McClellan based design approach for 3-D digital filters with minimization of the integral squared error

Guergana S. Mollova, Wolfgang F. G. Mecklenbräuker

***Abstract*—**This paper presents a new approach for the design of three-dimensional (3-D) FIR digital filters using the McClellan transformation method and an integral squared error (ISE) criterion. We concentrate our investigations on the design of 3-D filters having cone-shaped magnitude responses. Analytical closed-form relations for transform parameters and 1-D cut-off frequency are developed. In order to find an optimal solution for the transform parameters, we apply minimization of the ISE function based on the double integration in the frequency domain. All derived relations are expressed in terms of the desired angle of inclination of the cone filter. Design of a scaled transformation function is also discussed. Several examples are presented to show the effectiveness of the proposed technique.

***Keywords*—**Integral squared error criterion, McClellan transformation, Scaled transform function, 3-D FIR digital filters.

I. Introduction

IT is known, that the McClellan transformation is an efficient tool in designing multidimensional (M-D) digital filters. Especially methods for two-dimensional (2-D) filters are very well developed in the literature. This transform is easy to apply; it gives also an efficient 2-D implementation structure with minimum number of multiplications and good round-off noise performance [1]. The original transform method uses a linear-phase 1-D prototype filter, which is mapped by a transformation function to designed 2-D filter. Different optimality criteria have been applied in realization of these methods, e.g. the integral squared error (ISE) criterion for 2-D zero-phase FIR fan filters [2], [3]. Pei and Shyu [4] proposed a 2-D least-squares (LS) contour mapping with a simultaneous determination of the optimal 1-D cut-off frequency. Other 2-D approaches using McClellan transform and LS criterion are discussed in [5]-[7]. The coefficients of the M-D transform are obtained in [8] using LS optimization along to a series of contour points. A 3-D ellipsoidal frequency response is constructed as an example.

Many application areas require 3-D digital filters instead of 2-D filters (e.g. 3-D visual systems, medical diagnoses, 3-D seismic data processing, etc.). Some of the methods for design of 3-D filters are created as an extension of 2-D approaches. Several McClellan transform based methods for 3-D filters have been developed, as well [9]-[11].

In this work, we propose a new design technique for 3-D FIR lowpass filters with cone-shaped characteristics using the ISE criterion and 3-D first order McClellan transform. As a result, the new analytical formulas for the transform parameters will be derived leading to the effective computation procedure.

This paper is organized as follows. Section II presents the theoretical background of the new approach. We consider the calculation of transform parameters and 1-D cut-off frequency and describe the minimization of the ISE function. Next section deals with the design of a scaled transform. Simulation results and some accuracy issues are discussed in Section IV. Finally, in Section V we present the conclusions.

II. Design approach for 3-D FIR filters

A. The McClellan transform

The M-D McClellan transformation uses the following relation:

$$\cos(\omega) = F(\omega_1, \omega_2, ..., \omega_M) \, , \tag{1}$$

where $F(\omega_1,\omega_2,...,\omega_M)$ is M-D transform function and ω is the 1-D frequency variable.

Let us define the following first order 3-D transform function:

$$\begin{aligned} &F_3(\omega_1, \omega_2, \omega_3) = \\ &= t_{000} + t_{001}\cos(\omega_3) + t_{010}\cos(\omega_2) + \\ &+ t_{011}\cos(\omega_2)\cos(\omega_3) + t_{100}\cos(\omega_1) + t_{101}\cos(\omega_1)\cos(\omega_3) + \\ &+ t_{110}\cos(\omega_1)\cos(\omega_2) + t_{111}\cos(\omega_1)\cos(\omega_2)\cos(\omega_3) \, , \end{aligned} \tag{2}$$

which is one choice from the original McClellan transform of (I,J,K)-order:

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G. S. Mollova is with the University of Architecture, Civil Engineering and Geodesy, Department of Computer-Aided Engineering, 1 Hr. Smirnenski Blvd., 1046 Sofia, Bulgaria (phone: 359 2 9635245/309; fax: 359 2 8656674; e-mail: g.mollova@gmx.net).

W. F. G. Mecklenbräuker is with the Institute of Communications and Radio-Frequency Engineering, Vienna University of Technology, Gusshausstrasse 25/389, A-1040 Vienna, Austria (e-mail: wmecklen@pop.tuwien.ac.at)

$$F_{OM}\ (\omega_1,\omega_2,\omega_3) = \\ = \sum_{i=0}^{I}\sum_{j=0}^{J}\sum_{k=0}^{K} t_{ijk}\ \cos(\ i\omega_1)\cos(\ j\omega_2)\cos(\ k\omega_3).$$

We consider in our approach a 1-D zero-phase FIR filter of odd length $2N$+1 with a frequency response:

$$H(\omega) = \sum_{n=0}^{N} a(n) T_n\left[\cos(\omega)\right] ,$$

where the coefficients $a(n)$ can be expressed in terms of the impulse response samples of the 1-D filter. With $T_n\left[x\right]$ is denoted the n-th order Chebyshev polynomial [12] defined by the following recursive relation:

$$T_0\left[x\right] = 1$$
$$T_1\left[x\right] = x$$
$$T_{n+1}\left[x\right] = 2xT_n\left[x\right] - T_{n-1}\left[x\right], \qquad n \geq 2, \qquad -1 \leq x \leq 1.$$

Then, the 3-D frequency response is obtained using the McClellan transformation (1):

$$H(\omega_1,\omega_2,\omega_3) = H(\omega)\big|_{\cos(\omega)=F_3(\omega_1,\omega_2,\omega_3)} = \\ = \sum_{n=0}^{N} a(n) T_n[F_3(\omega_1,\omega_2,\omega_3)].$$

The function $F_3(\omega_1,\omega_2,\omega_3)$ should satisfy the condition $|F_3(\omega_1,\omega_2,\omega_3)| \leq 1$ for all frequencies ω_1, ω_2, and ω_3 in the interval $[-\pi,\ \pi]$ (as the cosine function $|\cos(\omega)| \leq 1$). This transform is known as scaled or "well-behaved" transform.

The 3-D surfaces created by transformation (1) will be called isopotential surfaces. It is known that: (i) the function $F_3(\omega_1,\omega_2,\omega_3)$ determines the shapes of isopotential surfaces of the designed 3-D filter, and (ii) the values along these surfaces are fixed by means of the 1-D prototype frequency response.

The problem now is how to calculate the transform coefficients introduced in (2), so that the designed 3-D frequency response to approximate the ideal one. The ISE criterion will be used as an optimality criterion.

B. Determination of transform parameters and 1-D cut-off frequency

We focus our attention on 3-D filters with a particular symmetry class, namely with a conical type of the magnitude response. The ideal double cone filter oriented in ω_3-direction is shown in Fig. 1. On the surface of the cone we have:

$$\omega_1^2 + \omega_2^2 - \frac{\omega_3^2}{\gamma^2} = 0 \ , \tag{3}$$

where γ is the slope of the cone. Below, we denote the angle of inclination of the cone with θ:

$$\theta = \arctan(\gamma)$$

as the angle between the surface of the cone and (ω_1,ω_2)-plane. The angle θ is an important parameter in the input specifications of designed 3-D cone filters.

We impose below the following conditions on the function $F_3(\omega_1,\omega_2,\omega_3)$:

$$\begin{aligned} F_3(0,0,\pi) &= \cos(0) = 1 \\ F_3(\pi,\pi,\pi) &= \cos(\pi) = -1 \\ F_3(\pi,0,0) &= \cos(\pi) = -1 \end{aligned} \tag{4}$$

These conditions are necessary to realize a true mapping from 1-D lowpass prototype to 3-D lowpass filter. With the first one we transform 1-D point ω=0 into the point of the passband of the cone $(0,0,\pi)$. By analogy, the π-point (1-D plane) is mapped into the stopband point (π,π,π) from 3-D plane. As the point $(\pi,0,0)$ is outside of the cone surface, we assume the validity of the third condition from (4). Finally, the following system of equations is obtained having 8 unknown transform parameters:

$$t_{000} - t_{001} + t_{010} - t_{011} + t_{100} - t_{101} + t_{110} - t_{111} = 1$$
$$t_{000} + t_{001} + t_{010} + t_{011} - t_{100} - t_{101} - t_{110} - t_{111} = -1$$
$$t_{000} + t_{001} - t_{010} - t_{011} + t_{100} + t_{101} - t_{110} - t_{111} = -1$$

We can express now three of the parameters as a function of the rest five:

$$\begin{aligned} t_{010} &= t_{101} + t_{100} - t_{011} \\ t_{000} &= t_{011} + t_{111} - t_{100} \\ t_{001} &= -1 + t_{100} - t_{011} + t_{110} \end{aligned} \tag{5}$$

By substituting (5) in (2), we rewrite the transform function as a function of 5 parameters (t_{011}, t_{111}, t_{100}, t_{110}, and t_{101}):

$$\begin{aligned} & F_3(\omega_1,\omega_2,\omega_3) = \\ & = t_{011}(1 - \cos(\omega_3) - \cos(\omega_2) + \cos(\omega_2)\cos(\omega_3)) + \\ & + t_{111}(1 + \cos(\omega_1)\cos(\omega_2)\cos(\omega_3)) + \\ & + t_{100}(-1 + \cos(\omega_3) + \cos(\omega_2) + \cos(\omega_1)) + \\ & + t_{110}(\cos(\omega_3) + \cos(\omega_1)\cos(\omega_2)) + \\ & + t_{101}(\cos(\omega_2) + \cos(\omega_1)\cos(\omega_3)) - \cos(\omega_3)\,. \end{aligned} \tag{6}$$

Let us denote the cut-off frequency of 1-D prototype with ω_0. We formulate our approximation problem as follows: to determine transform parameters from (6) and optimal frequency ω_0 under given angle of the cone $\theta \in (0,\pi/2)$, such that the isopotential surface corresponding to ω_0 approximates the cone surface defined by (3). The similar design problem was considered in [2], but applied for 2-D FIR fan filters.

To solve the above problem, at first we define the following

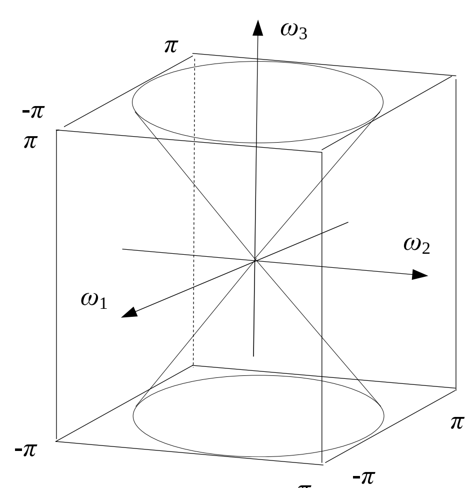

Fig. 1 The ideal frequency response of a cone filter (with γ=1)

deviation function on the cut-off isopotential surface:

$$E(\omega_1,\omega_2,\omega_3) = F_3(\omega_1,\omega_2,\omega_3) - c_0\,, \tag{7}$$

considering only small values of cosine arguments:

$$\cos(x) \approx 1 - \frac{x^2}{2}\ . \tag{8}$$

The parameter c_0 from (7) corresponds to the cut-off isopotential surface and therefore determines the frequency ω_0. The approximation (8) applied in our approach leads to better accuracy of designed filters for small values of the frequencies (see Fig. 2 for graphical illustration).

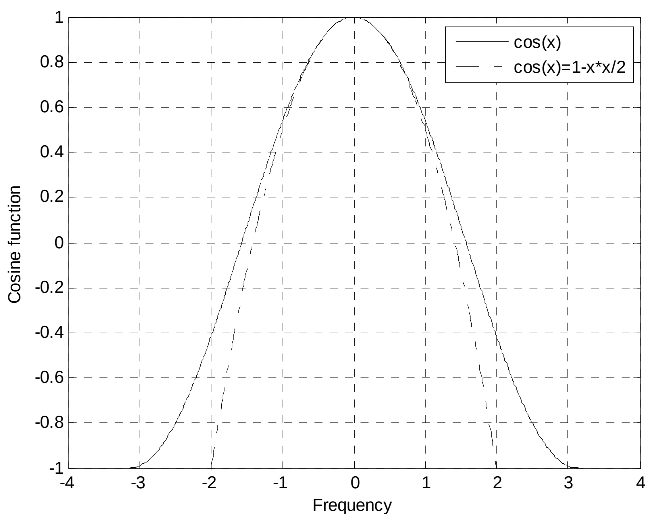

Fig. 2 The cosine function cos(x) plotted in the interval [-π, π] and its approximation

Using expressions (6)–(8) and after some calculations, we get the following relation from the equation $E(\omega_1,\omega_2,\omega_3)$=0:

$$\omega_1^2 + \omega_2^2 - \frac{\omega_3^2}{r/(1-r)} = \frac{2}{r}\left[\left(2r-1-c_0\right)+H\right]\ , \tag{9}$$

where the values of r and H are calculated as:

$$r = t_{111} + t_{110} + t_{100} + t_{101}\,,$$

$$H = \frac{\omega_1^2\omega_2^2}{4}(t_{111}+t_{110}) + \frac{\omega_2^2\omega_3^2}{4}(t_{011}+t_{111}) + \\ + \frac{\omega_1^2\omega_3^2}{4}(t_{111}+t_{101}) - t_{111}\frac{\omega_1^2\omega_2^2\omega_3^2}{8}.$$

As we stated in our approximation task, we wish the cut-off isopotential surface (i.e. the surface for ω=ω_0) to be as close as possible to the equation (3) describing the cone surface. Assuming this equivalence between (3) and (9), we get:

$$2r - 1 - c_0 = 0 \qquad \gamma^2 = \frac{r}{1-r}$$

and the parameters c_0 and r are finally determined as:

$$c_0 = -\cos(2\theta) \qquad r = \frac{1-\cos(2\theta)}{2}. \tag{10}$$

In order to reduce the approximation error (considering the expression for H) and also by using the above result for r, we found that:

$$\begin{aligned} t_{110} &= t_{011} = t_{101} = -t_{111} \\ t_{100} &= t_{111} + \frac{1-\cos(2\theta)}{2}. \end{aligned} \tag{11}$$

Therefore, by substituting relations (11) in (6), we can express the transform function in terms of only one unknown parameter (t_{111}) and angle θ:

$$\begin{aligned} &F_3(\omega_1,\omega_2,\omega_3) = \\ &= (t_{111}+r)(\cos(\omega_1)+\cos(\omega_2)-1) + \\ &+ (t_{111}-1+r)\cos(\omega_3) - t_{111}\cos(\omega_1)\cos(\omega_3) + \\ &+ t_{111}\cos(\omega_2)\cos(\omega_3)(\cos(\omega_1)-1) - t_{111}\cos(\omega_1)\cos(\omega_2), \end{aligned} \tag{12}$$

where the value of r is given by (10).

The equation of cut-off isopotential surface can be obtained by solving $F_3(\omega_1,\omega_2,\omega_3)$=$c_0$ for the frequency ω_3 as a function of ω_1 and ω_2:

$$\omega_3 = \arccos\left(\frac{c_0 + (t_{111}+r)(1-A) + t_{111}B}{t_{111}(1-A+B)+r-1}\right)\ , \tag{13}$$

where: $A = \cos(\omega_1) + \cos(\omega_2)$

$B = \cos(\omega_1)\cos(\omega_2)$.

This expression can be further optimized as follow:

$$\omega_3 = \arccos\left(\frac{t_{111}(1-A+B) + r(3-A) - 1}{t_{111}(1-A+B)+r-1}\right), \tag{14}$$

taking into account that c_0=−cos(2θ)=2r−1. We prove also that the following relation holds:

$$F_3(0,0,0) = c_0\,.$$

This may be explained by the fact that the isopotential cut-off surface passes through the origin of a 3-D plane. Several cut-off isopotential surfaces obtained for different angles θ will be plotted in Section IV.

C. *An application of the ISE criterion*

According to the results obtained in the previous section, the transform function (12) is expressed in terms of the parameter t_{111} (for small values of the cosine arguments). In order to find an analytical solution for this coefficient, we define the ISE function $M(t_{111})$ as:

$$M(t_{111}) = \int_0^{\pi}\int_0^{\pi} E^2(\omega_1, \omega_2 = f(\omega_1,\omega_3), \omega_3)\, d\omega_1 d\omega_3\ , \tag{15}$$

where $E(\omega_1, \omega_2 = f(\omega_1,\omega_3), \omega_3)$ is our deviation function from (7) determined in terms of ω_1 and ω_3 with the following value:

$$\begin{aligned} &E(\omega_1, \omega_2 = f(\omega_1,\omega_3), \omega_3) = 1 - 2r - \cos(\omega_3) + \\ &+ (\cos(\omega_1)+\cos(\omega_3))\left(r - t_{111}\omega_1^2 + t_{111}\frac{1-r}{r}\omega_3^2\right) + \\ &+ (r + t_{111} + t_{111}\cos(\omega_1)\cos(\omega_3))\left(\omega_1^2 - \frac{1-r}{r}\omega_3^2\right). \end{aligned} \tag{16}$$

In above consideration, we expressed cos(ω_2) as a function of ω_1 and ω_3 using equations (3) and (8):

$$\cos(\omega_2) \approx 1 + \frac{\omega_1^2}{2} - \frac{(1-r)\omega_3^2}{2r}.$$

Now we can reformulate our 3-D cone filter optimization problem as follows: to find the value of t_{111} that minimizes the

ISE function defined by (15) under given angle θ of the cone. The minimization is with respect to one parameter and therefore the analytical solution can be found. In order to determine the double integral in (15), we use the following statement:

$$M(t_{111}) = \int_0^{\pi}\left[\int_0^{\pi} E^2(\omega_1, \omega_2 = f(\omega_1,\omega_3), \omega_3)\, d\omega_3\right] d\omega_1\,, \tag{17}$$

which allow us to decompose the above task into solving a set of single integrals (with respect to ω_3 and ω_1). After analytical evaluation of all elementary integrals included in (17), we establish the following final result:

$$M(t_{111}) = P_1 + P_2 + P_3 + 2(P_4 + P_5 + P_6)\,, \tag{18}$$

where:

$$P_1 = \pi^2 + r\left(\frac{2\pi^4}{3} - 4\pi^2\right) + r^2\left(\frac{\pi^6}{5} - \frac{4\pi^4}{3} + \frac{\pi^2}{2}\right) +$$

$$+ t_{111}\left[\begin{array}{l}\dfrac{2\pi^4}{3} + 4\pi^2 + \\ + r\left(\dfrac{2\pi^6}{5} + \dfrac{19\pi^4}{3} - \dfrac{121\pi^2}{2}\right)\end{array}\right] + t_{111}^2\left(\frac{3\pi^6}{10} + \frac{17\pi^4}{2} - \frac{195\pi^2}{4}\right)$$

$$P_2 = (r-1)^2\left(\frac{\pi^6}{5} - \frac{7\pi^2}{2}\right) + t_{111}\frac{(r-1)^2}{r}\left(\frac{2\pi^6}{5} - 4\pi^2\right) +$$

$$+ t_{111}^2\frac{(r-1)^2}{r^2}\frac{3\pi^6}{10}$$

$$P_3 = t_{111}^2\left[\begin{array}{l}\dfrac{3\pi^6}{20} + \dfrac{17\pi^4}{4} - \dfrac{195\pi^2}{8} + \dfrac{(r-1)}{r}\left(\dfrac{\pi^6}{6} + \dfrac{5\pi^4}{3} + \dfrac{17\pi^2}{8}\right) + \\ + \dfrac{(1-r)^2}{r^2}\left(\dfrac{3\pi^6}{20} + \dfrac{3\pi^4}{4} - \dfrac{9\pi^2}{8}\right)\end{array}\right]$$

$$P_4 = t_{111}\left[(r-1)\left(3\pi^2 + \frac{\pi^4}{2}\right) + \frac{(r-1)^2}{r}\left(\frac{23\pi^4}{6} - \frac{97\pi^2}{4}\right)\right] +$$

$$+ t_{111}^2\left[\frac{r-1}{r}\left(\pi^4 + \frac{17\pi^2}{2}\right) + \frac{(r-1)^2}{r^2}\left(6\pi^4 - 36\pi^2\right)\right]$$

$$P_5 = \frac{(1-r)\pi^3}{3r}\left[2r^2\pi - r\pi - \frac{r^2\pi^3}{3} - t_{111}\left(\frac{2r\pi^3}{3} + \frac{3r\pi}{2} + \pi\right) - t_{111}^2\left(\frac{\pi^3}{2} + \frac{17\pi}{4}\right)\right]$$

$$P_6 = \frac{2\pi(1-r)}{r}\left[t_{111}\left(\frac{r\pi}{2} - \frac{r\pi^3}{3} - \pi\right) - t_{111}^2\left(\frac{\pi^3}{2} + \frac{17\pi}{4}\right)\right].$$

The final step in our derivations is to determine this optimal value of the coefficient t_{111} for which the first derivative of M is equal to zero:

$$\frac{\partial M(t_{111})}{\partial t_{111}} = 0\,.$$

Thus, considering (18) the following closed-form relation for t_{111} is found:

$$t_{111} = \frac{\dfrac{4\pi^4}{9} + \dfrac{8\pi^2}{3} - r\left(\dfrac{38\pi^4}{45} + \dfrac{29\pi^2}{3} - \dfrac{113}{2}\right) - \dfrac{r-1}{r}\left(\dfrac{2\pi^2}{3} + 4\right) - \dfrac{(r-1)^2}{r}I_1}{I_2 + \dfrac{r-1}{r}\left(\pi^4 + 17\pi^2 + \dfrac{289}{4}\right) + \dfrac{(r-1)^2}{r^2}I_2}\,, \tag{19}$$

where:

$$I_1 = \frac{2\pi^4}{5} + \frac{23\pi^2}{3} - \frac{105}{2}\ ;\ I_2 = \frac{9\pi^4}{10} + \frac{51\pi^2}{2} - \frac{585}{4}\ ;\ r = \frac{1-\cos(2\theta)}{2}.$$

This value of t_{111} is expressed as a function of desired angle θ and is a solution of our optimization task (15). Therefore, the transform function (12) is fully determined.

The transform coefficient t_{111} plotted as a function of the angle θ is given in Fig. 3. Its value varies in a range between zero and −0.2666. We would like to have values of t_{111} closer to zero (see the last term of expression for H). According Fig. 3, we can expect that our method will show better error performance for smaller values of θ. A detailed consideration on the accuracy of the method is presented in Section IV.

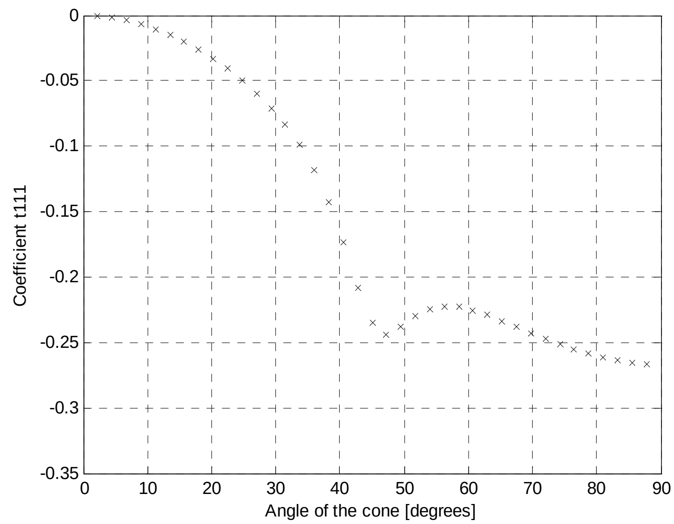

Fig. 3 Transform coefficient t_{111} as a function of the angle θ

III. Scaled Transform Function

As we discussed in the previous section, we are looking for the properly scaled transform function. Our investigations show that $F_3(\omega_1,\omega_2,\omega_3)$ specified by (12) needs a special scaling procedure, because its extremal values are:

$$F_{3\max} = 1 \qquad F_{3\min} = -2 + \cos(2\theta)\,. \tag{20}$$

That means that $|F_{3\min}|>1$ and the condition for scaled function $|F_3(\omega_1,\omega_2,\omega_3)| \le 1$ is not fulfilled. We apply below the *scaling scheme* proposed by Mersereau *et al*. [13] to get a scaled transform function:

$$F_3^S(\omega_1,\omega_2,\omega_3) = C_1 F_3(\omega_1,\omega_2,\omega_3) - C_2\,,$$

where:

$$C_1 = 2/(F_{3\max} - F_{3\min}) \qquad C_2 = C_1 F_{3\max} - 1. \tag{21}$$

Using relations (20) and (21) in our case we determine:

$$C_1 = \frac{2}{3-\cos(2\theta)} \qquad C_2 = \frac{\cos(2\theta)-1}{3-\cos(2\theta)}$$

and the scaled transform function is calculated as:

$$F_3^S(\omega_1,\omega_2,\omega_3) = \frac{1}{r+1}\left(\begin{array}{l} -t_{111} + (t_{111}+r)(\cos(\omega_1)+\cos(\omega_2)) + \\ + (t_{111}-1+r)\cos(\omega_3) + \\ + t_{111}\cos(\omega_2)\cos(\omega_3)(\cos(\omega_1)-1) - \\ - t_{111}\cos(\omega_1)\cos(\omega_3) - t_{111}\cos(\omega_1)\cos(\omega_2)\end{array}\right), \tag{22}$$

where the values of t_{111} and r are obtained in the previous section.

The scaling method [13] is originally developed for 2-D filters, but our investigations proved, that it is also applicable for 3-D case. The shapes of the surfaces obtained with the scaled function (22) are the same as those with non-scaled function $F_3(\omega_1,\omega_2,\omega_3)$ defined by (12). Hence, the equation of isopotential cut-off surface obtained with the function (22) is the same as this one with the non-scaled function (14).

The 1-D cut-off frequency associated with the scaled function can be determined as [13]:

$$\omega_0 = \cos^{-1}(C_1 c_0 - C_2) = \cos^{-1}\left(\frac{1-3\cos(2\theta)}{3-\cos(2\theta)}\right). \tag{23}$$

The relation (23) indicates the optimal value of the frequency ω_0 and will be used in our design methodology.

We have to point out that the method presented in this work differs from the approach [14] in the following main items: (i) three input conditions are imposed instead of four as in [14], (ii) the ISE criterion is applied, and (iii) the new relations for transform parameters are proposed.

IV. Design Methodology and Examples

All results given in this section are obtained using a Matlab simulation based on the derived expressions. The graphical view of the isopotential cut-off surfaces (for different angles θ) is shown in Fig. 4(a),(b). The plots are generated using expression (14). Fig. 4(a) presents also a comparison between the surfaces obtained with our method and method [14] without LS-optimization. The results show a better approximation to the cone surface for angles below 45° (π/4 radians) with the new method. The shapes of the surfaces for both methods are very close for angles above 45°.

The following *methodology* has been used for design of 3-D FIR cone filters:

- Determination of the 1-D cut-off frequency using relation (23) under given angle θ;
- Design a zero-phase 1-D FIR lowpass filter (we apply a McClellan-Parks algorithm in our examples). In this step, we create at first a linear-phase FIR filter with the desired magnitude response, and then by shifting the impulse response

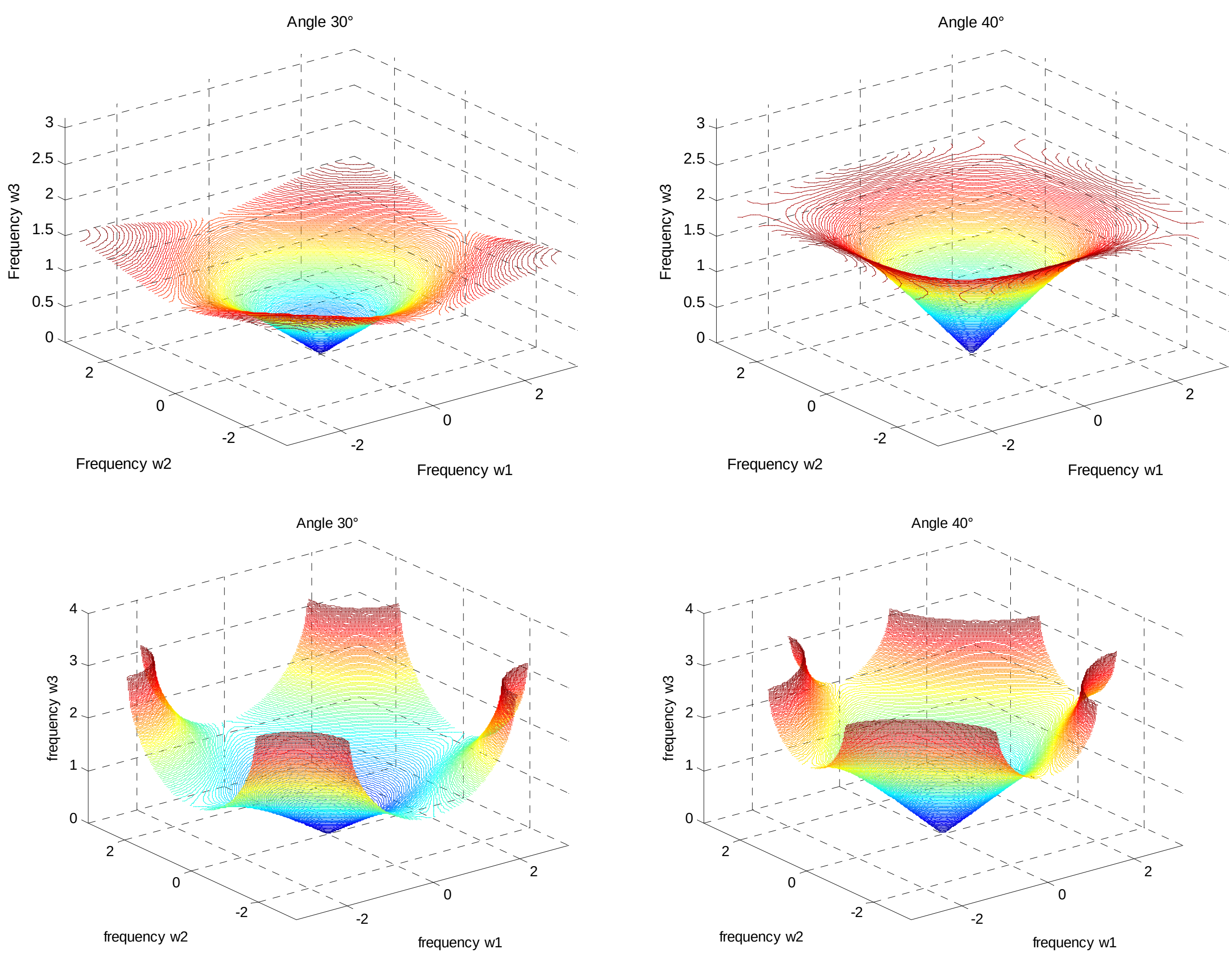

Fig. 4(a) The isopotential cut-off surfaces plotted for angles 30° and 40° (0.167π and 0.222π). The plots on the first row are obtained with the new method; second row – method [14]

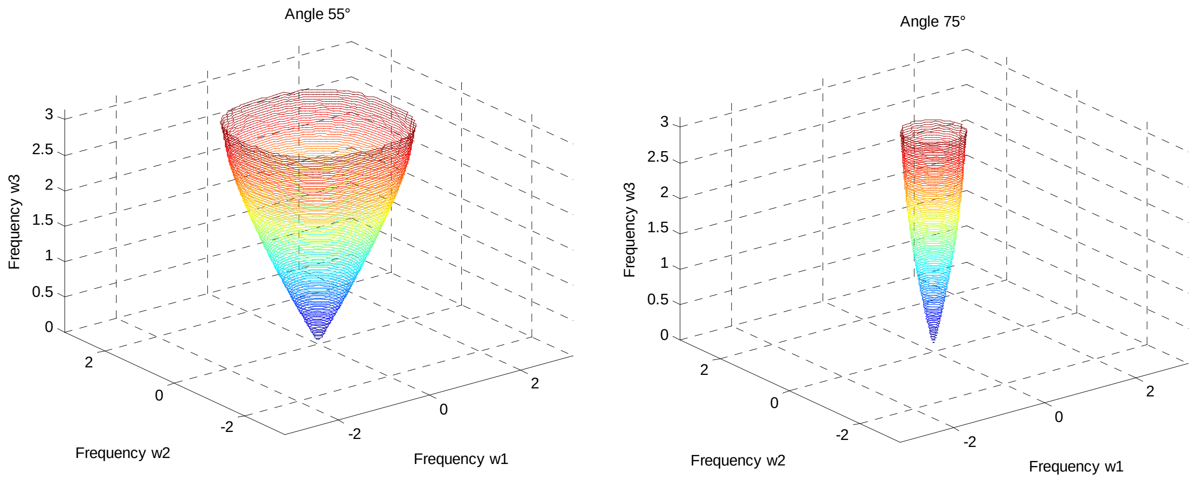

Fig. 4(b) The isopotential cut-off surfaces plotted for angles 55° and 75° (0.306π and 0.417π) with the new approach. The plots with method [14] – approximately the same

we get the corresponding zero-phase filter;

- Calculation of transform parameter t_{111} (see the expression (19)) in order to obtain the scaled transform function (22);

- Using the presentation of a 1-D frequency response in terms of the Chebyshev polynomial (given in Section II) and relation (1), we design our 3-D cone filter response. The result for the transform function from the previous step is applied.

Fig. 5 presents several 3-D magnitude responses plotted for ω_3=*constant* (as an example we choose $\omega_3=\omega_0$). Different angles of the cone filter are examined. The cut-off frequencies of the corresponding 1-D prototypes have the following values: $\omega_0=0.4238\pi$ (θ=42°), 0.2649π (θ=58°), 0.2028π (θ=65°), and 0.1192π (θ=75°). The coefficient t_{111} computed by using (19) is: $t_{111}=-0.1969$ (θ=42°), -0.2226 (θ=58°), -0.2330 (θ=65°), and -0.2530 (θ=75°). These results are obtained with 1-D prototypes of length $2N+1=33$ and transition band 0.1π. As we expected, the 3-D magnitude responses are equiripple (because 1-D equiripple prototypes determine the values along isopotential surfaces).

In order to evaluate the accuracy of the method, we draw the deviation function E for the scaled transform function:

$$E(\omega_1,\omega_2,\omega_3) = F_3^S(\omega_1,\omega_2,\omega_3) - \cos(\omega_0) \qquad (24)$$

in terms of ω_1 and ω_2 (for ω_3=*constant*). The graphical results (angles 25° and 40°) for both methods under consideration are shown in Fig. 6. A better error performance with the new method can be detected, essentially for bigger frequencies. This can be explained with the application of the ISE criterion (15) including double integration for the full frequency range of ω_1 and ω_3.

We observe that both methods lead to very close deviation functions only for small frequencies. It was proved that the following relations hold for these two methods:

$$E(\omega_1,\omega_2,\omega_3)\big|_{\omega_1=\omega_2=0} = \frac{(1+\cos(2\theta))(1-\cos(\omega_3))}{3-\cos(2\theta)}$$

$$E(0,0,0) = 0\,, \quad \text{for } \forall\,\theta\,.$$

In order to estimate more precisely the accuracy, we calculated also the mean deviation function as a mean value between 50 different functions E:

$$E_{mean} = \frac{1}{N_m}\sum_{k=1}^{N_m} E(\omega_1,\omega_2,\omega_3)\big|_{\omega_3=\omega_k}\,, \qquad (25)$$

where N_m=50 and the ω_k are equally spaced frequencies in the interval $[-\pi, \pi]$. Several plots of E_{mean} as a function of ω_1 and ω_2 determined for given angle θ are presented in Fig. 7. These graphics confirm all conclusions for the accuracy of our method stated above.

V. Conclusion

In this paper, the application of the ISE criterion for design of the McClellan based 3-D cone FIR filters is shown. The analytical closed-form expressions for transform coefficients and 1-D cut-off frequency are derived. Several design examples are presented using the new relations.

The graphical results (for cut-off isopotential surfaces, 3-D magnitude responses, and deviation function) show a good error performance of the method. The comparison between the new approach and method [14] proves the improvement of the accuracy (in particular for filters with angles below 45°).

The proposed method enjoys very short computation time without time-consuming iterative procedures. It can be also extended to design of other types of 3-D FIR filters (e.g. with spherical and elliptical responses).

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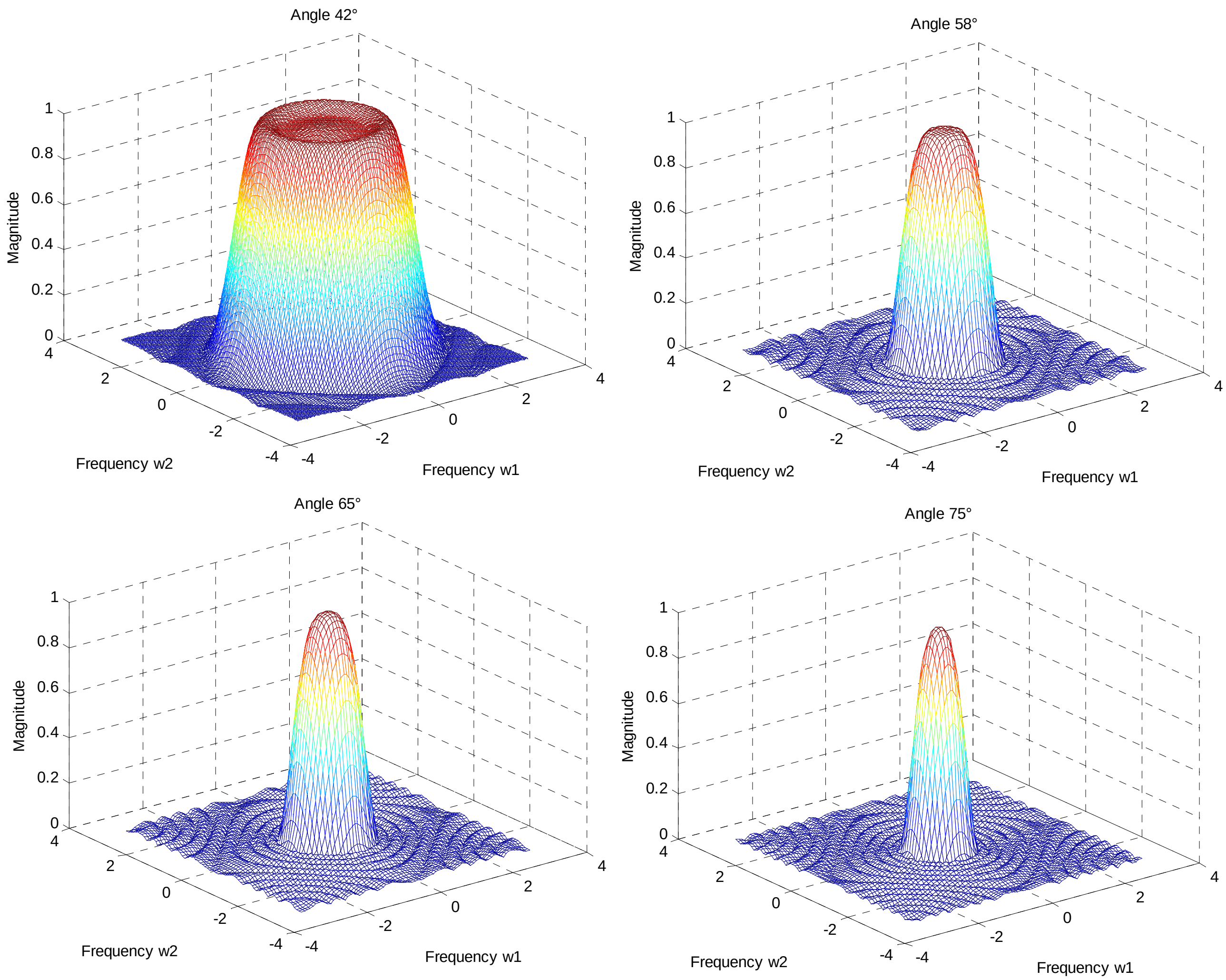

Fig. 5 The 3-D magnitude responses obtained for $\omega_3=\omega_0$ and different angles θ (42°, 58°, 65°, and 75°)

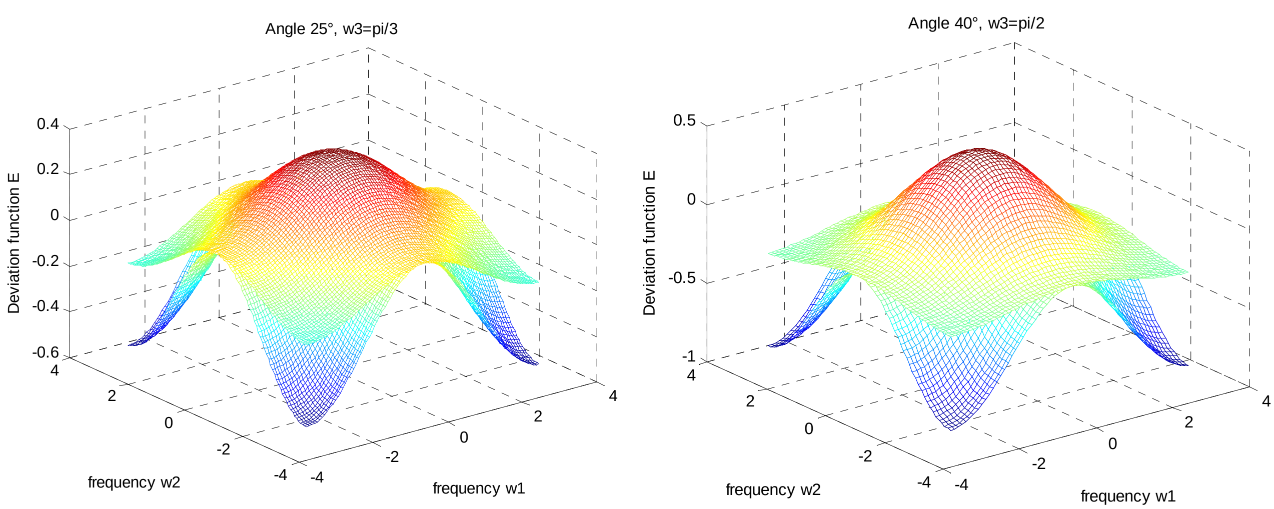

Fig. 6 Deviation function E for both methods. Angles 25° and 40° (new method – upper plots; method [14] – lower plots)

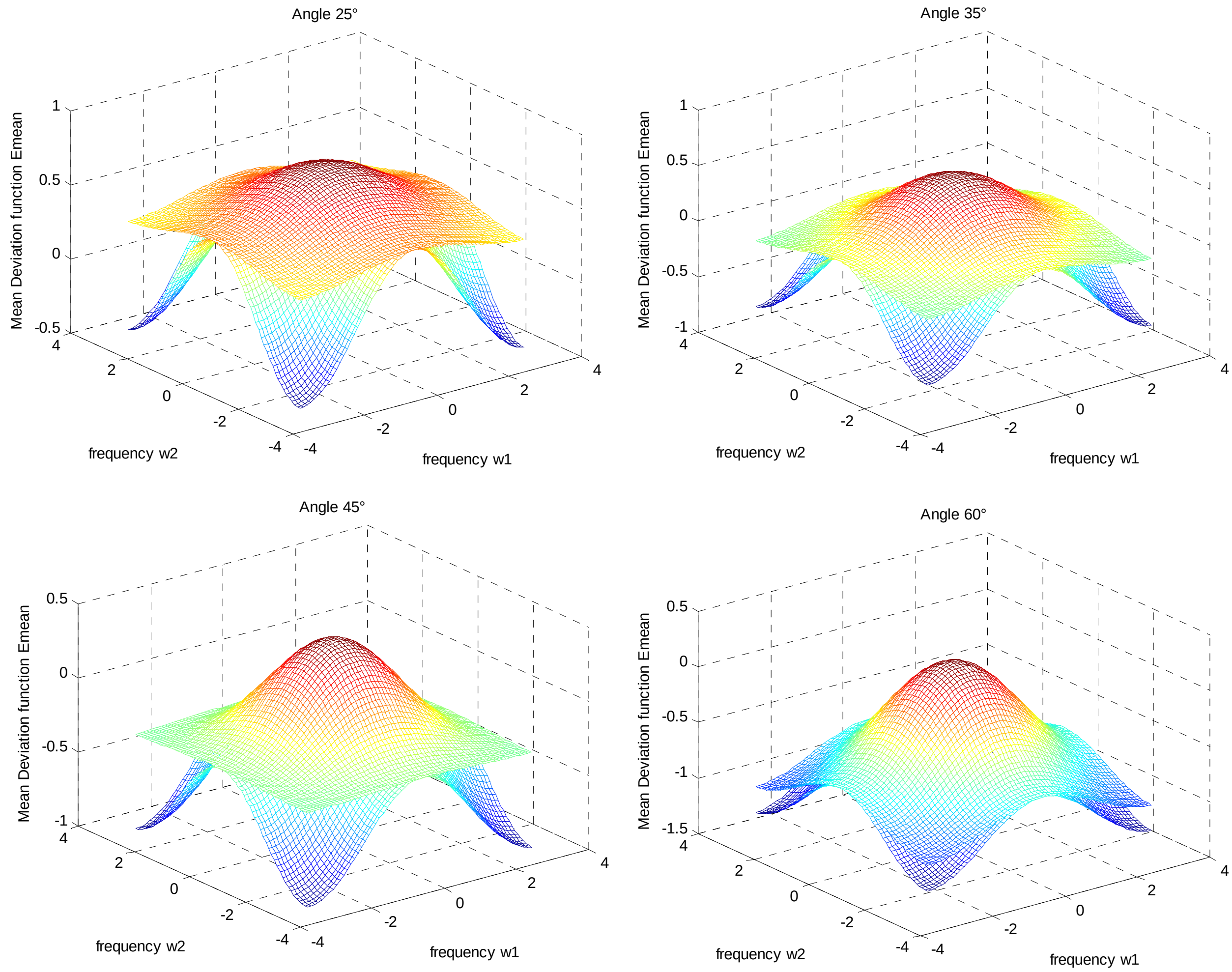

Fig. 7 The mean deviation function E_{mean} for different angles θ
(new method – upper plots; method [14] – lower plots)

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