Revisit Controlled Lagrangians for Spherical Inverted Pendulum

Guangyu Liu, Subhash Challa, and Longguang Yu

Abstract— **Recently, passivity based control theory (energy approaches) has undergone a breakthrough in dealing with underactated mechanical systems with two successful and similar tools, controlled Lagrangians and controlled Hamiltanians (IDA-PBC). Because of the complexity, successful case studies are lacking, for example, MIMO, underactuated and unstable systems. The seminal paper of controlled Lagrangians proposed by Bloch and his colleagues presented a benchmark example–a simplified spherical inverted pendulum on a cart but the detailed design and its verification were neglected. To compensate this ignorance, the note revisits their design idea by addressing explicit control functions for a similar device motivated by real applications. Some observations are given through computer simulation. At the courtesy of the original idea, the case study is known to be the first MIMO, underactuated and unstable system stabilized in full state space via energy approaches.**

Index Terms— **Full energy shaping, spherical inverted pendulum, MIMO**

I. Introduction

THE method of controlled Lagrangians (CL) is a constructive approach to the derivation of stabilizing control laws for Lagrangians mechanical systems. The theory originated from [1] and was systematically introduced in [2], [3]. Various supplementary and additional results were also presented in the literature (e.g., [4]). The method of controlled Lagrangians was developed in two salient phases: (i) the kinetic energy shaping [2]; (ii) the full energy shaping [3]. The latter had advantages over the classical potential shaping [5] in dealing with underactauted systems. Meanwhile, there was a development of its Hamiltonian counterpart, port-controlled Hamiltonians [6], [7]. Two methods were equivalent for simple mechanical systems [8], [9].

The *full* energy shaping [3] claimed to be able to solve the *full* stabilization of the spherical inverted pendulum but the detailed design and its verification were ignored. Here, we elaborate the control idea from [3] and verify its effectiveness trough computer simulation for a slightly general mechanical system (see Figure 1) motivated by real-life applications: a vector thrusted body hovering in certain altitude and personal transports. To the best of our knowledge, it is the first MIMO, underactuated and unstable system that is stabilized in full state space via energy approaches as is inherit from [3].

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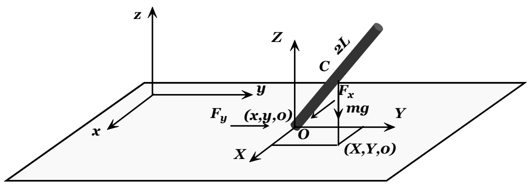

Fig. 1. The spherical inverted pendulum

Several controllers for the stabilisation and tracking of the spherical inverted pendulum were proposed in [10], [11], [12], [13], [14], [15], [16], [17], [18], [19]. A swung-up strategy was given in [10]. Although partial feedback linearization was incorporated, using the first approximation in [11] made the controller a local stabilizing result. By exploring the forwarding structure, adding nested saturating levels to the linear one led to a nonlinear controller that yielded a "global" domain of attraction (see [12] and its improved versions [15], [16]). The controller in [13] (see also [17]) that relied on singular perturbation idea yielded a "semi-global" stability region. The associated "strong" Lyapunov function could be used as a controlled Lyapunov function (c.l.f) to obtain the total stability of the perturbed closed loop system. So, robustness was implied. Similar to the energy ideas (e.g., [3]), the controller in [14] used a "weak" Lyapunov function to derive (non-local) asymptotic stability of the nominal closed loop system based on LaSalle's invariance principle. Because the "weak" Lyapunov function that could not be used directly as c.l.f, the robustness was not ensured. Way-point tracking and nonlinear output tracking controllers were given respectively in [18], [19].

Although some controllers (e.g., [15], [16], [17]) have some advantages, motivated by their physical insight, the energy approaches (e.g., [3]) are of practical interest of many people.

The remaining of the paper is organized as follows: Section II and III review respectively the theory [3] and the model [15]; the explicit control law is given in Section III; computer simulation is carried out in Section V; final observation is given in Section VI.

II. Preliminaries

A. The Notations

- $Q = S \times G$ is a n-dimensional manifold with the coordinates $q = (x^\alpha, \theta^a) \in \mathrm{R}^n$, where $x^\alpha \in S$ with index α going from 1 to $n - r$ are the shape variables and $\theta^a \in G$, with index a going from 1 to r are the group variables and the corresponding S, G are called the shape space and the Abelian group respectively. TQ denotes tangent bundle to Q.

- $T^{j_1,\ldots,j_r}_{i_1,\ldots,i_s}$ is a tensor of type (r,s) with r covectors and s vectors. g_{ab} represents a $m\times n$ matrix with index a going from 1 to m and b going from 1 to n, g^{ab} denotes the inverse of the matrix g_{ab} if $n=m$. $\tau^b_{\delta,\alpha}=\frac{\partial\tau^b_\delta}{\partial\alpha}$ denote the partial derivative. The summation convention over repeated indices is implied to the tensor product.
- x_e with a subscript e is the equilibrium.

B. Lagrangians and Controlled Lagrangians

The Lagrangians $\mathcal{L}:TQ\rightarrow \mathrm{R}$ of the mechanical system is defined as

$$\mathcal{L}\left(x^\alpha,\dot{x}^\beta,\theta^a,\dot{\theta}^b\right) = \frac{1}{2}g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta+g_{\alpha a}\dot{x}^\alpha\dot{\theta}^a+\frac{1}{2}g_{ab}\dot{\theta}^a\dot{\theta}^b-V(x^\alpha,\theta^a) \tag{1}$$

where g_{ij} is the metric tensor, $\frac{1}{2}g_{ij}\dot{q}^i\dot{q}^j$ is the kinetic energy and $V(q)$ is the potential energy. Hence, the controlled Euler-Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}^\alpha}-\frac{\partial\mathcal{L}}{\partial x^\alpha}=0,\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\theta}^a}-\frac{\partial V}{\partial\theta^a}=u_a, \tag{2}$$

where the controls u_a only act in the θ^a directions.

The controlled Lagrangians $\tilde{\mathcal{L}}$ involves changing the metric tensor and potential energy by introducing quantities $\tau,\sigma,\rho,\epsilon$ that is denoted by

$$\begin{aligned}\tilde{\mathcal{L}} &\triangleq \mathcal{L}\left(x^\alpha,\dot{x}^\beta,\theta^a,\dot{\theta}^a+\tau^a_\alpha\dot{x}^\alpha\right)+\frac{1}{2}\sigma g_{ab}\tau^a_\alpha\tau^b_\beta\dot{x}^\alpha\dot{x}^\beta\\ &+\frac{1}{2}(\rho-1)g_{ab}\left(\dot{\theta}^a+g^{ac}g_{\alpha c}\dot{x}^\alpha+\tau^a_\alpha\dot{x}^\alpha\right)\\ &\times\left(\dot{\theta}^b+g^{bd}g_{\beta d}\dot{x}^\beta+\tau^b_\beta\dot{x}^\beta\right)-V_\epsilon(x^\alpha,\theta^a).\end{aligned} \tag{3}$$

where V_ϵ is a new function. The controlled Lagrangians implies a new potential energy function $V'(x^\alpha,\theta^a)=V(x^\alpha,\theta^a)+V_\epsilon(x^\alpha,\theta^a)$. Quantities τ,σ will be defined by the matching conditions and the values σ,ρ,ϵ are determined by stability theorems which is reviewed next.

We summarize the simplified matching conditions in [2], [3]:

SM-1: $\sigma_{ab}=\sigma g_{ab}$ for a constant σ;
SM-2: g_{ab} is independent of x^α;
SM-3: $\tau^b_\alpha=-(1/\sigma)g^{ab}g_{\alpha a}$;
SM-4: $g_{\alpha a,\delta}=g_{\delta a,\alpha}$;
SM-5: $\frac{\partial^2 V}{\partial x^\alpha\partial\theta^a}g^{ad}g_{\beta d}=\frac{\partial^2 V}{\partial x^\beta\partial\theta^a}g^{ad}g_{\alpha d}$.

This leads to a control law u^{con}_a. To achieve asymptotic stability, a dissipative control u^{diss}_a is added such that

$$u_a\triangleq u^{con}_a+\frac{1}{\rho}u^{diss}_a. \tag{4}$$

is the desired control law. Euler-Lagrange equations in terms of $\tilde{\mathcal{L}}$ are

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{\mathcal{L}}}{\partial\dot{x}^\alpha}-\frac{\partial\tilde{\mathcal{L}}}{\partial x^\alpha}&=\left(-\frac{1}{\sigma}+\frac{\rho-1}{\rho}\right)g^{ad}g_{\alpha d}u^{diss}_a\\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{\mathcal{L}}}{\partial\dot{\theta}^a}-\frac{\partial\tilde{\mathcal{L}}}{\partial\theta^a}&=u^{diss}_a.\end{aligned} \tag{5}$$

The assumption **SM-5** can be replaced by similar ones after introducing a new coordinate chart

$$(x^\alpha,\eta^a)\triangleq(x^\alpha,\theta^a+h^a(x^\alpha)), \tag{6}$$

where the function $h:U\rightarrow g$ for an open subset U in S is the solution of the first order partial differential equation $\frac{\partial h^a}{\partial x^\alpha}=\left(\frac{\rho-1}{\rho}-\frac{1}{\sigma}\right)g^{ac}g_{\alpha c}$ with $h^a(x_e)=0$. Two extra assumptions are:

SM-5': The potential $V(x^\alpha,\theta^a)$ is of the form $V(x^\alpha,\theta^a)=V_1(x^\alpha)+V_2(\theta^a)$ where V_1 has a maximum at $(x^\alpha)=(x^\alpha_e)$ ((SM-5') is a particular case of (SM-5)).

SM-6: The matrix $(g_{a\alpha}(x^\alpha_e))$ is one-to-one (injective).

In the new coordinates (x^α,η^a), $V(x^\alpha,\theta^a)=V_1(x^\alpha)+V_2(\theta^a))$ becomes $V(x^\alpha,\eta^a)=V_1(x^\alpha)+V_2(y^a-h^a(x^\alpha))$. Then, the solution V_ϵ is given by

$$V_\epsilon=-V_2(y^a-h^a(x^\alpha))+\tilde{V}_\epsilon(\eta^a), \tag{7}$$

where $\tilde{V}_\epsilon(\eta^a)$ is an arbitrary function and the total modified potential energy function is given by

$$V'_\epsilon(x^\alpha,\eta^a)=V_1(x^\alpha)+\tilde{V}_\epsilon(\eta^a), \tag{8}$$

We express the kinetic energy as follows

$$\tilde{K}=\frac{1}{2}A_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta+\frac{1}{2}\rho g_{ab}\dot{\zeta}^a\dot{\zeta}^b \tag{9}$$

where $\dot{\zeta}^a=\dot{y}^a+(1/\rho)g^{ab}g_{\alpha b}\dot{x}^\alpha$ and $A_{\alpha\beta}=g_{\alpha\beta}-(1-1/\sigma)g_{\alpha d}g^{da}g_{a\beta}$. The controlled energy, $\tilde{E}$, is written in new coordinates as

$$\tilde{E}=\tilde{K}+V_1(x^\alpha)+\tilde{V}_\epsilon(\eta^a). \tag{10}$$

In the new coordinates $(x^\alpha,\dot{x}^\alpha,\eta^a,\dot{\eta}^a)$, the controlled Lagrangians takes the form

$$\begin{aligned}\tilde{\mathcal{L}}&\triangleq\frac{1}{2}\left(g_{\alpha\beta}-\left(\frac{\rho-1}{\rho}-\frac{1}{\sigma}\right)g^{ab}g_{\alpha a}g_{\beta b}\right)\dot{x}^\alpha\dot{x}^\beta+\\ &g_{\alpha a}\dot{x}^\alpha\dot{\eta}^a+\frac{1}{2}\rho g_{ab}\dot{\eta}^a\dot{\eta}^b-V_1(x^\alpha)-\tilde{V}_\epsilon(\eta^a)\end{aligned} \tag{11}$$

and the Euler-Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{\mathcal{L}}}{\partial\dot{x}^\alpha}-\frac{\partial\tilde{\mathcal{L}}}{\partial x^\alpha}=0,\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{\mathcal{L}}}{\partial\dot{\eta}^a}-\frac{\partial\tilde{\mathcal{L}}}{\partial\eta^a}=u^{diss}_a. \tag{12}$$

LaSalle's invariance principle gives the asymptotic stability of the equilibrium as follows.

Theorem 2.1: (Asymptotic Stabilization-Specific Case [3]): Assume that conditions (SM-1)-(SM-4), (SM-5') and (SM-6) hold. Let (x^α_e) be the maximum point of V_1 of interest. Then, there is an explicit feedback control such that $(x^\alpha_e,\theta^a_e,0,0)$ becomes an asymptotically stable equilibrium such that

$$\frac{d}{dt}\tilde{E}=c^b_a g_{bd}\dot{\eta}^a\dot{\eta}^b\geq 0 \tag{13}$$

and the total control law (4) is written as follows

$$\begin{aligned}u_a&=-\kappa\left(g_{\beta a,\gamma}-g_{\delta a}A^{\delta\alpha}\left(g_{\alpha\beta,\gamma}-\frac{1}{2}g_{\beta\gamma,\alpha}-\right.\right.\\ &\left.\left.(1+\kappa)g_{\alpha d}g^{da}g_{\beta a,\gamma}\right)\right)\dot{x}^\beta\dot{x}^\gamma+\kappa g_{\delta a}A^{\delta\alpha}\frac{\partial V}{\partial x^\alpha}\\ &+\kappa g_{\delta a}A^{\delta\alpha}\frac{1}{\rho}g_{\alpha d}g^{db}\left(-\frac{\partial V'}{\partial\theta^b}+u^{diss}_b\right)\\ &+\frac{\rho-1}{\rho}\frac{\partial V}{\partial\theta^a}-\frac{1}{\rho}\frac{\partial V_\epsilon}{\partial\theta^a}+\frac{1}{\rho}u^{diss}_a\end{aligned} \tag{14}$$

where $\kappa \triangleq -1/\sigma$, $A_{\alpha\beta} \triangleq g_{\alpha\beta} - (1+\kappa) g_{\alpha d} g^{da} g_{\beta a}$, $u_a^{diss} = c_a^d g_{bd} \left(\frac{\rho-1}{\rho} - \kappa\right) g^{ac} g_{\alpha c} \dot{x}^\alpha$ with c_a^d a positive definite matrix and parameters to satisfy the following three conditions: (1) $V_\epsilon(\eta^a)$ should have a maximum at $\eta_e^a = \theta_e^a$; (2) $\rho < 0$; (3) $\kappa > \max\left\{\lambda | \det(g_{\alpha\beta} - \lambda g_{\alpha a} g^{ab} g_{b\beta})|_{x^\alpha = x_e^\alpha} = 0\right\} - 1$.

III. MODELLING

With reference to Figure 1, we consider the spherical inverted pendulum be a slender rigid body sliding on a horizontal plane (see [15]), which is more realistic than that in [3]. The configuration space is denoted by $Q = S \times G$ with Cartesian coordinates $(x, y) \in G$ the local coordinates for translational coordinates and Cartesian coordinates $(X, Y) \in S$ the projections of the center of mass in the horizontal plane. (F_x, F_y) denotes a planar control signal acting on the base of the pendulum in the horizontal plane. Thus, $q = (x, y, X, Y) \in Q$ is the vector of the generalized coordinates. The kinetic energy is given by

$$T = \tfrac{1}{2} g_{ij} \dot{q}^i \dot{q}^j ,$$

where $g_{ij} = \begin{pmatrix} m & 0 & m & 0 \\ 0 & m & 0 & m \\ m & 0 & \frac{4m}{3}\frac{L^2-Y^2}{L^2-X^2-Y^2} & \frac{4m}{3}\frac{XY}{L^2-X^2-Y^2} \\ 0 & m & \frac{4m}{3}\frac{XY}{L^2-X^2-Y^2} & \frac{4m}{3}\frac{L^2-X^2}{L^2-X^2-Y^2} \end{pmatrix}$ is the metric tensor. The total potential energy is given by [1]

$$V \triangleq mg(\sqrt{L^2 - X^2 - Y^2} - L). \tag{15}$$

We define the Lagrangians of the pendulum $\mathcal{L} : TQ \mapsto Q$

$$\mathcal{L} = K(\dot{x}, \dot{y}, X, Y, \dot{X}, \dot{Y}) - V(X, Y), \tag{16}$$

which is independent of (x, y), the cyclic variables.

Then, applying Euler-Lagrange equations (2) to (16) gives the equations of dynamics, x^α with index α going from X to Y, θ^a with index a going from x to y and u_a with index α going from x to y, that is, $(u_x, u_y) \triangleq (F_x, F_y)$.

IV. CONTROL DESIGN

First, we apply Theorem 2.1 to the spherical inverted pendulum to obtain a control law. Then, we estimate the stability region of the closed loop system.

Checking matching conditions: As we can seen from the kinetic energy $T = \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j$ in (15), we read the sub-matrices of the metric tensor g_{ij} as follows $g_{ab} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$, $g_{\alpha a} = g_{a\beta} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$, $g_{\alpha\beta} = \begin{pmatrix} \frac{4m}{3}\frac{L^2-Y^2}{L^2-X^2-Y^2} & \frac{4m}{3}\frac{XY}{L^2-X^2-Y^2} \\ \frac{4m}{3}\frac{XY}{L^2-X^2-Y^2} & \frac{4m}{3}\frac{L^2-X^2}{L^2-X^2-Y^2} \end{pmatrix}$. So, we can define the controlled Lagrangians as (3). All matching conditions (SM-1)-(SM-4), (SM-5') and (SM-6) in Theorem 2.1 are satisfied:

(SM-1) is satisfied if we define $\sigma_{ab} \triangleq \sigma g_{ab} = \sigma m \delta_{ab}$ where σ is constant and δ_{ab} is the Kronecker $\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$.

(SM-2) is satisfied because g_{ab} is constant matrix, which is independent of (X, Y).

(SM-3) is satisfied if we define $\tau_\alpha^b \triangleq -(1/\sigma) g^{ab} g_{\alpha a}$ such that $\tau_X^x = \tau_Y^y = -(1/\sigma)$, $\tau_Y^x = \tau_X^y = 0$.

(SM-4) is satisfied because $g_{Xx,Y} = \frac{\partial m}{\partial Y} = 0$, $g_{Yx,X} = \frac{\partial 0}{\partial X} = 0$, $g_{Xy,Y} = \frac{\partial 0}{\partial Y} = 0$ and $g_{Yy,X} = \frac{\partial m}{\partial X} = 0$.

(SM-5') is satisfied because $V = V_1(x^\alpha) + 0 = mg(\sqrt{L^2 - X^2 - Y^2} - L)$ and V_1 has a maximum at the equilibrium $(X, Y) = (0, 0)$.

(SM-6) is satisfied since the mapping $g_{\alpha a}(x_e^\alpha) = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}_{(X,Y)=(0,0)}$ evaluated at the equilibrium is injective.

Modifying the potential energy: To modify the potential energy, we introduce the new coordinate chart. The solutions of the partial differential equations $\frac{\partial h^x}{\partial X} = \frac{\rho-1}{\rho} - \frac{1}{\sigma}$, $\frac{\partial h^x}{\partial Y} = 0$, $\frac{\partial h^y}{\partial Y} = \frac{\rho-1}{\rho} - \frac{1}{\sigma}$ and $\frac{\partial h^y}{\partial X} = 0$ with boundary conditions $h^x|_{(X,Y)=(0,0)} = 0$ and $h^y|_{(X,Y)=(0,0)} = 0$ in this case are simple to obtain, which are $h^x = \left(\frac{\rho-1}{\rho} - \frac{1}{\sigma}\right) X$, $h^y = \left(\frac{\rho-1}{\rho} - \frac{1}{\sigma}\right) Y$. We define the new coordinate chart $(\eta^a, x^\alpha) = (\eta^x, \eta^y, X, Y)$ according to (6), where $\eta^x = x + \left(\frac{\rho-1}{\rho} - \frac{1}{\sigma}\right) X$, $\eta^y = y + \left(\frac{\rho-1}{\rho} - \frac{1}{\sigma}\right) Y$.

Next, we define the potential V'_ϵ for the controlled Lagrangians. To this end, we define a negative definite function as $\tilde{V}_\epsilon(\eta^a) \triangleq -\epsilon mg\left((\eta^x)^2 + (\eta^y)^2\right)$ which has a maximum at the equilibrium $(\eta^x, \eta^y) = (0, 0)$ when $\epsilon > 0$. As shown in (8), the potential V'_ϵ for the controlled Lagrangians in the new coordinates is given by

$$\begin{aligned} V'_\epsilon \triangleq\ & mg(\sqrt{L^2 - X^2 - Y^2} - L) \\ & -\epsilon mg\left((\eta^x)^2 + (\eta^y)^2\right). \end{aligned} \tag{17}$$

Computing a control law: Applying the general formula (14) provides an explicit stabilising control law

$$u_x \triangleq \tilde{F}_x(q, \dot{q}, \kappa, \rho, \epsilon),\ u_y \triangleq \tilde{F}_y(q, \dot{q}, \kappa, \rho, \epsilon) \tag{18}$$

which is given in next page and where κ, ρ and ϵ are determined by applying Theorem 2.1. We choose $\epsilon > 0$ such that the appended potential energy function $\tilde{V}_\epsilon$ is negative definite. We also check that the following conditions are satisfied.

(1) $\tilde{V}_\epsilon(y^a)$ has a maximum at the equilibrium $(\theta_e^a) = (x_e, y_e) = (0, 0)$ because the equilibrium $(x_e^\alpha) \in Q/G$, i.e., $(X, Y) = (0, 0)$, are the maximum point of V_1 according to (SM-5') and the constructed function $\tilde{V}_\epsilon(y^a)$ have a maximum at the equilibrium $(y_e^a) = (0, 0)$.

(2) We assign ρ to be a negative real number.

(3) We assign a positive real number for κ that satisfies $\kappa > 1/3$. Consequently, we have $\kappa > 1/3$, $\rho < 0$ and $\epsilon > 0$.

Next, we show that a non-local (but not global) domain of attraction is yielded by the closed loop system. The function $h : U \to R$ is valid for $U = \{(X, Y) \in R^2 | \sqrt{X^2 + Y^2} < L\}$ which corresponds to the upper space as the case that the pendulum is above the horizontal plane. We use $R^2 \times U \subset Q$ as a domain of a local chart on Q and the new local chart on TQ is given as: $(x, y, X, Y, \dot{x}, \dot{y}, \dot{X}, \dot{Y}) \in (\mathrm{R}^2 \times U) \times \mathrm{R}^4$. Likewise, for the new local chart (x^α, η^a) on $R^2 \times U \subset Q$, its corresponding local chart on TQ is given as: $(x, y, X, Y, \dot{x}, \dot{y}, \dot{X}, \dot{Y}) \mapsto (\eta^x, \eta^y, X, Y, \dot{\eta}^x, \dot{\eta}^y, \dot{X}, \dot{Y}) \in (\mathrm{R}^2 \times U) \times \mathrm{R}^4$. By (10), the energy function in the new chart

[1] Noting that we do not use the same potential energy as [3] and the potential energy in our formulation is non-positive in order to make the closed loop energy function a Lyapunov candidate.

Explicit formula for (18) and $\kappa_\rho \triangleq \frac{\rho-1}{\rho} + \kappa$.

$$
\begin{aligned}
u_x = \; & \frac{m\kappa}{L^2(1-3\kappa)+3(X^2+Y^2)(1+\kappa)} \Bigg(-3gX\sqrt{L^2-X^2-Y^2} - \frac{4(Y^2-L^2)X\dot{X}^2+4(X^2-L^2)X\dot{Y}^2-8X^2Y\dot{X}\dot{Y}}{L^2-X^2-Y^2} \\
& +3\left(\frac{L^2-X^2}{\rho}+\frac{3Y^2(1+\kappa)}{\rho(1-3\kappa)}\right)\left(2\epsilon g\left(x+\kappa_\rho X\right)+c_x^x\left(\dot{x}+\kappa_\rho\dot{X}\right)+c_x^y\left(\dot{y}+\kappa_\rho\dot{Y}\right)\right)+\frac{12XY}{\rho(1-3\kappa)}\times \\
& \left(2\epsilon g\left(y+\kappa_\rho Y\right)+c_y^x\left(\dot{x}+\kappa_\rho\dot{X}\right)+c_y^y\left(\dot{y}+\kappa_\rho\dot{Y}\right)\right)\Bigg)+\frac{m}{\rho}\left(2\epsilon g\left(x+\kappa_\rho X\right)+c_x^x\left(\dot{x}+\kappa_\rho\dot{X}\right)+c_x^y\left(\dot{y}+\kappa_\rho\dot{Y}\right)\right).
\end{aligned}
$$

$$
\begin{aligned}
u_y = \; & \frac{m\kappa}{L^2(1-3\kappa)+3(Y^2+X^2)(1+\kappa)} \Bigg(-3gY\sqrt{L^2-Y^2-X^2} - \frac{4(X^2-L^2)Y\dot{Y}^2+4(Y^2-L^2)Y\dot{X}^2-8Y^2X\dot{Y}\dot{X}}{L^2-Y^2-X^2} \\
& +3\left(\frac{L^2-Y^2}{\rho}+\frac{3X^2(1+\kappa)}{\rho(1-3\kappa)}\right)\left(2\epsilon g\left(y+\kappa_\rho Y\right)+c_y^y\left(\dot{y}+\kappa_\rho\dot{Y}\right)+c_y^x\left(\dot{x}+\kappa_\rho\dot{X}\right)\right)+\frac{12YX}{\rho(1-3\kappa)}\times \\
& \left(2\epsilon g\left(x+\kappa_\rho X\right)+c_x^y\left(\dot{y}+\kappa_\rho\dot{Y}\right)+c_x^x\left(\dot{x}+\kappa_\rho\dot{X}\right)\right)\Bigg)+\frac{m}{\rho}\left(2\epsilon g\left(y+\kappa_\rho Y\right)+c_y^y\left(\dot{y}+\kappa_\rho\dot{Y}\right)+c_y^x\left(\dot{x}+\kappa_\rho\dot{X}\right)\right).
\end{aligned}
$$

is written as

$$
\begin{aligned}
\tilde{E} = \; & \frac{1}{2}\begin{pmatrix}\dot{X}\\ \dot{Y}\end{pmatrix}^T A_{\alpha\beta}\begin{pmatrix}\dot{X}\\ \dot{Y}\end{pmatrix} + \\
& \frac{1}{2}m\rho\left((\dot{\eta}^x)^2+(\dot{\eta}^y)^2\right)-\epsilon mg\left((\eta^x)^2+(\eta^y)^2\right) \\
& -mg(L-\sqrt{L^2-X^2-Y^2}).
\end{aligned} \tag{19}
$$

where $A_{\alpha\beta} = \begin{pmatrix}\alpha_1-m(1+\kappa) & \alpha_2\\ \alpha_2 & \alpha_3-m(1+\kappa)\end{pmatrix}$ with $\alpha_1 = \frac{4m}{3}\frac{L^2-Y^2}{L^2-X^2-Y^2}$, $\alpha_2 = \frac{4m}{3}\frac{XY}{L^2-X^2-Y^2}$ and $\alpha_3 = \frac{4m}{3}\frac{L^2-X^2}{L^2-X^2-Y^2}$. Theorem 2.1 implies the following result.

Corollary 4.1: Suppose that all conditions in Theorem 2.1 are satisfied. For any compact subset $\tilde{U} \in U$, there exists a $\kappa^* > 1/3$ such that for $\kappa > \kappa^*$, the controlled energy (19) is negative definite and the corresponding Lyapunov function $\mathcal{V} = -\tilde{E}$ is positive definite for all states

$$(\eta^x, \eta^y, X, Y, \dot{\eta}^x, \dot{\eta}^y, \dot{X}, \dot{Y}) \in (R^2 \times \tilde{U}) \times \mathrm{R}^4.$$

Furthermore, let the set $\Omega_c = \{(x^\alpha, \eta^a, \dot{x}^\alpha, \dot{\eta}^a) \in TQ | \mathcal{V} \le c\} \subset (\mathrm{R}^2 \times \tilde{U}) \times \mathrm{R}^4$ be a positively invariant set for some $c \in R$. Let the set $\mathcal{E} = \{(x^\alpha, \eta^a, \dot{x}^\alpha, \dot{\eta}^a) \in \Omega_c | \dot{\eta}^a = 0$ or $\frac{d}{dt}\tilde{E} = 0\}$ and $\mathcal{M}$ is the largest invariant subset of $\mathcal{E}$. Then, Ω_c is an estimate of domain of attraction and $\mathcal{M} = (x_e^a, \dot{x}_e^a, \eta_e^\alpha, \dot{\eta}_e^\alpha) = (0, 0, 0, 0)$.

Proof: Our argument is based on the result for the general case in [3] where the LaSalle's invariance principle is used to establish the stability. The proof is carried out in two steps: at step one, we find κ^* such that for $\kappa > \kappa^*$, the energy function is negative definite; at step two, we show that the set Ω_c is an estimate of domain of attraction.

At the first stage, we try to make the energy (19) negative definite and zero at zero.

We check that:
(1) $-mg(L-\sqrt{L^2-X^2-Y^2}) \le 0$ for $(X,Y) \in U$ where the equality holds if and only if $X = Y = 0$;
(2) $-\epsilon mg\left((\eta^x)^2+(\eta^y)^2\right) \le 0$ for $(\eta^x, \eta^y) \in \mathrm{R}^2$ where the equality holds if and only if $\eta^x = \eta^y = 0$;
(3) $\frac{1}{2}m\rho\left((\dot{\eta}^x)^2+(\dot{\eta}^y)^2\right) \le 0$ where the equality holds if and only if $\dot{\eta}^x = \dot{\eta}^y = 0$.

To make the energy function (19) negative definite, $A_{\alpha\beta}$ must be negative definite, that is, $\alpha_1 - m(1+\kappa) < 0$ and $(\alpha_1 - m(1+\kappa))(\alpha_3 - m(1+\kappa)) - \alpha_2^2 < 0$ are satisfied. To this end, we let κ satisfy

$$
\begin{aligned}
& \kappa > \frac{1}{3}, \quad \kappa > \frac{\|\alpha_1\|_\infty}{m} - 1 \\
& \kappa > 2m\left(\|\alpha_1\|_\infty + \|\alpha_3\|_\infty + \left(\|\alpha_1\|_\infty^2 + 4\|\alpha_2\|_\infty^2 + \right.\right. \\
& \qquad \left.\left. \|\alpha_1\|_\infty\|\alpha_3\|_\infty + \|\alpha_3\|_\infty^2\right)^{1/2}\right) - 1
\end{aligned} \tag{20}
$$

simultaneously where we introduce infinity norm $\|\cdot\|_\infty$ such that $\alpha_i \le \|\alpha_i\|_\infty$, $i = 1, 2, 3$ for all $(X, Y) \in \tilde{U} \subset U$. We define $\kappa^* = \inf_{\kappa \in R^+}\{\kappa \text{ satisfies } (20)\}$.

For $\kappa > \kappa^*$, the controlled energy (19) is negative definite with a maximum at $(0,0,0,0,0,0,0,0) \in (\tilde{U} \times \mathrm{R}^2) \times \mathrm{R}^4$. We conclude that the corresponding Lyapunov function $\mathcal{V}$ is positive definite at a domain $(R^2 \times \tilde{U}) \times \mathrm{R}^4$. Seeing from (20), we conclude that as $\kappa^* \to \infty$ implies $\|\alpha_i\|_\infty \to \infty$ and $\sqrt{X^2+Y^2} \to L$, the set $\tilde{U}$ expands to U.

At the second stage, we show that Ω_c is an estimate of domain of attraction by applying LaSalle's invariance principle. Here, we relax the conditions in [3] for general cases. Specifically, we do not shrink Ω_c as the domain of attraction.

In the last step, $\mathcal{V}$ is positive definite in a domain $R^2 \times \tilde{U} \times R^4$. By Theorem 2.1, the time derivative of the Lyapunov function satisfy $\frac{d\mathcal{V}}{dt} \le 0$. So, Ω_c is a positively invariant set such that $(x^\alpha(0), \eta^a(0), \dot{x}^\alpha(0), \dot{\eta}^a)(0) \in \Omega_c$ implies $(x^\alpha(t), \eta^a(t), \dot{x}^\alpha(t), \dot{\eta}^a(t)) \in \Omega_c$ for $t \ge 0$.

The set $\mathcal{E}$ is a subset of Ω_c where $\frac{d\mathcal{V}}{dt} = 0$. As $\mathcal{M}$ is the largest invariant subset of $\mathcal{E}$, we suppose $z(t) = (x^\alpha(t), \eta^a(t), \dot{x}^\alpha(t), \dot{\eta}^a(t)) \in \mathcal{M}$ for all $t \ge 0$ and then, in $\mathcal{M}$, we have $\eta^a(t) = \eta^a(0) = \eta_e^a = 0$, $\dot{\eta}^a(t) = 0$, $g^{ac}g_{\alpha c}\dot{x}^\alpha = 0$ (i.e., $\dot{x}^\alpha = 0$) for all $t \ge 0$, where we use some results: equations (40) and (43) in [3, page 1563]. So, we have $z(t) = (x^\alpha(t), 0, 0, 0) \in \mathcal{M}$ for all $t \ge 0$. Substituting these conditions into Euler-Lagrange equations (12) for x^α variables, we know that $z(t) = (x^\alpha(t), \eta_e^a(t), 0, 0) \in \mathcal{M}$ complies with the following equation (the general form is given by equation (45) in [3, page 1563]):

$$\begin{pmatrix}\ddot{X}\\ \ddot{Y}\end{pmatrix} = \begin{pmatrix}\frac{3gX\sqrt{L^2-X^2-Y^2}}{4L^2}\\ \frac{3gY\sqrt{L^2-X^2-Y^2}}{4L^2}\end{pmatrix}. \tag{21}$$

In [3], the authors shrink Ω_c to study the linearized dynamics of the general form which includes (21) to conclude a general stability result. Here, we directly investigate the nonlinear dynamics (21). Since $\frac{3g\sqrt{L^2-X^2-Y^2}}{4L^2} > 0$ in U, there is only one equilibrium $(x^\alpha, \dot{x}^\alpha) = (X_e, Y_e, \dot{X}_e, \dot{Y}_e) = (0,0,0,0)$ of the dynamics (21) such that any trajectory $(X(t), Y(t), \dot{X}(t), \dot{Y})(t)$ starting in $U \times R^2$ will escape from $U \times R^2$ except when the trajectory is the equilibrium. Thus, we have the invariant equilibrium $z(t) = (0,0,0,0) \in \mathcal{M}$. The above argument implies that the largest invariant set in $\mathcal{E}$ is the origin: $\mathcal{M} \stackrel{\triangle}{=} (x_e^\alpha, \dot{x}_e^\beta, \eta_e^a, \dot{\eta}_e^b) = (0,0,0,0)$.

Then, we conclude that any states starting in Ω_c approach an invariant set $\mathcal{M}$ which contains only the origin as $t \to \infty$. $\square$

V. Computer Simulation

To ease the visualization of the projections: X and Y, we give the total projection of the pendulum in the horizontal plane, that is, $2r = 2\sqrt{X^2+Y^2}$. Let the pendulum length be $2L = 0.6\,(m)$, mass $m = 0.35\,(kg)$ and the gravitational acceleration $g = 9.8\,(N/s^2)$. The dimension of all forces: both control inputs and disturbnaces are Newton (i.e., N). These imply appropriate dimensions for other parameters which are omitted for brevity.

By trials and errors, we start with all absolute values of parameters, 1 and then change those values with increasing some values or decreasing some values and finally select the design parameters as $\kappa = 100$, $\rho = -0.02$, $\epsilon = 1 \times 10^{-4}$, and $c_x^x = c_y^y = 0.01$, $c_x^y = c_y^x = 0$.

Remark 1: Admittedly, one has the freedom to tune the parameters in the control function (18) such as κ, ρ, ϵ, c_x^x, c_y^y, c_x^y and c_y^x. The design process is, however, not systematic and the tuning rules are lacking to optimize the design parameters. Many of our choices lead to oscillatory trajectories. For example, with an increase in $\epsilon = 1 \times 10^{-3}$ and other design parameters as before, the trajectory oscillates heavily before converging to the origin (see Figure 2).

Case 1: Let the exogenous disturbance and unmodelled dynamics be zero. Figure 3 shows the simulation result with the initial values

$$(x, \dot{x}, y, \dot{y}, X, \dot{X}, Y, \dot{Y}) = (20, 2, -20, 2, 0.1, 0.1, -0.1, 0.1).$$

which indicates a large domain of attraction.

Analytically, there exists a compact set $\Omega_c \subset U$, the domain of attraction for the given parameters. However, it is unclear how the domain of attraction increases with those design parameters. Here, we approximately estimate some projections of the domain of attraction associated with the nominal controlled system based on the (quantitative) simulation study. To reduce the complexity of analysis, let $(\dot{x}(0), \dot{y}(0), \dot{X}(0), \dot{Y}(0)) = (0,0,0,0)$ be initial conditions for the rates. Figure 4 shows the projections in two scenarios: first, let $(x(0), y(0)) = 0$ and all initial angles inside the outer layer converge to the origin and diverge outside the outer layer; second, let $\sqrt{x(0)^2 + y(0)^2} = 570(m)$ (this implies many cases for $(x(0), y(0))$) and only initial angles inside the inner layer, a very small neighborhood about the origin, converge to the origin. Therefore, the method of controlled Lagrangain yields some bounded (non-local) domain of attraction.

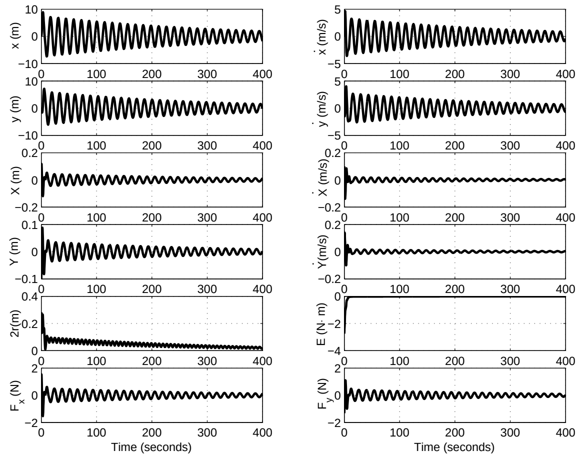

Fig. 2. The oscillatory trajectory results from inappropriate design parameters

Case 2: Introduce an exogenous input to the system (2) such that its right hand side becomes $(-C_X\dot{X},\ -C_Y\dot{Y},\ u_x - C_x\dot{x}, u_y - C_y\dot{y})$, where $C_x = C_y = 10^{-4}\,(N\!\cdot\! s/m)$ and $C_X = C_Y = 5\times 10^{-4}\,(N\!\cdot\! s/m)$. Figure 5 shows the simulation result with the initial values

$$(x, \dot{x}, y, \dot{y}, X, \dot{X}, Y, \dot{Y}) = (2, 0, 2, 0, 0, 0, 0, 0),$$

where the pendulum falls over eventually. The controlled Lagrangian design yields poor robustness for this set of design parameters.

However, our claims in the domain of attraction and the robustness are based on a simulation study and should be interpreted tentatively since we have not explored all degrees of freedom in the simulations. A better alternative would be to analytically analyze robustness but the the best of our knowledge this problem remains open in the literature.

Remark 2: Controlled Lagrangians and controlled Hamiltonians solve the matching conditions for an open loop system without physical damping. It has been shown that physical damping can affect stability in the closed loop because whenever the kinetic energy is modified, physical damping terms do not always enter as dissipation with respect to the closed energy function [20], [21].

Remark 3: The approach is also summarized in [22] which is tentatively compared with other approaches [11], [15], [17] based on computer simulation.

VI. Conclusion

Motivated by the physical insight from those energy or passivity based control tools, an explicit controller using the idea of controlled Lagrangians [3] is computed to stabilize the spherical inverted pendulum in full state space. The associated closed loop yields a non-local stability region based on LaSalle's invariance principle. This is verified through computer simulation. However, due to the lack of tuning rules and tools like control Lyapunov functions, robustness is an open issue and further research on the general energy ideas is required.

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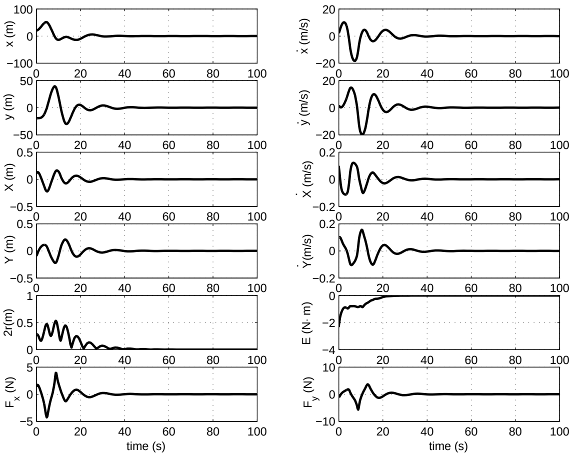

Fig. 3. Simulation results in *Case 1*

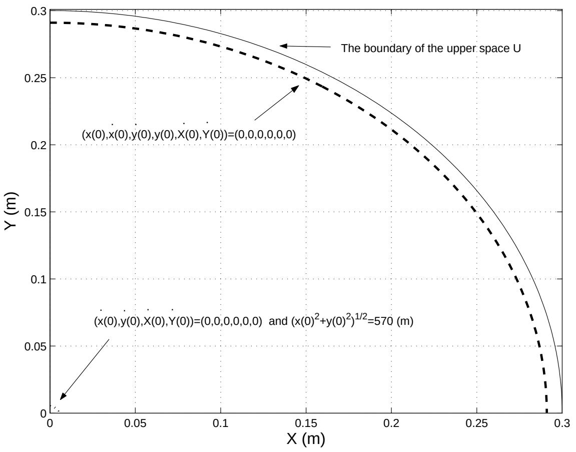

Fig. 4. The estimate of some projections of D.O.A based on simulations

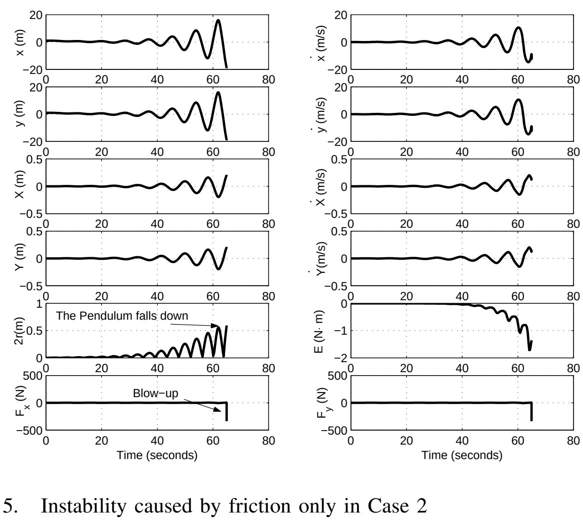

Fig. 5. Instability caused by friction only in Case 2

REFERENCES

[1] A. Bloch, P.Krishnaprasad, J. Marsden, and G. S. de Alvarez, "Stabilization of rigid body dynamics by internal and external torques," *Automatica*, vol. 28, pp. 745–756, 1992.

[2] A. Bloch, N. Leonard, and J. Marsden, "Controlled lagragians and the stabilisation of mechanical systems i:the first matching theorem," *IEEE T. Automat. Contr.* vol. 45, pp. 2253–2269, 2000.

[3] A. Bloch, D. Chang, N. Leonard, and J. Marsden, "Controlled lagragians and the stabilisation of mechanical systems ii: potential shaping, *IEEE T. Automat. Contr.* vol. 46, pp. 1556–1571, 2001.

[4] D. Auckly and L. Kapitanski, "On the λ-equations for matching control laws," *SIAM Journal on control and optimization*, vol. 37, pp. 1372–1388, 2002.

[5] R. Ortega, A. Loria, P. J. Nicklasson, and H. Sira-Ramirez, *Passivity–based control of Euler–Lagrange systems*, Springer-Verlag, 1998.

[6] A. V. D. Schaft, *L_2 gain and passivity techniques in nonlinear cntrol, Communication & Control Engineering Series*, Springer-Verlag, 2000.

[7] R. Ortega, W. Spong, F. Gomez-Estern, and G. Blankenstein, "Stabilization of a class of underactuated mechanical systems via interconnection and damping assignment," *IEEE T. Automat. Contr.* vol. 47, pp. 1218–1233, 2002.

[8] G. Blankenstein, R. Ortega, and A. V. D. Schaft, "The matching conditions of controlled Lagrangians and ida-passivity based control," *Int. J. Control.* vol. 75, pp. 645–665, 2002.

[9] D. Chang and J. Marsden, "The equivalence of of controlled Lagrangians and controlled hamiltonian systems," *ESAIM Contr. Optim. Calculus. Variations.* vol. 8, pp. 393–422, 2002.

[10] X. Albouy and L. Praly, "On the use of dynamic invariants and forwarding for swinging up a spherical inverted pendulum," in *Proceedings of 39th Conference on Decision & Control*, Sydney, Australia, pp. 1667–1672, 2000.

[11] R. Yang, Y. Kuen, and Z. Li, " Stabilisation of a 2-dof spherical pendulum on x-y table, " in *Proc. of the 2000 IEEE international conference on control application*, Anchorage, Alaska, USA, pp. 724–729, 2000.

[12] G. Liu, D. Nešić, and I. Mareels, "Modelling and stabilisation of a spherical inverted pendulum," in *Proceeding of 16th IFAC World Congress*, (in CD-ROM, no pages) 2005.

[13] G. Liu, I. Mareels, and D. Nešić, "Control of a class of nonlinear interconnected systems II: the case study," in *Proc. of 6th Asian Control Conference*, Bali, Indonesia, pp. 107-114 2006.

[14] C. Aguilar-Ibanez, O. Gutierrez, and H. Sossa-Azuela, "Lyapunov Approach for the stabilization of the Inverted Spherical Pendulum," in *Proc. of 45th IEEE Conference on Decision and Control*, Pretoria, South Africa, pp. 6133–6137 , 2006.

[15] G. Liu, D. Nešić and I. Mareels, "Nonlinear stable-inversion based output tracking for the spherical inverted pendulum," *International Journal of Control.* vol. 81, pp. 116–133, 2008.

[16] G. Liu, and L. Marconi, "Revisit the high and low gain approach of a spherical inverted pendulum," in *Proc. of 17th IFAC World Congress*, Seoul, Korea, accepted, 2008.

[17] G. Liu, I. Mareels, and D. Nešić, "Decentralized control design of interconnected chains of integrators: a case study ," *Automatica*, accepted.

[18] ——, "Way-point tracking of a spherical inverted pendulum," in *Proc. of The 6th IEEE international conference on control and automation,* Guangzhou, China, pp. 654–659, 2007.

[19] G. Liu, D. Nešić, and I. Mareels, "Nonlinear stable-inversion based output tracking for the spherical inverted pendulum," *International Journal of Control*, vol. 83, pp. 116-133, 2008.

[20] C. Reddy, W. Whitacre, and C. Woolsey, "Controlled Lagrangians with gyroscopic forcing: an experimental application," *Proceedings of the 2004 ACC*, Boston, Massachuseetts, pp. 511–516, 2004.

[21] F. Gómez-Estern and A. van der Schaft, "Physical damping in ida-pbc controlled underactuated mechanical systems," *Eur. J. Control: Spec. Issue. Hamiltonian. Lagrangians. Methods. Nonlinear. Control.* vol. 10, pp. 451–468, 2004.

[22] G. Liu, I. Mareels, and D. Nešić, "A note on the control of a spherical inverted pendulum," in *Proc. of 7th IFAC Symposium on Nonlinear Control Systems*, Pretoria, South Africa, pp. 838–843, 2007.