quad (47)$$

$$\sigma_i^+ = 1/\sqrt{2}\sqrt{(\sigma_x^+ - q)^2 + (\sigma_x^+ - \sigma_x^+)^2 + (\sigma_0^+ + q)^2}$$

Algorithm for the problem solution consists in iterative process of the integral equations of Hammerstein's type solution and in the case if unilateral contact take place additional iterative algorithm is used. Algorithm has been elaborated in [6], [7]. In the problems under consideration algorithm is convergent and convergence is enough fast.

We will consider here thermomechanical state of the shell which arise due to two factors: a. no homogeneous distribution of external temperature load; b. presence of macro roughness on the rigid body, which can be described by no homogeneous initial gap.

Calculation have been done for the data:, geometrical parameters $r = 0.5m$, $h = 0.01m$, , $h_0 = 0.5h$, $l_b = r$, material properties: $E = 2.5\cdot 10^5$ MPa, $\nu = 0.25$ $\alpha_\tau = 2.5\cdot 10^{-5}\,1/{}^oC$, $\lambda_1 = 2V/m^oC$, $\lambda_* = 1V/m^oC$

Example 1. We consider axisymmentrical cylindrical shell of infinite length placed into the rigid stirrup with homogeneous initial gap.

Temperature on the stirrup surface of is equal to

$$T^+(x) = T_m + T_b \sin\pi x/l_b, T_b = 600^o C \ T_m = 100^o C .$$

On the shell surface act homogeneous load $p(x) = 10MPa$, and temperature $T^- = 0^o\ C$.

In the Fig.1. are presented: Mizes stresses on external σ^+ and internal σ^- surfaces of the shell, force of contact interaction q, normalized bending $W = w/h_0$ and temperature on contact surface T_k. The dashed lines correspond to solution for perfect thermal contact without counting influence of the shell deformation on the heat exchange and the solid lines correspond the presented here solution.

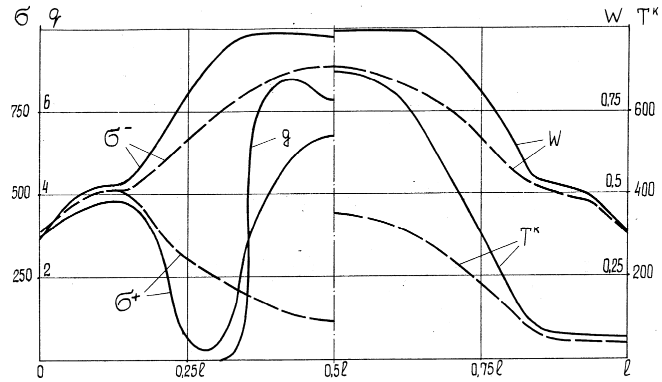

Fig 1.

Analysis of these data shows, that no homogeneous temperature distribution cause significant shell deformations and close mechanical contact with rigid stirrup. As result in the shell significant stress occur. Calculations with considering perfect thermal contact lead to significant inaccuracy, which is not only quantitative but also qualitative.

Example 2. Here we consider axisymmentrical cylindrical shell of infinite length placed into the rigid stirrup with no homogeneous initial gap. The gap is given by the function

$$h_*(x) = h_o + h_b \sin \pi x / l_b \text{ , } h_0 = 0.5h \text{ , } h_b = h_0 / 2 \text{ .}$$

In the Fig.2. are presented: Mizes stresses on external σ^+ and internal σ^- surfaces of the shell, , normalized bending $W = w / h_0$ and temperature on contact surface T_k. The dashed lines correspond to solution for perfect thermal contact without counting influence of the shell deformation on the heat exchange and the solid lines correspond the presented here solution.

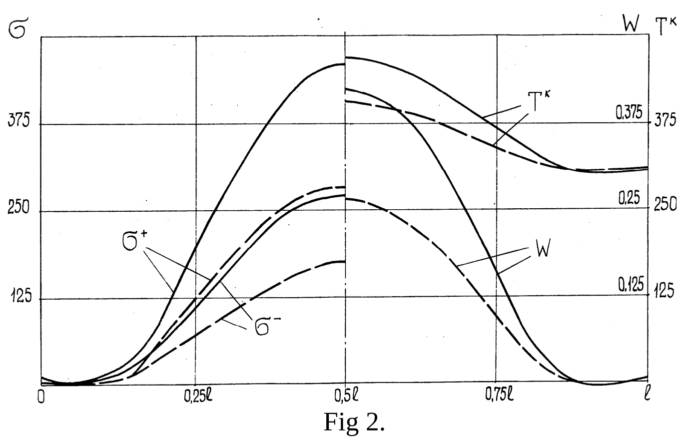

Fig 2.

Analysis of these data shows, that no homogeneous initial gap cause significant shell deformations. As result in the shell occur significant no homogeneous stress and temperature distribution. Calculations with considering perfect thermal contact lead to significant inaccuracy. Some values of thermomechanical parameters differ twice.

VI. Conclusion

The results presented here and in previous our publications show that the in thermoelastic contact problems for thin-walled structures mutual influence temperature and deformation may be significant and it have to be taken into account in engineering design.

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