NUMERICALLY MODELLING THE EXTENDED PLASMA FLUID SYSTEM WITH A MODIFIED SEMI DISCRETE SCHEME.

R. Naidoo

***Abstract*— An electrostatic plasma model of a one dimensional fluid system was studied for smooth and shock wave solutions with only the conservation of mass and continuity equations (2 fluid system). In this study the plasma system is extended to include the energy equations (3 fluid system). A modified Riemann solver free semi-discrete scheme (SD3-Kurganov-Levy scheme) was modified to include general source terms, limiters and boundary conditions for the 3-fluid plasma system. For simple initial density perturbations and shocks of the electrons and ions, we illustrate how the method captures the formation and evolution of solitons and shockwaves. The SD3 scheme exhibited stable numerical smooth and shock solutions of the electron and ion subsystems with no oscillations. The electron and ion sub-solutions exhibited different time scales with the electron waves travelling faster than the ion waves.**

***Keywords*—hyperbolic,plasma,semi-discrete,shocks,solitons**.

I. INTRODUCTION

Theory, experiments and simulations [3] show that solitons and shocks occur both in the laboratory and space. Previously we simulated solitons using NNT (modified fully discrete Nessyahu and Tadmor) and SD3 (modified semi-discrete Kurganov Levy) schemes to numerically integrate a plasma two fluid hyperbolic system [7].The plasma system is now extended to include the energy equation for electron and ions which is now called plasma three fluid system. However another thrust to the study is to examine the impact of the extension of the plasma system to include the energy equations in terms of smooth and shock wave solutions. The SD3 scheme which is Riemann solver free third order scheme is nonstaggered was used in [1,8]. One advantage is that it can be applied on non-staggered grids and thus ease the implementation of boundary conditions. We have adapted the SD3 scheme so that it can be applied to systems with stiff source and flux terms. The concept of the limiter in a fully discrete NNT scheme was blended into the SD3 scheme.

II. THE NUMERICAL SCHEME

The system of equation can be written in conservative form. The conservation form of these equations in Cartesian space variable is

$$\frac{\partial U}{\partial t}+\frac{\partial F(U)}{\partial x}=G \qquad (1)$$

,U is the field variable, F is the flux variable and G is the source term with the x-space and t the time coordinate.
The SD3 scheme due to Kurganov and Tadmor [1] is applied to (1) as outlined in [8]. In applying this method we employ uniform spatial and temporal grids with spacing,

$\Delta x = x_{j+1} - x_j$;

$\Delta t = t^{n+1} - t^n$ (with j and n being suitable integer indices) together with the semi-discrete scheme ("SD3") [8],

$$\frac{d\bar{u}_j}{dt} = -\frac{1}{2\Delta x}\left[f\left(u^+_{j+\frac{1}{2}}(t)\right)+f\left(u^-_{j-\frac{1}{2}}(t)\right)-f\left(u^+_{j-\frac{1}{2}}(t)\right)-f\left(u^-_{j-\frac{1}{2}}(t)\right)\right]$$

$$-\frac{a_{j+\frac{1}{2}}(t)}{2\Delta x}\left[u^+_{j+\frac{1}{2}}(t)-u^-_{j+\frac{1}{2}}(t)\right]-\frac{a_{j-\frac{1}{2}}(t)}{2\Delta x}\left[u^+_{j-\frac{1}{2}}(t)-u^-_{j-\frac{1}{2}}(t)\right]+g(u_j(t)) \qquad (2)$$

The construction of this scheme is described in detail in [1] and [8].We note in particular that the solution is updated by fitting on already computed or known cell average values $\{\bar{U}^n_j\}$ at time level n, piecewise polynomials of degree two on cells of size Δx central at x_j namely

$$P_j(x,t^n) = A_j + B_j(x-x_j) + C_j(x-x_j)^2 \qquad (3)$$

Where the constants are $(A_j.....)$ are specified later.

Here $u^+_{j\pm\frac{1}{2}} := P_{j+1}(x_{j\pm\frac{1}{2}},t^n); u^-_{j\pm\frac{1}{2}} := P_j\left(x_{j\pm\frac{1}{2}},t^n\right)$

$$a^n_{j\pm\frac{1}{2}} = \max\left(\rho\left(\frac{\partial f}{\partial u}(u^-_{j\pm\frac{1}{2}}(t))\right),\rho\left(\frac{\partial f}{\partial u}(u^+_{j\pm\frac{1}{2}}(t))\right)\right) \qquad (4)$$

where the forms (4) are respectively the left and right intermediate values at $x_{j+\frac{1}{2}}$ and $\rho(.)$ denotes the spectral radii of the respective flux Jacobian, defining the maximum local propagation speeds $a^n_{j\pm\frac{1}{2}}$.

This scheme has been tested on problems involving shock propagation in various gas fluids [8] and is

known to give accurate results. Here we are interested in applying the SD3 scheme (2), together with a modification detailed below.

III. IMPLEMENTATION DETAILS OF THE NUMERICAL SCHEME

The SD3 scheme is explicit in time. Thus the implementation of it follows closely the prescription given in [1] and [8] where in particular we use for the non-oscillatory piece-wise polynomial (3) the "CWENO" reconstruction [3] in which form (3) is determined by

$$A_j = \bar{u}_j^n - \frac{W_c}{12}(\bar{u}_{j+1}^n - 2\bar{u}_j^n + \bar{u}_{j-1}^n), \qquad (5)$$

$$B_j = \frac{1}{\Delta x}\left[W_R(\bar{u}_{j+1}^n - \bar{u}_j^n) + \frac{W_c}{2}(\bar{u}_{j+1}^n - \bar{u}_{j-1}^n) + W_L(\bar{u}_j^n - \bar{u}_{j-1}^n)\right] \qquad (6)$$

$$\text{And } C_j = \frac{W_j}{\Delta x^2}(\bar{u}_{j-1}^n - 2\bar{u}_j^n + \bar{u}_{j+1}^n) \qquad (7)$$

The constants W_L, W_C, W_R and are calculated as in [1] and involve heuristic factors which have a bearing on the sharpness of the slopes near discontinuities.

In addition, it is required to compute at every time step the spectral radii of the Jacobian of the flux terms, which we obtained exactly for the case to follow.

In some cases (at or near discontinuities) we shall find that solutions can be improved, by employing the so called non-linear limiters, in the calculation of the derivative terms. Thus we employ the min-mod limiter function $MM(.)$ given by [6,7]

$$MM(s_1, s_2, ..) = \begin{Bmatrix} \min\{s_j\}\ if\ s_j > 0\ \forall_j \\ \max\{s_j\}\ if\ s_j < 0\ \forall_j \\ 0, otherwise \end{Bmatrix} \qquad (8)$$

Then using (8) in (6) we obtain the improved form,

$$B_j = \frac{1}{\Delta x}\left[W_R MM(\bar{u}_{j+1}^n - \bar{u}_j^n, \bar{u}_j^n - \bar{u}_{j-1}^n) + \frac{W_c}{2} MM(\bar{u}_{j+1}^n - \bar{u}_j^n, \bar{u}_j^n - \bar{u}_{j-1}^n)\right]$$

$$+ \frac{1}{\Delta x} W_L [MM(\bar{u}_j^n - \bar{u}_{j-1}^n, \bar{u}_{j-1}^n - \bar{u}_{j-2}^n)] \qquad (9)$$

IV. THE EXTENDED PLASMA FLUID EQUATONS

The one-dimensional Poisson equations for an unmagnetized electrostatic system consist of electrons and ions taken as ideal fluids and together with the ideal gas law.

We set up the system of plasma equations taking into account conservation of mass, continuity and energy. The system can be written as

$$\frac{\partial U(x,t)}{\partial t} + A(U)\frac{\partial U}{\partial x} = G(U,\phi) \text{ and} \qquad (10)$$

$$\frac{\partial U(x,t)}{\partial t} + \frac{\partial F(U)}{\partial x} = G(U,\phi) \qquad (11)$$

In the above , $U = [u_1, u_2, u_3, u_4, u_5, u_6]^T$ is the 6-vector of conserved quantities (densities, momentum and energies), $F(U)$ is the flux vector function with $A(U)$ its Jacobian and $G(U,\phi)$ the vector function of the RHS.

$$\frac{\partial ne}{\partial t} + \frac{\partial}{\partial x}(n_e v_e) = 0 \qquad (12)$$

$$\frac{\partial ni}{\partial t} + \frac{\partial}{\partial x}(n_i v_i) = 0 \qquad (13)$$

$$\frac{\partial}{\partial t}(n_e v_e) + \frac{\partial}{\partial x}\left[\frac{(\alpha-1)}{Rm}E_e - \frac{1}{2}(\alpha-3)\frac{(n_e v_e)^2}{n_e}\right] = \frac{-e}{Rm} n_e \mathrm{E} \qquad (14)$$

$$\frac{\partial}{\partial t}(n_i v_i) + \frac{\partial}{\partial x}\left[(\alpha-1)E_i - \frac{1}{2}(\alpha-3)\frac{(n_i v_i)^2}{n_i}\right] = e n_i \mathrm{E} \qquad (15)$$

$$\frac{\partial E_e}{\partial t} + \frac{\partial}{\partial n}\left[\frac{n_e v_e}{n_e}\alpha E_e - \frac{1}{2}(\alpha-1)R_m\frac{(n_e v_e)^3}{n_e^2}\right] = -e n_e v_e \mathrm{E} \qquad (16)$$

$$\frac{\partial Ei}{\partial t} + \frac{\partial}{\partial x}\left[\alpha\frac{n_i v_i}{n_i}E_i - \frac{1}{2}(\alpha-1)\left(\frac{(n_i v_i)^3}{n_i^2}\right)\right] = e n_i v_i \mathrm{E} \qquad (17)$$

The index k=e(i) denotes electrons (ions) respectively and $n_k, v_k, p_k, y_k, m_k\ q_k, E_k, E$ are the respective component densities, flow velocities, partial pressures, adiabatic indices (=1 for electrons and =3 ions),particle masses,charges, Energy, Electric field and ϕ is the electric potential.The $R_m = \frac{m_e}{m_i}$ is the electron to ion mass ratio. The electric field (E) and potential field (ϕ) is calculated using the Poisson's equations is given by

$$E = -\frac{\partial \phi}{\partial x} \ , \tag{18}$$

$$\frac{\partial^2 \phi}{\partial x^2} = -4\pi \sum_{k=e,i} q_k n_k \tag{19}$$

$$\frac{\partial^2 \phi}{\partial x^2} = u_1 - u_2 \equiv S(U) \tag{20}$$

To write the Jacobian of the plasma system as electron and ion submatrices we write the systems as:

$$U = \begin{pmatrix} n_e \\ n_e v_e \\ E_e \\ n_i \\ n_i v_i \\ E_i \end{pmatrix} \qquad F = \begin{pmatrix} n_e v_e \\ \frac{(\alpha-1)}{Rm} E_e - \frac{1}{2}(\alpha-3)\frac{(n_e v_e)^2}{n_e} \\ \frac{n_e v_e}{n_e} \alpha E_e - \frac{1}{2}(\alpha-1) R_m \frac{(n_e v_e)^3}{n_e^2} \\ n_i v_i \\ (\alpha-1) E_i - \frac{1}{2}(\alpha-3)\frac{(n_i v_i)^2}{n_i} \\ \alpha \frac{n_i v_i}{n_i} E_i - \frac{1}{2}(\alpha-1)\left(\frac{(n_i v_i)^3}{n_i^2}\right) \end{pmatrix}$$

$$G = \begin{pmatrix} 0 \\ \frac{-e n_e E}{R_m} \\ -e n_e v_e E \\ 0 \\ e n_i E \\ e n_i v_i E \end{pmatrix} \tag{21}$$

Due to the Relative mass ratio R_m the electron subsystem can be described as being a stiff system. The ion subsystem is a nonstiff system.

The Jacobian matrix is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{(\gamma-3)m_1^2}{2\rho_1^2} & \frac{(\gamma-3)m_1}{\rho_1} & \frac{(\gamma-1)}{R_m} & 0 & 0 & 0 \\ \frac{m_1}{\rho_1^2}\gamma E_1 + (\gamma-1)R_m \frac{m_1^3}{\rho_1^3} & \frac{\gamma E_1}{\rho_1} - \frac{3}{2}(\gamma-1)R_m \frac{m_1^2}{\rho_1^2} & \frac{\gamma m_1}{\rho_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{(\gamma-3)m_2^2}{2\rho_2^2} & \frac{(\gamma-3)m_2}{\rho_2} & (\gamma-1) \\ 0 & 0 & 0 & -\frac{m_2}{\rho_2^2}\gamma E_2 + (\gamma-1)\frac{m_2^3}{\rho_2^3} & \frac{\gamma E_2}{\rho_2} - \frac{3}{2}(\gamma-1)\frac{m_2^2}{\rho_2^2} & \frac{\gamma m_2}{\rho_2} \end{pmatrix}$$

where $m_1 = n_e v_e$ and $m_2 = n_i v_i$ and $\rho_1 = n_e, \rho_2 = n_i, E_1 = E_e, E_2 = E_i$

The eigenvalues for the electron subsystem are:

$$\lambda 1 = \frac{m1}{\rho 1}, \lambda_2 = \frac{m_1}{\rho_1} - \sqrt{\frac{\gamma(\gamma-1)}{2R_m\rho_1}}\sqrt{2E_1\rho_1 - m_1^2 R_m} \ ,$$

$$\lambda_3 = \frac{m_1}{\rho_1} + \sqrt{\frac{\gamma(\gamma-1)}{2R_m\rho_1}}\sqrt{2E_1\rho_1 - m_1^2 R_m} \tag{22}$$

The eigenvalues for the ion subsystem are:

$$\lambda 4 = \frac{m2}{\rho 2}, \lambda_5 = \frac{m_2}{\rho_2} - \sqrt{\frac{\gamma(\gamma-1)}{2\rho_2}}\sqrt{2E_2\rho_2 - m_2^2} \ ,$$

$$\lambda_6 = \frac{m_2}{\rho_2} + \sqrt{\frac{\gamma(\gamma-1)}{2\rho_2}}\sqrt{2E_2\rho_2 - m_2^2} \tag{23}$$

The electron subsystem will have real eigenvalues providing

$$-\sqrt{\frac{2E_1\rho_1}{R_m}} < m_1 < \sqrt{\frac{2E_1\rho_1}{R_m}} \ , \tag{24}$$

Similarly the ion subsystem will have real eigenvalues providing

$$-\sqrt{2E_2\rho_2} < m_2 < \sqrt{2E_2\rho_2} \tag{25}$$

The negative values in (24) and (25) are significant in that momentum of the electrons and ions can be negative. There are six real and unequal eigenvalues associated with the plasma system providing that (24) and (25) are satisfied. In equation (4) we use the above eigenvalues as characteristic speeds.The system is hyperbolic if the above condition is satisfied. Furthermore the electron and ion subsystems are Euler. The system will be expected to exhibit a contact discontinuity and shock discontinuities as in a Riemann gas situation described in [5].

V. GRID AND COMPUTATIONAL SPECIFICATION

The coupled 3-fluid-Poisson equations are solved as a system. For the numerical integration we employ a system length $L_x = 256\lambda_{de}$, with the number of grid points per Debye length $N_{px} = 10$, giving $\Delta x = 0.1$, $R_m = 0.02$ and $\Delta t = 0.001$. This choice satisfies the CFL condition [4] for linear stability given in terms of the spectral radius Λ_m of the Jacobian $A(U)$, $\Lambda_m \frac{\Delta t}{\Delta x} \le 1$ where Λ_m is taken from eigenvalues of (22) and (23). In our application, we illustrate how solitons can be generated from an initial Gaussian density perturbation of the form

$$n_k(x,0) = 1.0 + 2.0\exp(-\frac{1}{2}(x - x_c)^2; 0 \le x \le L_x, \quad (26)$$

where x_c is the system centre and L_x is its length. The initial velocities of the ions and electrons are set to zero for all x and reflective boundary conditions are employed [2].
The initial conditions for both the electron and ion subsystems shock waves are:
$t = 0$

$$Rim3: \begin{cases} \rho = 2.5, u = 1.0, p = 1.0, x < 50 \\ \rho = 1.0, u = 0.4, p = 0.4, x > 50 \end{cases} \quad (27)$$

The Rim 3 initial conditions satisfies (24) and (25) above.

To determine CPU times (s) for the SD3 and SD3 with limiter computational time the solitons were computed for 20000 time units. The results are indicated in the table below.

Scheme	Time(s)
SD3	718.40
SD3 with limiter	778.81

VI. DISCUSSION

Using the initial conditions for the SD3 scheme as above and using equations (5-9) in the CWENO reconstruction in the SD3 we allowed the system to evolve from the initial equilibrium state, with a Gaussian density profile as given above. In figures 1 and 2 the SD3 scheme with limiter shows stability over a long time period. The non physical oscillations have been considerably reduced in the SD3 limiter scheme as compared to previous studies [7]. These results agree in the physics with previous simulations and no dissipation over long integration times
In the figure 1 below we observe the situation when some 4000 time steps have been reached. Two effects that are noticeable are that there is no significant dissipation in the soliton structures. In previous studies inherent dissipation was noted in [6] of such schemes, an effect which manifests when $\Delta t \sim (\Delta x)^2$ as is the case here. It is thus expected that the SD3 scheme with limiter will be of great advantage in the numerical investigation of other non-linear 3-fluid plasma structures. An interesting observation in Fig 1 is that a density hump of the electron wave remains stationary at the center whilst the ion wave density at the center remains constant. This phenomenon was not observed in the plasma two fluid system in [7].

Furthermore in figures 1 and 2 we have demonstrated how the modified SD3 scheme maybe used to solve a plasma fluid and electrostatic filed equations. The subsystems of electron and ion are Euler equations the solutions as depicted in figure 1 and 2 reveal real non-linear waves. In figures 1 and 2 the smooth and shock solutions of the electron evolve faster than the ion waves indicating that two different time scales. The momentum graphs in figure 1 exhibit negative and positive momentum due to conditions (24) and (25). As time progresses the amplitudes decreases.In figure 2 the shock wave solution exhibit shockwaves and contact shockwaves as in the Eulerian gas in [8]. However the ion shockwaves takes a longer time to form into shock waves.
In the Table above the CPU times for SD with limiter scheme is 8% larger than SD3 schemes indicated that the SD3 scheme with limiter is more expensive.

Figures

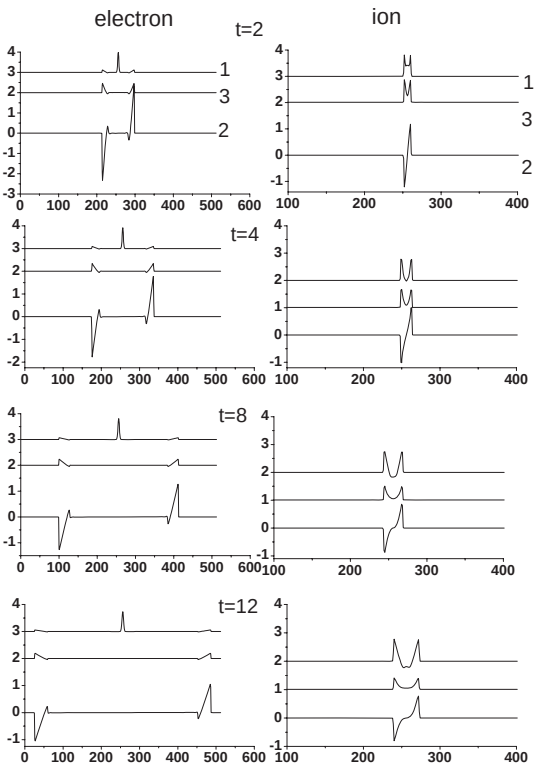

Fig 1 Smooth solution for electron and ion in plasma.
Electron:1~ $_e$+2,2~m_e,3~ $_e$+1.Ion:1~ $_i$+1,2~m_i,3~ $_i$+1
Grid values: x=0.1, t=0.001 CFL=0.01

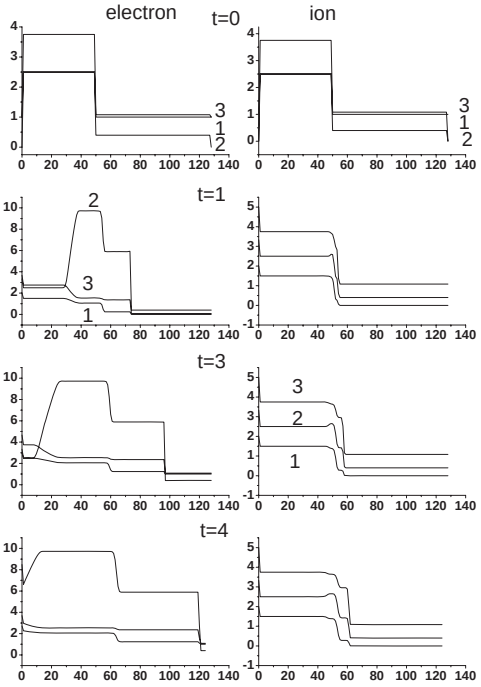

Fig 2 Shock wave solution for electron and ion in plasma.
Electron:1~ $_e$,2~m_e,3~ $_e$.Ion:1~ $_i$-1,2~m_i,3~ $_i$
Grid values: x=0.1, t=0.001 CFL=0.01

A. References

1. A.Kurganov,E.Tadmor. New high-resolution central schemes for non-linearConservation laws and convection-diffusion equations. J. Comput. Phys. 160 (2000) 1461—1488.
2. Baboolal,S. and Naidoo,R. 2003. Local Boundary Conditions with High Resolution Numerical Scheme for Non-Oscillatory Shock Absorption and Reflection. J. Comput and Applied Mechanics, Vol. 4., No., (2003), pp97-104.
3. F.Chen, Introduction to Plasma Physics, Plenum Press, New York, London, 1974.
4. Hirsch,C. 1997. Numerical Computation for Internal and External Flows. Vol 2 Computational Methods for Inviscid and Viscous Flows. John Wiley and Sons (New York)
5. LeVeque, R. 1992. Numerical Methods for Conservation Laws. Birkhauser Verlag (Berlin)
6. Nessyahu and Tadmor. Non-oscillatory central differencing for hyperbolic conservation laws. J.Comput.Phys. 87 (1990)
7. R.Naidoo and S.Baboolal. Numerical integration of the plasma fluid equations with a modification of the second-order Nessyahu-Tadmor central scheme and soliton modelling. Mathematics and Computers in Simulation 69 (2005) 457-466.
8. R.Naidoo and S.Baboolal. Application of the Kurganov-Levy semi-discrete numerical scheme to hyperbolic problems with non-linear source terms. Future Generation Computer Systems 20 (2004) 465-573.