Quick sampling method for cubic Bezier curves by chordal error

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Abstract**—**In this paper, we will present a simple and quick sampling method to digitize the cubic Bezier curves via controlled the chordal error

Keywords**—**sampling method, chordal error, cubic Bezier curve.

I. Introduction

Computers can only display the discrete data and models. However, most of important geometric models in mathematics or CAD are continuous or differentiable. How to demonstrate these geometric models in computer is a difficult and important task. The best idea to solve this problem is to simple the curves and present differentiable curves by poly-line i.e. a set of vertices. For this reason, sampling method becomes the most important problem in computational geometry. How to find a quick algorithm and control the error tolerance are the main issues of this field.

Digitization problem has two forms. The first: fixed the number of points, we want to find a best partition to digitize the original curves such that the maximum distance between original curve and digital models is minimal. In 2003, Victoria presented a good method to improve this kind of problem. Second: we want to find a minimum number of points to digitize the curves when we fix an error of tolerance. There are many different kinds of simplification methods presented in the version of computational geometry, digital geometry and CAD. However, most of these methods are based on the numerical analysis and differential equations. In these methods, the error of tolerances play a more important role than the complexity of times and spaces (memory). But these methods are still very important since they can be preformed in general surfaces.

Bezier curves and Bezier surfaces are significant models in computer aided geometric design. In this field, any general surfaces or curves models are approximated by the piecewise Bezier curves and surfaces. Many of people investigate the properties and applications of Bezier models. In 1999, Oscar presented a digitization of Bezier curves via the analysis of normal vector. Holliday and Farin interpreted the diagonal of a tensor product Bezier volume from the viewpoint of geometric in 1999. These are important properties of Bezier models and we will develop our results from some properties of Bezier curve..

In this paper, we will present a digital method via the subdivision property of Bezier models. The subdivision property of Bezier models is a powerful tool in CAGD. But using this property, we still need a sampling method to determine when we must stop. And we will use the chordal error to improve it.

In general, the shape of Bezier curves with high degree is too complexity, difficult to estimate its chordal error. And people usually use the Bezier curves not greater than three. Hence, we want to develop a sampling method via controlling chordal error on cubic Bezier models, the maximum distance between the digital models and original models are under control. Furthermore, this method only depends on the Bezier control points. Even though our method does not, so far, apply to general surfaces, it is still a very efficient digital method on cubic Bezier curves and surfaces. This sampling method is the simplest and the quickest of these digital methods. But it only uses on cubic Bezier models.

II. Preliminaries

In this paper, we need some properties of Bezier curves. We will introduce these properties in this section: The Bezier formula be presented in 1960. Bezier presented the Bezier curve from the linear interpolation and the Bernstein polynomial:

$$b(t) = \sum_{i=0}^{n} B_i^n(t) b_i \quad t \in [0,1]. \qquad (1)$$

And de Casteljau also presented the Bezier curve form de Casteljau algorithm:

Set $b_i^0 = b_i$, $i = 1,2,\cdots,n$

$$b_i^r = (1-t)b_1^{r-1} + t b_{i+1}^{r-1}, \begin{cases} r = 1,2,\cdots n \\ i = 0,2,\cdots n-r \end{cases}. \qquad (2)$$

We call b_i^r be the DC points.

From the viewpoint of de Casteljau algorithm, we can subdivide the Bezier curve to two Bezier curves easily. After subdividing, the control points of left curve are b_0^r, $r = 0,1,2,\cdots,n$ and the control points of right curve are b_{n-r}^r, $r = 0,1,2,\cdots,n$.

Consider a Bezier curve $b(t)$ with control points b_i in R^3

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and L the straight line with endpoints b_0, b_1. The chordal error of Bezier curve is a real number defined by $CE = \max_{t\in[0,1]} d(b(t), L)$, where $d(b(t), L)$ is the distance from $b(t)$ to the straight line L..

III. The Chordal Error of Cubic Bezier Curves

Chordal error is important in CAD and other applications. When we do not admit the error greater than a scalar number, we must to control the chordal error. Although, this value is difficult to estimate in general Bezier models. We can estimate the chordal error via controlling the volume of model. In fact, the volume of Bezier model supplied an upper bound of chordal error. Moreover, we will obtain a simple and natural formula to estimate the chordal error of quadric and cubic Bezier curve. Now let us estimate the chordal errors of quadric and cubic Bezier curve as follows:

First, we consider the cubic Bezier curve in R^2 (or in a plane). Without lost the general, we assume that the control points of cubic Bezier curves are

$$\begin{cases} b_0 = (0,0) \\ b_1 = (x_1, y_1) \\ b_2 = (x_2, y_2) \\ b_3 = (1,0) \end{cases} \tag{3}$$

From the definition of chordal error of cubic Bezier curve, it is only depend on y_1 and y_2. Hence, the chordal error of $b(t)$ is

$$CE = \frac{1}{9}\left(\frac{(y_2 - 2y_1 + D)(-y_1 + 2y_2 - D)(y_1 + y_2 + D)}{(y_2 - y_1)^2}\right) \tag{4}$$

when $\frac{1}{3}\left(\frac{y_2 - 2y_1 + D}{(y_2 - y_1)}\right) \in (0,1)$

or

$$CE = \frac{1}{9}\left(\frac{(y_2 - 2y_1 - D)(-y_1 + 2y_2 + D)(y_1 + y_2 - D)}{(y_2 - y_1)^2}\right) \tag{5}$$

if $\frac{1}{3}\left(\frac{y_2 - 2y_1 - D}{(y_2 - y_1)}\right) \in (0,1)$,

where $D = \sqrt{{y_1}^2 - y_1 y_2 + {y_2}^2}$.

In spaced cubic Bezier curve, we assume that the control points

$$\begin{cases} b_0 = (0,0,0) \\ b_1 = (x_1, y_1, z_1) \\ b_2 = (x_2, y_2, z_2) \\ b_3 = (1,0,0) \end{cases} \tag{6}$$

we consider this problem in xy-plane and xz-plane. Suppose the chordal error of $b(t)$ in xy-plane is CE_{xy}, CE_{yz} in yz-plane and CE_{xz} in xz-plane. The chordal error of such cubic Bezier curve can be controlled by $\max\{CE_{xy}, CE_{yz}, CE_{xz}\}$ or $\sqrt{{CE_{xy}}^2 + {CE_{yz}}^2 + {CE_{xz}}^2}$.

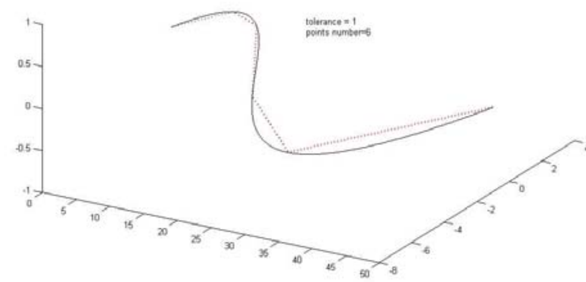

Fig 1. tolerance =1.

IV. Computational Results

In this section, we will test our sampling method for general cubic Bezier curves. From figure 1 to figure 3, we sampling a cubic Bezier curve by different chordal errors, 1, 10^{-1} and 10^{-2}. Next, we sample the random cubic Bezier curve with control points $b_i \in [-2,2]^3 \subset R^3$ and we test 10^4 random cubic Bezier curves for each different chordal errors. The y-axis shows the average of the number of points of digital curve. In figure 4, we show the relation between the number of points and the tolerance of error. From this table, it is easy to observe that our sampling method has linear growth.

Of course, our sampling method can be extended to improve the cubic B-spline curves. In figure 5, we choose a general cubic B-spline curve in R^3 and the results of digitalization with two kinds of chordal errors presented in figure 6 and 7.

Acknowledgment

This work is partially supported by NSC, Taiwan.

References

[1] Antoni Zygmund and Richard L. Wheeden, Measure and integral - an introduction to real analysis. 1977.
[2] Azriel Rosenfeld and Reinhard Klette, Digital geomery, Information Sciences, p123-127, 148(2002).
[3] D.J. Holliday, G.E. Farin, A geometric interpretation of the digonal of a tensor product Bezier volume. CAGD, 16, 1999, p837-p840.
[4] Gravesen, Jens. Adaptive subdivision and the length and energy of Bezier curves. Computational Geometry, vol 8; issue 1, June, 1997, p13-31.
[5] M. Do Carmo, Differential Geometry of Curves and Surfaces. Prentice Hall, 1976.
[6] G. Farin, Curves and Surfaces for Computer aided geometric design, 4nd Edition. Academic Press, 1996.
[7] Gallier, Jean, Curves and Surfaces in Geometic Modeling Theory and Algorithms. Morgan Kaufmann, 2000.
[8] D.J. Holliday, G.E. Farin, A geometric interpretation of the diagonal of a tensor product Bezier volume. CAGD, 16, 1999, p837-p840.

[9] Jianmin Zheng, Thomas W. Sederberg. Gaussian and mean curvatures of rational Bezier patches. CAGD 20(2003) 297-301.

[10] Oscar Figueiredo, Jean-Pierre Reveilles, Roger D. Hersch. Digitization of Bezier curves and patches using discrete geometry, DGCI'99, LNCS 1568, Springer-Verlag, p388-398, 1999.

[11] Hernandez-Mederos, Victoria and Estrada-Sarlabous, Jorge. Sampling points on regular parametric curves with control their distrubtion. CAGD, vol 2-, issue 6, Sep. 2003, p363-p382.

[12] Antoni Zygmund and Richard L. Wheeden, Measure and integral - an introduction to real analysis. 1977.

[13]

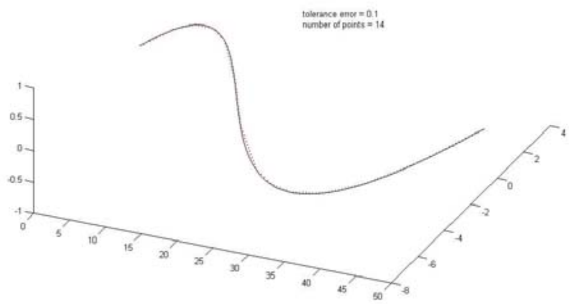

Fig 2. tolerance =0.1

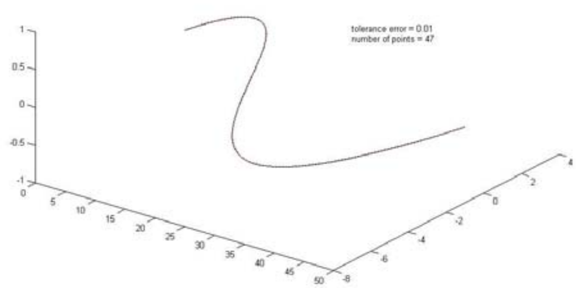

Fig 3. tolerance =0.01

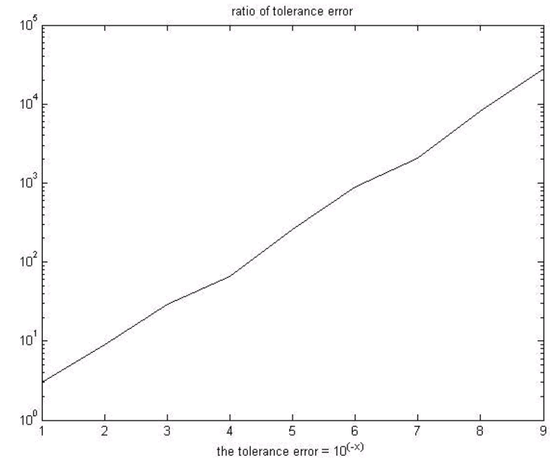

Fig 4. random cubic Bezier curves.

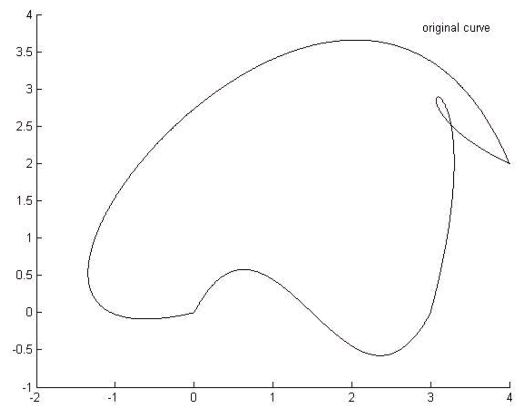

Fig 5. the original cubic B-spliner curve

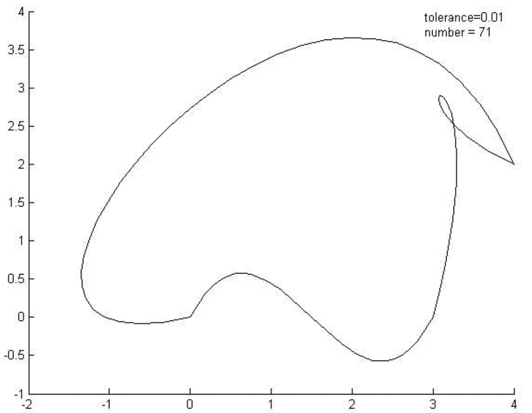

Fig 7. tolerance =0.001

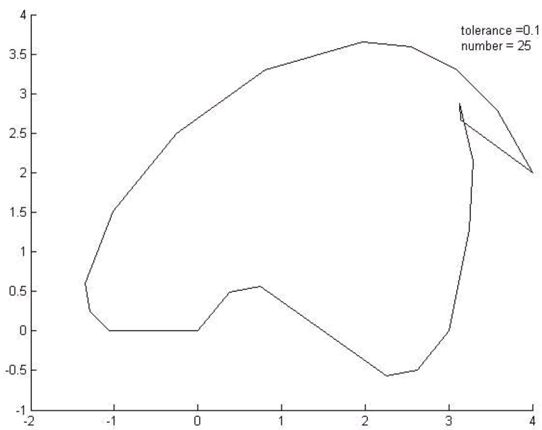

Fig6 tolerance = 0.1