An EMD Based Simulation of Fractional Gaussian Noise

Pei-Wei Shan and Ming Li

***Abstract*—**This paper introduces a system for synthesizing fractional Gaussian noise (FGN) based on two approaches in the platform of MATLAB. One is empirical mode decomposition (EMD) and the other correlation method. The system can be used to simulate FGN and fractional Brownian motion (fBm), respectively. A comparison of theoretical and real autocorrelation of a synthesized realizations of FGN is illustrated between EMD-based simulation and correlation-based one.

***Keywords*—**EMD, Fractional Gaussian noise, Simulation, Correlation

I. INTRODUCTION

MANY natural processes such as weather data, electrical or physiological measurements, and man made phenomena such as traffic flow data exhibit spectrums that have been observed to follow the $1/f$ law. In fact, fractional Brownian motion (fBm) is the only known correlation model that satisfies Wornell's definition of $1/f$ processes [1]. One characterizing feature of fBm is its statistical self-similar property [2], and it absolutely depends on the Hurst index.

In practice, however, it is difficult to generate fractional Gaussian noise (FGN) since FGN is too narrow for being accurately consistent with real traffic data on the Internet in the general sense [3] [4] [5].

Ref. [6] presents a model of synthesizing FGN based on correlation. Ref [7] proposes an algorithm for generating FGN using Empirical Mode Decomposition (EMD). In this paper we introduces a simulation system designed and implemented by us that is based on MATLAB. The system can be used to synthesize FGN and fBm separately based on correlation and EMD with given Hurst index.

We will introduce the structure and functions of our system in Section 2, present the simulation method based on EMD and correlation in Section 3 and in Section 4 respectively. A case of generating fractional processes is shown in Section 5 and conclusions in Section 6.

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II. SYSTEM STRUCTURE AND FUNCTION

A. System Structure

The designed system integrates two newly raised methods: the EMD-based simulation and the correlation-based one. As known, the GUIDE groupware of MATLAB provides strong support on Graphical User Interface and perfect support on users' demand. The system contains seven modules that are white noise simulating, method, distribution option, H value setting, fractional processes generating, sampling length setting and data exporting.

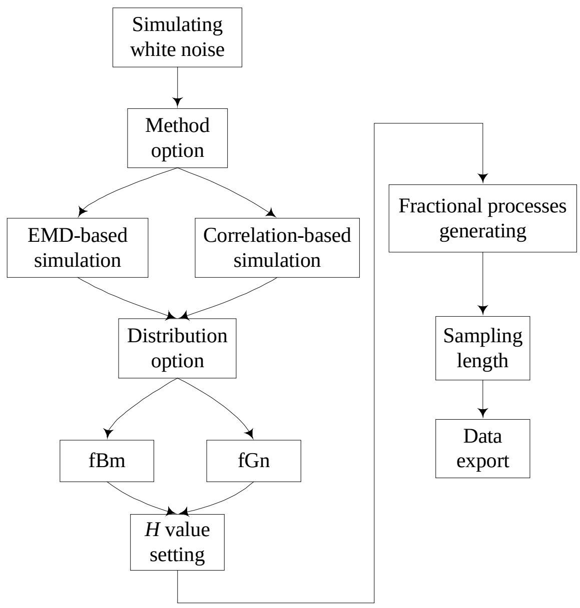


Fig. 1. System structure.

Fig.1. is the flow chart of simulation. Using GUI-based MATLAB, we implement a system interface of the simulation of fractional Gaussian noise. Please refer to Fig.2.

B. System Function

Fig.2 shows the main interface of the simulation system. With white noise simulated, the user has alternative methods of EMD-based and correlation-based. The system can then be used to synthesize FGN and fBm with given H index. Users can set the sampling length via the module of sampling length setting and export generated processes through the module of data exporting. Several formats of files, such as '.txt, .dat, .xls', are supported.

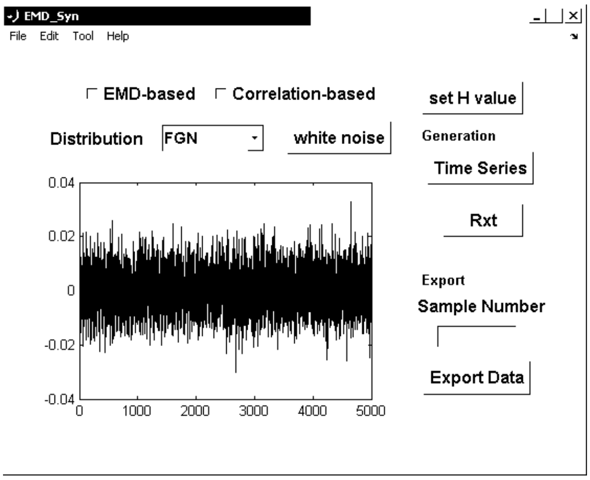


Fig. 2. System interface.

Fig.2. represents a white noise generated by the simulation system. The data length is 5000 points.

III. EMD-BASED SIMULATION OF FRACTIONAL PROCESSES

A. Empirical Mode Decomposition

FGN can be viewed as the increment process of fBm. It is self-similar with index $H\in(0,1)$. Thus, the autocorrelation of FGN in the continuous case is given by

$$r_H[k]=\frac{\sigma^2}{2}[|k+1|^{2H}-2|k|^{2H}+|k-1|^{2H}] \tag{1}$$

White noise increments correspond to the special case of FGN realization for which $H = 1/2$.

For any signal, $s(t)$, EMD ends up with the following representation [8]

$$s(t)=\sum_{k=1}^{K}C^{\langle k\rangle}(t)+r^{\langle K\rangle}(t) \tag{2}$$

where $C^{\langle k\rangle}(t)$ is the kth mode(or Intrinsic Mode Function(IMF)) of the signal, and $r^{\langle K\rangle}(t)$ stands for residual trend.

Fig.3. is an example of decomposition. The signal is a White noise realization of data length $N = 1024$. The white noise has seven IMFs and a residual signal representing a negligible trend.

It is shown in [9] that the IMF variance of FGN is expected to be an exponentially decreasing function of the IMF index. Thus, we have

$$Var\{C_H^{\langle k\rangle}(t)\}\propto\rho_H^{k(2H-2)} \tag{3}$$

where $C_H^{\langle k\rangle}(t)$ is the kth IMF of process $s(t)$, $\rho_H\approx 2$ for all H [9]. Hence, (3) leads to

$$Var\{C_H^{\langle k\rangle}(t)\}\propto 2^{k(2H-2)} \tag{4}$$

Using this property, we can get the EMD-based method for generating fractional processes.

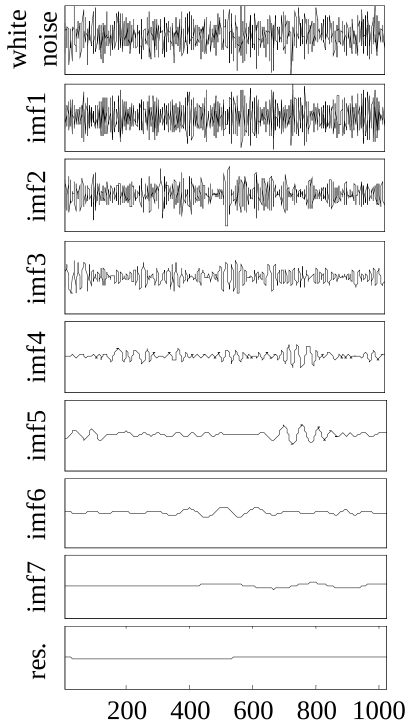


Fig. 3. EMD of a white noise realization ($N = 2^{10}$), white noise denote $\xi(t)$ ($H = 1/2$).

The decomposition of ordinary Gaussian noise (as in Fig.1), $H = 1/2$, by use of EMD ends up with IMFs with variance progression. So we get: $Var\{C_{1/2}^{\langle k\rangle}(t)\}\propto\rho_H^{-k}$ [9]. Then, we can generate fractional Gaussian noise by summing all IMFs with appropriately weighting factors. The method above is called 'H-dependent EMD reconstruction'.

B. H-dependent EMD Reconstruction

Steps of the algorithm,

1. Initialization. Generate ordinary Gaussian noise, $\xi(t)$, H = 1/2.
2. Decomposition. Decompose $\xi(t)$ with EMD method to generate the K IMFs $C_{1/2}^{\langle k\rangle}(t)$, $k = 1, \ldots, K$ of $\xi(t)$.
3. Reconstruction. Weight each IMF of rank k by factor $2^{k(\beta)}/\sigma<k>$, where H is the expected Hurst index, and reconstruct new fractional process $\xi_H(t)$ by the following EMD reconstruction, where $\sigma^{\langle k\rangle}$ stands for standard deviation of the kth IMF:

$$\xi_H(t)=\sum_{k=1}^{K}\frac{2^{k(\beta-1)}}{\sigma^{\langle k\rangle}}C_{1/2}^{\langle k\rangle}(t) \tag{5}$$

with

$$\beta=\begin{cases}H & for\ \ fGn\\ H+1 & for\ \ fBm\end{cases} \tag{6}$$

Parameter β is related with index H for respectively FGN or fBm processes synthesis.

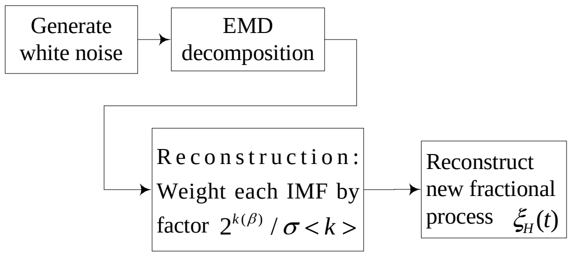


Fig. 4. Flow chart of EMD reconstruction.

IV. Correlation-based Model for Synthesizing Fractional Gaussian Noise

A. Simulating White Noise

Let $w(t)$, $W(\omega)$ and $S_w(\omega)$ be a white-noise function, its spectrum and the power spectrum, respectively. Then,

$$W(\omega) = F[w(t)] = \int_{-\infty}^{\infty} w(t) e^{-j\omega t} dt \tag{7}$$

where F is the operator of Fourier transform.
Let r_w be w's autocorrelation. Then,

$$r_w(\tau) = E[w(t)w(t+\tau)] \tag{8}$$

Thus, we have,

$$S_w(\omega) = W\overline{W} = F[r_w(\tau)] \tag{9}$$

Suppose $w(t)$ is the unit white noise, $S_w(\omega) = 1$. Then,

$$r_w(\tau) = \delta(\tau) \tag{10}$$

Let $h(t)$ and $H(\omega)$ be the impulse function and system function of a linear filter respectively.
Then,

$$y(t) = w(t) * h(t) \tag{11}$$

where * mean the operation of convolution.

Thus, the steps of synthesizing fractional processes by filtering white noise are: simulating white noise first and then synthesizing $h(t)$ based on a given correlation structure.

Let θ be a real random function with arbitrary distribution. Then,

$$W(\omega) = e^{j\theta(\omega)} \tag{12}$$

is a Fourier transform of white noise. Thus, a white-noise function can be given by

$$w(t) = F^{-1}[W(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\theta} e^{j\omega t} d\omega \tag{13}$$

In the discrete case,

$$w(n) = \mathrm{IFFT}[W(\omega)] \tag{14}$$

where IFFT represents the inverse of fast Fourier transform (FFT).

Thus, according to (14), we can accurately simulate a white noise $w(n)$.

B. Generating FGN from White Noise

Simulating x with a desired autocorrelation r_x can be regarded as synthesizing a simulator h [6],

$$r_y = r_x \tag{15}$$

From (11), we can get, $S_y(\omega) = |H(\omega)|^2$. Let $S_y(\omega) = S_x(\omega)$. Then,

$$|H(\omega)| = [S_x(\omega)]^{0.5} \tag{16}$$

Therefore,

$$h = F^{-1}\{[F(r_x)]^{0.5}\} \tag{17}$$

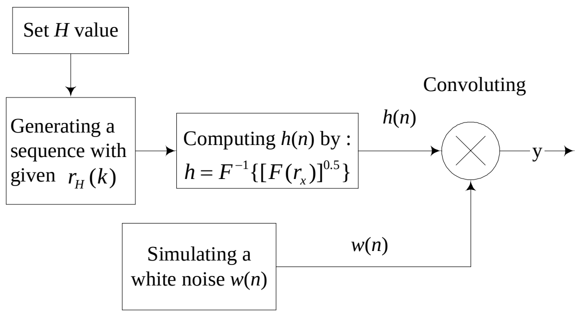


Fig. 5. Flow chart of correlation-based model.

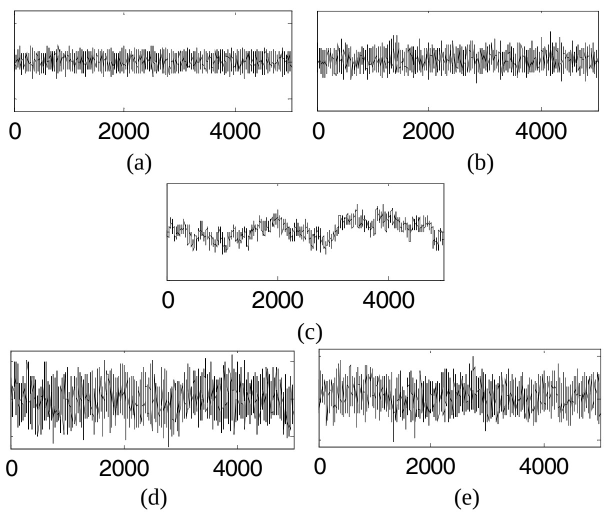
where F^{-1} is the inverse of F. Consequently, the output of simulator is given by

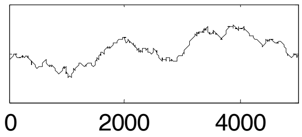
$$y = w * F^{-1}\{[F(r_x)]^{0.5}\} \tag{18}$$

Equation (18) is the solution of synthesizing FGN according to a given correlation structure.

V. A Comparison between the Simulated Fractional Processes

Fig.4 and Fig.5 show the computation procedures of EMD-based simulation and correlation-based computational model, respectively. Therefore, we generate several fractional processes with our designed system that is implemented in the platform of MATLAB.

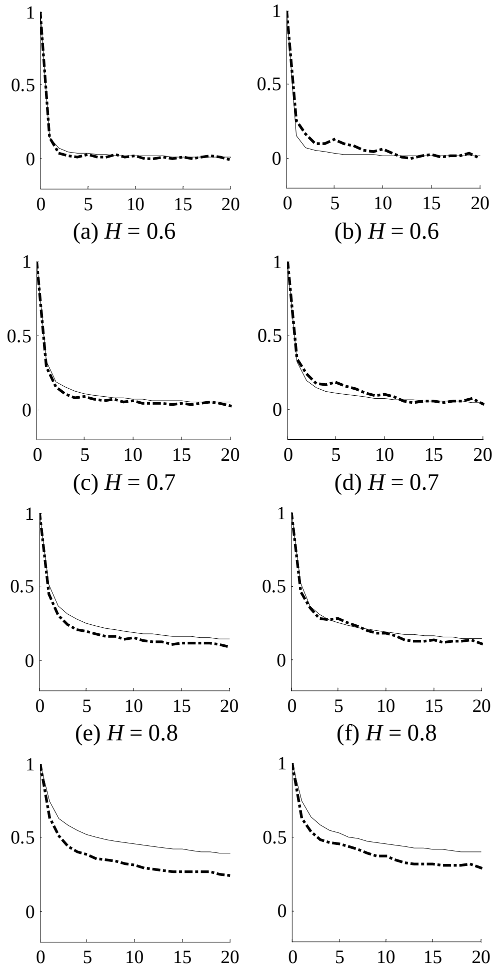




(f)

Fig. 6. Examples of generated fractional processes (5000 data points). (a). FGN simulated by EMD, H = 0.2. (b). FGN simulated by correlation, H = 0.2. (c). fBm simulated by EMD, H = 0.2. (d). FGN simulated by EMD, H = 0.8. (e). FGN simulated by correlation, H = 0.8. (f). fBm simulated by EMD, H = 0.8.

According to (1), Fig.7 represents the comparison of theoretical and real autocorrelation of FGNs with different values of H index $H = \{0.6, 0.7, 0.8, 0.9\}$. These FGNs are respectively generated by EMD-based simulation and correlation-based computation model.



(a) H = 0.6 (b) H = 0.6

(c) H = 0.7 (d) H = 0.7

(e) H = 0.8 (f) H = 0.8

(g) H=0.9 (h) H=0.9

Fig. 7. Examples of theoretical and real autocorrelation of FGNs generated by EMD-based simulation and correlation-based computation model. (a). The autocorrelation of FGNs simulated by EMD, H = 0.6. (b). The autocorrelation of FGNs simulated by correlation, H = 0.6. (c). The autocorrelation of FGNs simulated by EMD, H = 0.7. (d). The autocorrelation of FGNs simulated by correlation, H = 0.7. (e). The autocorrelation of FGNs simulated by EMD, H = 0.8. (f) The autocorrelation of FGNs simulated by correlation, H = 0.8. (g). The autocorrelation of FGNs simulated by EMD, H = 0.9. (h). The autocorrelation of FGNs simulated by correlation, H = 0.9.

In Fig.7, the solid lines imply the theoretical autocorrelations and dotted lines the autocorrelations of FGNs obtained by EMD and correlation, respectively.

VI. Conclusion

We have addressed a simulation system of FGN in the platform of MATLAB. The methods based on EMD and a given function of correlation have been interpreted. The simulation results have been demonstrated. The system can be used to simulate FGN and fBm with the selection by users. The present system may be a promising tool for FGN simulation.

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