Local Times of Processes Driven by Factional Brownian Motion

YU SUN, LIANG ZHOU, CHANGCHUN GAO

Abstract—Considing the processes associated with fractional Bessel processes driven by factional Brownian Motion with Hurst parameter 0<*H*<1, we study the properties and show the local times exist and get Tanaka formula of the processes as well as the local time. For 1-dimensional linear self-attracting diffusion process we study the convergence and local time.

Key-Word—fractional Brownian motion, fractional Bessel processes, local time, Tanaka formula

I. Introduction

FIRST we conside factional Bownian motion (fBm).
Definition 1 (fBm) Let :
$H \in (0,1)$ be a constant. The (1-parameter) fractional Brownian motion (fBm) with Hurst parameter H is the Gaussian process

$$B_H(t) = B_H(t,\omega), t \in R, \omega \in \Omega \quad ,$$

Satisfying

$$B_H(0) = E[B_H(t)] = 0 \, ,$$

for all $t \in R$, and

$$E[B_H(s)B_H(t)] = \frac{1}{2}\{|s|^{2H} + |t|^{2H} - |s-t|^{2H}\}; s,t \in R$$

Where E denotes the expectation with respect to the probability law P for

$$\{B_H(t,\omega); t \in R, \omega \in \Omega\}$$

where (Ω, F) is a measurable space.

If H = 1/2 then $B_H(t)$ coincides with the classical Brownian motion, denoted by B(t).

If H > 1/2 then $B_H(t)$ has long range dependence , in the sense that

$$\rho_n = E[B_H(1)\cdot B_H(n+1) - B_H(n)] > 0$$

for all n = 1, 2, . . . , and

$$\sum_{n=1}^{\infty} \rho_n = \infty$$

If H < 1/2 then $B_H(t)$ is anti-persistent, in the sense that $\rho_n < 0$ for all n = 1, 2, . . .

in this case $\sum_{n=1}^{\infty} \rho_n < \infty$ (Shiryaev [5], p. 233)

Another important property of fBm is self-similarity: For any $H \in (0,1)$ and $\alpha > 0$ the law of $\{B_H(\alpha t)\}_{t\in R}$ is the same as the law of $\{\alpha^h B_H(t)\}_{t \in R}$.

Next we will give the definition of Bessel processes and Fractional Brownian Motion.

For every δ ≥ 0 and x ≥ 0, the solution to the equation

$$X_t = x + \delta t + 2\int_0^t \sqrt{|X_s|}\, dW_s$$

is unique and strong. In the case δ =0, x = 0, the solution X_t is identically zero and applying the comparison theorem (see Revuz–Yor [11] Theorem IX.(3.7)) we conclude $X_t \geq 0$ for all $\delta \geq 0$.

Definition 1.1 ($BESQ^\sigma$) For every $\delta \geq 0$ and x ≥ 0 the unique strong solution to the equation

$$X_t = x + \delta t + 2\int_0^t \sqrt{|X_s|}\, dW_s$$

is called the square of a δ-dimensional Bessel process started at x and is denoted by $BESQ^\sigma$.

Remark: the law of $BESQ^\sigma(x)$ on C(R_+, R)by Q_x^σ . We call the numberδ the dimension of BESQ . This notation arises from the fact that a $BESQ^\sigma$ process X_t can be represented by the square of the Euclidean norm of δ-dimensional Brownian motion

B_t : $X_t = | |B_t|^2 |$.

The number $\nu \equiv \delta/2 - 1$ is called the index of the process $BESQ^\sigma$.

Definition 1.2 (BES^σ) The square root of $BESQ^\sigma(a^2)$, $\delta \geq 0$, $a \geq 0$ is called the Bessel process of dimension δ started at a and is denoted by $BES^\sigma(a)$.

Remark: the law of $BES^\sigma(a)$ by P_a^σ

In the case δ ≥ 2, $BES^\sigma(a)$, a > 0, will never reach 0.

For δ > 1 a $BES^\sigma(a)$ process Z_t satisfies

$$E[\int_0^t (ds/Z_s)] < \infty$$

and is the solution to the equation

$$Z_t = a + \frac{\delta - 1}{2}\int_0^t \frac{ds}{Z_s} + W_t$$

For δ ≤ 1 the situation is less simple. For δ = 1 we have with $It\hat{o}$ Tanaka's formula

$$Z_t = |W_t| = \tilde{W}_t + L_t$$

 Manuscript received April 3, 2007; Revised received Nov. 30, 2007

where

$$\widetilde{W}_t \equiv \int_0^t sign(W_s) dW_s$$

is a standard Brownian motion, and Lt is the local time of Brownian motion. Refer to Revuz–Yor [11] and Pitman–Yor [9, 10] for the more study of Bessel processes.

Definition 1.3 Denote the fractional Bessel process by

$$R_H = \sqrt{B_H(1)^2 + B_H(2)^2 + ... + B_H(d)^2}$$

where

$$B_H = (B_H(1), B_H(2), ..., B_H(d))$$

be a d-dimensional fractional Brownian motion with Hurst parameter H $\in$(0, 1).

We hope to obtain a stochastic calculus for fBm and to use its properties into application.

However, if $H \neq 1/2$ then $B_H(t)$ is not a semimartingale, so we cannot use the general theory of stochastic calculus for semimartingales on $B_H(t)$.

For example, as $H \neq 1/2$ the fractional Brownian motion $B_H(t)$ has not $L\hat{e}vy$ type characteristic, i.e., the process (see Hu [7])

$$X_H = \int_0^t sign(B_H(s)) dB_H(s), \frac{1}{2} < H < 1 \quad (1)$$

is not a fBm. Furthermore, the process

$$Y_H(t) = \sum_{j=1}^{d} \int_0^t \frac{B_H^j}{R_H(s)} dB_H^j(s) \quad (2)$$

is the fractional Bessel process. Thus, it is interesting to investigate the properties of these processes. Hu and Nualart obtained some properties of these processes in [7].

The purpose of this paper is to prove the local times of these processes based on $B_H(t)$ exist,

1/2 < H < 1. Moreover, we give a Tanaka formula of the process X_H given by (1) and (2).

For 1-dimensional linear self-attracting diffusion process we study the convergence and local time.

Consider the path dependent stochastic differential equation of the form

$$X_t^H = B_t^H + \int_0^t \int_0^s \Phi(X_s^H - X_u^H) du ds \quad (1)$$

where B^H is a d-dimensional fractional Brownian motion with Hurst index $H \in (0,1)$ and Φ Lipschitz continuous. Then it is not difficult to show that the above equation admits a unique strong solution. We will call the solution the fractional self-attracting diffusion driven by fBm. We will consider only a particular case as follows,the linear fractional self-attracting diffusion:

$$X_t^H = B_t^H - a\int_0^t \int_0^s (X_s^H - X_u^H) du ds + \upsilon t \quad (2)$$

with $a > 0$ and $\upsilon \in \mathbb{R}^d$. Our aims are to study the convergence and local times of the processes given by above formula with $d = 1$.

II. Fractional $It\hat{o}$ type stochastic integral

For 1/2 < H < 1, an alternative integration theory based on the Wick product ◇ was introduced by [3], as follows:

$$\int_0^t u(s) dB_H(s) := \lim_{|\pi_n| \to 0} \sum_k u(t_k) \diamond (B_H(t_{k+1}) - B_H(t_k))$$

Where

$$\pi_n : 0 \le t_0 \le t_1 \le ... \le t_n = t$$

is an arbitrary partition of [0, t],

$$\pi_n := \max_k \{t_{k+1} - t_k\}$$

and $\lim |x_n| \to 0$ means the limit in $L^2(\mu)$.

The definition of the integrals has been extended by [8] (see also [1]) to all 0 < H < 1 as follows:

$$\int_0^t u(s) dB_H(s) := \int_0^t u(s) \diamond W_{(H)}(s) ds$$

where

$$W_{(H)}(t) = \frac{dB_H(t)}{dt} \in (S)^*$$

with $(S)^*$ the Hida space of stochastic distributions if u : $R_+ \to (S)^*$ satisfies that $\mu(t) <> W^H(t)$ is dt-integrable in $(S)^*$. These fractional $It\hat{o}$ integrals have many properties of the classical $It\hat{o}$ integral.

Definition 2.1 Let $F : \Omega \to R$ and choose $\gamma \in \Omega$. Then we say F has a directional M-derivative in the direction γ if :

$$D_\gamma^{(H)} F(\omega) := \lim_{\xi \to 0} \frac{1}{\xi} [F(\omega + \xi M\gamma) - F(\omega)]$$

Exists almost surely in $(S)^*$. In that case we call

$$D_\gamma^{(H)} F(\omega)$$

the directional M-derivative of F in the direction γ.

Definition 2.2 We say that $F : \Omega \to R$ is differentiable if there exists a function:

$$\Psi : R \to (S)^*$$

Such that

$$D_\gamma^{(H)} F(\omega) = \int_R M\Psi(t) M\gamma(t) dt$$

for all

$$\gamma \in L_H^2(R)$$

Then we write

$$D_t^{(H)} F := \frac{\partial(H)}{\partial \omega} F(t, \omega) = \Psi(t)$$

And we call $D_t^{(H)}F$ the Malliavin derivative or the stochastic gradient of F at t. In the classical case ($H=\frac{1}{2}$) we use the notation D_t for the corresponding Malliavin derivative.

Proposition 2.3 Let $F \in (S)^*$.Then

$$D_t F = M D_t^{(H)} F$$

for $a.a.t \in R$

Proposition 2.4 Suppose:

$$Y: R \to (S)^*$$

is $dB^{(H)}$-integrable. Then

$$D_t^{(H)}(\int_R Y(s)dB^{(H)}(s)) = \int_R D_t^{(H)} Y(s) dB^{(H)}(s) + Y(t)$$

Proposition 2.5 Let $D_{1,2}^{(H)}$ be the set of all $F \in L^2(\mu)$ such that the Malliavin derivative $D_t^{(H)}F$ exists and

$$E[\int_R [D_t^{(H)}F]^2 dt] < \infty$$

The following result has been obtained with a different proof in Lemma 2 of [M]

Proposition 2.6 Suppose:

$g \in L_H^2(R)$ is deterministic and let $F \in D_{1,2}^{(H)}$.

Then

Consi

$$F o \int_R g(t) dB^{(H)}(t) =$$

$$F \cdot \int_R g(t) dB^{(H)}(t) - \langle g, D^{(H)} \cdot F \rangle$$

Recall that the Malliavin Φ-derivative of the function U : $\Omega \to R$ defined in [3] as follows:

$$D_s^\phi U = \int_0^\infty \phi(r,s) D_r U dr$$

where $D_r U$ is the fractional Malliavin derivative at r. Define the space $L_\phi^{1,2}$ to be the set of measurable processes u such that $D_s^\phi u(s)$ exists for a.a. $s \geq 0$ and

$$E[(\int_0^\infty D_s^\phi u(s) ds)^2 + \int_0^\infty \int_0^\infty u(s_1) u(s_2) \phi(s_1, s_2) ds_1 ds_2] < \infty$$

Then the integral $\int_0^\infty u(s) dB_H(s)$ can be well defined as an element of $L^2(\mu)$

Theorem 2.7 ([3]). Let $\{u(t), t \geq 0\}$ be a stochastic process in $L_\phi^{1,2}$. Then for the process

$$\eta(t) = \int_0^\infty u(s) dB_H(s), t \geq 0$$

we have

$$D_s^\phi \eta(t) = \int_0^t u(r) dB_H(r) + \int_0^t u_r \phi(s,r) dr$$

In particular, if u is deterministic, then

$$D_s^\phi \eta(t) = \int_0^t u(r) \phi(s,r) dr$$

Theorem 2.8 ([3]). Let $F \in C^{1,2}$ ($R_+ \times R$) with bounded second order derivatives and let the process X be given as follows:

$$X(t) = x + \int_0^t v(s) ds + \int_0^t u(s) dB^H(s),$$

$$t \geq 0, x \in R$$

With $u \in L_\varnothing^{1,2}$. Then we have

$$F(X(t)) = F(x) + \int_0^t \frac{d}{dx} F(X(s)) dX(S) + \int_0^t \frac{d^2}{dx^2} F(X(s)) u(s) D_s^\varnothing X(s) ds$$

for all $t \geq 0$.

III. Local Time and Tanaka Formula for processes associated with fractional Bessel processes

Refer to [9], the weighted local time $L(B_H)$ of fractional Brownian motion are established:

$$L(B_H) = 2H \int_0^t \delta(B_H(s) - x) s^{2H-1} ds$$

The Tanaka formula is given as:

$$(B_H(t) - x)^+ = x^+ + \int_0^t 1_{\{B_H(s) > x\}} dB_H(s) + \frac{1}{2} L_t^x (B_H)$$

$$| B_H(t) - x | = | x | + \int_0^t sign(B_H(s)) dB_H(s) + L_t^x (B_H)$$

In this section we show that the local times of the process

$$X_H = \int_0^t sign(B_H(s)) dB_H(s), t \geq 0$$

$$Y_t^H = \sum_{j=1}^d \int_0^t \frac{B_s^H}{R_s^H} dB_s^H(j)$$

exist and obtain their Tanaka formula. We will also find a relationship between the weighted local time of fractioanal Brownian motion and the local time of the process X_H for d=1.

First we conside some properties of process X_H

Propostion 3.1. The process X={Xt , t ≥0} is H-self-similar.

Propostion 3.2. For any $0 < H < 1$

$$\int_0^t sign(B_s)dB_s = \sum_{k=1}^{\infty} c_k I_{2k}(h_{2k})$$

Where

$$c_k = \frac{(-1)^{k-1}}{\sqrt{2\pi}(2k-1)(k-1)!2^{k-2}}$$

$$h_{2k(s1,\ldots\ldots,s2k)} = (s1 \vee s2 \vee \cdots \vee s2k)^{-(2k-1)H}$$

A consequence of this proposition is the following

Propostion 3.3. For any 0<H<1, the random variable sign(BH) belongs to the Sobolev space Dα,2
For any α<1/2.

Lemma 3.4. (Hu [7])

$$E[sign(B_H(s))sign(B_H(u))]$$

$$= \sum_{k=0}^{\infty} \frac{4(2k)!(s^{2H} + u^{2H} - |s-u|^{2k+1})}{(2k+1)^2 2\pi (k!2^k)^2 (su)^{(2k+1)}}, t \geq 0$$

We can get the proof of this Lemma in [7]. By using this Lemma its easy to show the following result holds

Lemma 3.5. Let $\frac{1}{2} < H < 1$, then

$$sign(B_H(t))D_H X_H(t) \geq 0, a.s$$

for all $t \geq 0$.

Theorem 3.6. Let $\Phi : R^+ \to R$ be a convex function having polynomial growth and let
the process X_H be defined by

$$X_H(t) = \int_0^t sign(B_H(s))dB_H(s), t \geq 0$$

Then there exists a continuous increasing process A^Φ such that:

$$\Phi(X_H(t)) = \Phi(0) +$$

$$\int_0^t D^-\Phi(X_H(s))sign(B_H(s))dB_H(s) + \frac{1}{2}A_t^\Phi, t \geq 0$$

where $D^-\Phi$ denotes the left-hand derivative of Φ.

Proof: If $\Phi \in C^2$, then this is the $It\hat{o}$ formula and

$$A_t^\Phi = \int_0^t \Phi''(X_s)sign(B_H(s))D_H X_H(s)ds$$

and Lemma 3.1 implies that the process A^Φ is increasing.

Let now $\Phi \notin C^2$. For $\varepsilon > 0$ and $x \in R$ we set

$$p_\varepsilon(\varepsilon) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{1}{2\varepsilon}x^2}$$

and

$$\Phi_\varepsilon(x) = \int_R p_\varepsilon(x-y)\Phi(y)dy, (\varepsilon > 0)$$

Then $\Phi_\tau(x)$ has polynomial growth and $\Phi_\tau \in C^2$. It follows that for all $\varepsilon > 0$ there exists a continuous increasing process A^Φ such that

$$\Phi_\varepsilon(X_H(t)) = \Phi_\varepsilon(0) +$$

$$\int_0^t \Phi'_\varepsilon(X(s))sign(B_H(s))dB_H(s) + \frac{1}{2}A_t^{\Phi_\varepsilon}$$

and

$$A_t^{\Phi_\varepsilon} = \int_0^t \Phi_\varepsilon''(X_H(s))sign(B_H(s))D_H X_H(s)ds$$

$$= \int_R \Phi_\varepsilon''(x)$$

$$(\int_0^t \delta(X_H(s)-x)(sign(B_H(s))D_H X_H(s)ds)dx$$

Noting that for all $x \in R$

$$\lim_{\varepsilon \downarrow 0} \Phi_\varepsilon(x) = \Phi(x)$$

$$\lim_{\varepsilon \downarrow 0} \Phi'_\varepsilon(x) = D^-\Phi(x)$$

So as $\varepsilon \to 0$

$$\int_0^t \Phi'_\varepsilon(X_H(s))sign(B_H(s))dB_H(s)$$

$$\to \int_0^t D^-\Phi(X_H(s))sign(B_H(s))dB_H(s)$$

in probability. As a result, $A_t^{\Phi t}$ converges also to a process A^Φ which, as a limit of increasing processes, is itself an increasing process and

$$\Phi(X_H(t)) = \Phi(0) +$$

$$\int_0^t D^-\Phi(X_H(s))sign(B_H(s))dB_H(s) + \frac{1}{2}A_t^{\Phi_\varepsilon}$$

The process A^Φ can now obviously be chosen to be a.s. continuous. This completes the proof.

Corollary 3.7. For any real number x, there exists an increasing continuous process $L^x(X_H)$ called the local time of the process X_H in x such that,

$$|X_H(t) - x|$$

$$= |x| + \int_0^t sign(X_H(s)-x)dX_H(s) + L_t^x(X^H)$$

Combining this corollary with [3, 9], we get the following

Corollary 3.8. Let $L(X)$ denote the local time of the process X and let

$$L_t^x(B_H) = 2H\int_0^t \delta(B_H(s)-x)s^{2H-1}ds$$

be the weighted local time of fractional Brownian motion . Then we have

$$L_t^x(X_H) - L_t^x(B_H)$$
$$= |X_H(t) - x| - |B_H(t) - x|$$
$$+ 2\int_0^t 1_{\{X_s \le x\}} sign(B_H(s) - x) dB_H(s)$$

Corollary 3.9. For any real number x and $t \ge 0$, we have

$$L_t^x(X_H)$$
$$= \int_0^t \delta(X_H(s) - x) sign(B_H(s)) D_H(s) X_H(s) ds.$$

Moreover, for any convex function having polynomial growth $\Phi : R^+ \to R$ the following Ito-Tanaka type formula holds:

$$\Phi(X_H(t))$$
$$= \Phi(0) + \int_0^t D^- \Phi(X_H(s)) sign(B_H(s)) dB_H(s)$$
$$+ \frac{1}{2}\int_R L_t^x(X_H) \mu_\Phi(dx)$$

where $D^-\Phi$ denotes the left derivative of Φ and the signed measure μ_Φ is defined by

$$\mu_\Phi([a,b]) = D^-\Phi(b) - D^-\Phi(a), a < b, a, b \in R$$

So we have got the relationship between local time and weight local time.

Finally, by the same method on can show that the local time of the process

$$Y_H(t) = \sum_{j=1}^d \int_0^t \frac{B_H^j}{R_H(s)} dB_H^j(s)$$

holds, where

$$B_H = (B_H(1), B_H(2), ..., B_H(d))$$

is a $d(\ge 2)$ dimensional fractional Brownian motion with Hurst index 1/2 < H < 1 and

$$R_H = \sqrt{B_H(1)^2 + B_H(2)^2 + ... + B_H(d)^2}$$ is the fractional Bessel process.

IV. Convergence and Local time for linear self-attracting diffusion process

We consider convergence of the solution of the equation (2), the socall the linear fractional self-attracting diffusion. The method used here is essentially due to M. Cranston and Y. Le Jan [16].

Proposition 4.1 The solution to the equation (2) can be expresses as

$$E[L_t^x] = \frac{1}{\sqrt{2\pi}} \int_0^t \frac{1}{\sigma_s} e^{-\frac{1}{2}x^2\sigma_x^{-2}ds}, \qquad t \ge 0$$

Where

$$E[\ell_T^x] = \frac{H(2H-1)}{\sqrt{2\pi}} \int_0^t \frac{1}{\sigma_s} e^{-\frac{1}{2}x^2\sigma_s^{-2}\tilde{h}(s)ds}, \quad t \ge 0$$

for $s, t \ge 0$.

This proposition can also be obtained by the same method as Cranston and Le Jan [16].

In this section, we study the usual local time and weighted local time of the process and obtain the Meyer-Tanaka type formula of the weighted local time. We consider the linear fractional self-attracting diffusion

X^H $\{X_t^H, 0 \le t \le T\}$

with $\nu = 0$. It follows that the linear fractional self-attracting diffusion is a centered Gaussian process.

For $T \ge t \ge s \ge 0$, we put

$$\sigma_t^2 = E[(X_t^H)^2]$$
$$\sigma_{t,s}^2 = E[(X_t^H - X_s^H)^2]$$

Then

$$\sigma_t^2 = \int_0^t \int_0^t h(t,u) h(t,v) \phi(u,v) du dv, \quad 0 \le t \le T$$

And

$$\sigma_{t,s}^2 = \int_0^t \int_0^t [h(t,u) - h(s,u)][h(t,v) - h(s,v)] \phi(u,v) du dv,$$
$$0 \le s \le t \le T$$

Noting that:

$$\int_0^t \int_0^t \phi(u,v) du dv = t^{2H} and e^{\frac{a}{2}(t^2 - s^2)} \le h(t,s) \le 1$$

for all $t \ge s \ge 0$ we get

$$e^{\frac{a}{2}t^2} t^{2H} \le \sigma_t^2 = \int_0^t \int_0^t h(t,u) h(t,v) \phi(u,v) du dv \le t^{2H}.$$

Lemma 4.2 For all $t \ge s \ge 0$ we have

$$c_T (t-s)^{2H} \le \sigma_{t,s}^2 \le (1 + C_T)(t-s)^{2H},$$

for some constants C_T, $C_t > 0$ depending on T.

From the lemma above, we see that

$$\int_0^T \int_0^T E[(X_t^H - X_s^H)^2]^{-1/2} ds dt < \infty$$

holds for all $T \ge 0$, and furthermore, we can show that the process is local nondeterminism for every $0 \le T \le \infty$, i.e

$$Var(\sum_{j=2}^n u_j (X_{t_j}^H - X_{t_{j-1}}^H)) \ge k_0 \sum_{j=2}^n u_j^2 \sigma_{t_j, t_{j-1}}^2$$

with a constant $k_0 > 0$. Combining this with Berman, we obtain :

Proposition 4.3 If $\nu = 0$, then the solution X^H of the equation (2) has continuous local time such that

$$L_t^x = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{[x-\varepsilon, x+\varepsilon]}(X_x^H) ds = \int_0^t \delta(X_s^H - x) ds,$$

Where
$\delta(X_s^H - \cdot)$
denotes the delta function of X_s^H.

For $t \ge 0$, $x \in R$ we now set

$$\ell_t^x = 2H(2H-1)$$
$$\int_0^t \delta(X_s^H - x)ds \int_0^s h(s,m)(s-m)^{2H-2}dm.$$

Then l_t^x is well-defined and

The process $(l_t^x)_{t\ge 0}$ is called the weighted local time of X^H at $x \in R$

Lemma 4.4(Hu[27].) let Y be normally distributed with mean zero and variance $\sigma^2(\sigma > 0)$ Then the data function
$\delta(Y - \cdot)$
of Y exists uniquely and we have

$$\delta(Y-x) = \frac{1}{2\pi}\int_R e^{i\xi(Y-x)}d\xi, \qquad x \in R$$

As a consequence, we have

$$E[L_t^x] = \frac{1}{\sqrt{2\pi}}\int_0^t \frac{1}{\sigma_s} e^{-\frac{1}{2}x^2\sigma_x^{-2}ds}, \qquad t \ge 0$$

$$E[\ell_T^x] = \frac{H(2H-1)}{\sqrt{2\pi}}\int_0^t \frac{1}{\sigma_s} e^{-\frac{1}{2}x^2\sigma_s^{-2}\tilde{h}(s)ds}, \quad t \ge 0$$

Where and

$$\tilde{h}(s) = \int_0^s h(s,m)(s-m)^{2H-2}dm.$$

Proposition 4.5 Assume that $T \ge 0$ is given. Then ℓ_T^x and L_T^x are square integrable for all $x \in R$ and we have

$$E[(L_T^x)^2] \le \frac{C_H}{k\pi c_T} T^{2-2H}$$

$$E[(\ell_T^x)^2] \le \frac{C_H}{k\pi c_T} T^{2H}$$

Theorem 4.6 Let X^H be the solution to the equation(2)with Hurst index

$$\frac{1}{2} < H < 1,$$
$$X_0^H = z, v = 0$$

and let ℓ be the weighted local time of X^H. Suppose that $\Phi : R^+ \to R$ is a convex function having polynomial growth. Then:

$$\Phi(X_t^H) = \Phi(z) + \int_0^t D^-\Phi(X_s^H)dX_s^H + \int_R \ell_t^x \mu\Phi(dx),$$

Where $D^-\Phi$ denotes the left derivative of Φ and the signed measure μ_Φ is defined by

$$\mu_\Phi([a,b]) = D^-\Phi(b) - D^-\Phi(a), \qquad a < b, a, b \in R.$$

Proof. For $\varepsilon > 0$ and $x \in R$ we set

$$\Phi_\varepsilon(x) = \int_R p_\varepsilon(x-y)\Phi(y)dy \quad (\varepsilon > 0),$$

Where Then
$\Phi_\varepsilon \in C^2$
and we have
for all $x \in R$
It follows that for all $\varepsilon > 0$

$$\Phi_\varepsilon(X_t^H) = \Phi_\varepsilon(z) +$$
$$\int_0^t \Phi'_\varepsilon(X_s^H)dX_s^H + 2H(2H-1)\int_0^t \Phi''_\varepsilon(X_s^H)\tilde{h}(s)ds$$

On the other hand, it is easy to see that $\Phi_\varepsilon(X_t^H)$ converges to $\Phi(X_t^H)$ almost surely, and

$$\int_0^t \Phi'_\varepsilon(X_s^H)X_s^H ds \to \int_0^t D^-\Phi(X_s^H)X_s^H ds \ \ a.s.,$$

And furthermore

$$\int_0^t \Phi'_\varepsilon(X_s^H)dB_s^H \to \int_0^t D^-\Phi(X_s^H)dB_s^H$$

in $(S)^*$

Finally, we have as $\varepsilon \to 0$

$$\int_0^t \Phi''_\varepsilon(X_s^H)\tilde{h}(s)ds =$$
$$\int_0^t ds\tilde{h}(s)\int_R \Phi''_\varepsilon(x)\delta(X_s^H - x)dx$$
$$\to \frac{1}{2H(2H-1)}\int_R \ell_t^x \mu_\Phi(dx)$$

This completes the proof.

Corollary 4.7 Let X^H be the solution to the equation (2) with Hurst index

$$\frac{1}{2} < H < 1, \quad X_0^H = z, v = 0$$

and let l be the weighted local time of X^H. Then The Tanaka formula

$$|X_t^H - x| = |X_0^H - x| + \int_0^t sign(X_s^H - x)dX_s^H + \ell_t^x$$

Holds for all $x \in R.$

V. Conclusion

It can be seen from the above-mentioned analysis that the processes associated with fractional Bessel processes

$$X_H(t) = \int_0^t sign(B_H(s))dB_H(t), \frac{1}{2} < H < 1$$

$$Y_H(t) = \sum_{j=1}^{d} \int_0^t \frac{B_H^j}{R_H(s)} dB_H^j(s)$$

where

$$B_H = (B_H(1), B_H(2), ..., B_H(d))$$

converge, have the local times $L^x(X_H)$ and Ito-Tanaka type formula.

For 1-dimensional linear self-attracting diffusion process

$$X_t^H = B_t^H - a\int_0^t \int_0^s (X_s^H - X_u^H) duds + \upsilon t$$

We study the convergence and obtain the weight local time as showed above.

$$\Phi(X_H(t))$$

$$= \Phi(0) + \int_0^t D^- \Phi(X_H(s)) sign(B_H(s)) dB_H(s)$$

$$+ \frac{1}{2}\int_R L_t^x(X_H)\mu_\Phi(dx)$$

holds.

Acknowledgment

We thanks her advisor Changchun Gao and Litanyan, for their encouragment and guidance for the research. We are grateful to the reviewers and the editors for valuable and detailed comments and suggestoins.

Yu Sun deeply appreciate her partents for their enduring patience and care and is grateful to all her friends for encouraging and kind care: Liang Zhou, Guoxin Wu.

References

[1] Bender, C., An Itˆo formula for generalized functionals of a fractional Brownian motion with arbitrary Hurst parameter. Stochastic process. Appl, Vol.104, 2003, pp. 81-106.

[2] Hu, Y., Øksendal, B., Salopek, D.M., Weighted local time for fractional Brownian motion and applications to finance, Stochastic Anal, Appl, Vol.23, 2005, pp. 15-30.

[3] Duncan, T.E.,Hu, Y., Duncan, B.P., Stochastic calculus for fractional Brownian motion, I Theory. SIAM J. Control Optim, Vol.38, 2000, pp. 582-612.

[4] Nualart, D., Stochastic integration with respect to fractional Brownian motion and applications. Contemporary Math. Vol.336, 2003, pp.3-39.

[5] A. Shiryaev: On arbitrage and replication for fractal models. In A. Shiryaev and A.Sulem (eds.): Workshop on Mathematical Finance, INRIA, Paris 1998.

[6] A. Shiryaev: Essentials of Stochastic Finance. World Scientific 1999.

[7] Hu, Y., Nualart, D., Some processes associated with fractional Bessel processes. J. Theoret. Prob. Vol.18, 2005, pp. 377-397.

[8] Elliott, R.J., Van der Hoek, J., A general fractional white noise theory and applications to finance. Math. Finance. Vol.13, 2003, pp. 301-330.

[9] Pitman, J. and Yor, M. Bessel processes and infinitely divisible laws. In Stochastic Integrals, Volume 851 of Lecture Notes in Mathematics, 285–370. Springer-Verlag Berlin Heidelberg, 1980.

[10] Pitman, J. and Yor, M. A decomposition of Bessel bridges. Z. Wahrsch. Verw. Gebiete, Vol.59, 1982, pp.425–457.

[11] Revuz. D. and Yor, M. Continuous Martingales and Motion. Springer-Verlag Berlin Heidelberg, 1991.

[12] Berman, S.M. Local times and sample function properties of stationary Gaussian processes, Trans. Amer. Math. Soc. 137, 277-299, 1969.

[13] Berman, S.M. Local nondeterminism and local times of Gaussian processes, Indiana Univ. Math. J. 23, 69-74, 1973.

[14] Chakravarti N. and Sebastian K.L. Fractional Brownian Motion Model for polymers, Chemical Physics Letters 267, 9-13, 1997.

[15] Cherayil J. and Biswas P. Path integral description of polymers using fractional Brownian walks, The Journal of Chemical Physics 11, 9230-9236, 1993.

[16] Cranston M. andLe Jan Y. Self-attracting diffusions: two case studies, Math. Ann. 303, 87-93, 1995.

[17] Durrett R. and Rogers L.C.G. A symptotic behavior of Brownian polymer, Prob. Theory Rel. Fields 92, 337-349, 1991.

[18] Elliott R.J. andVan der Hoek J. A general fractional white noise theory and applications to finance, Math. Finance 13, 301-330, 2003.

[19] Herrmann S. and Roynette B. Boundedness and convergence of some self-attracting diffusions. Math. Ann. 325, 81-96, 2003.

[20] Herrmanna S. and Scheutzow M. Rate of convergence of some selfattracting diffusions, Stoc. Proc. Appl. 111, 41-55, 2004.

[21] Hu Y. Integral Transformations and Anticipative Calculus for Fractional Brownian Motions, Memoirs Amer. Math. Soc. Vol. 175, No. 825, 2005.

[22] Hu Y. Self-intersection local time for fractional Brownian motions — via chaos expansion, J. Math. Kyoto Univ. 41 2001, 233-250.

[23] Hu Y. and Nualart D. Renormalized self-intersection local time for fractional Brownian motion, Ann. Prob. 33, 948-983, 2005.

[24] Hu Y. and Øksendal B. Chaos expansion of local time of fractional Brownian motions, Stochastic Anal. Appl. 20, 815-837, 2002.

[25] Hu Y. and Øksendal B. Fractional white nosie calculus and applications to finance, Infinite Dimensional Analysis, Quantum Probability and Related Topics 6, 1-32, 2003.

[26] Mandelbrot B.B. and Van Ness J.W. Fractional Brownian motion, fractional noises and applications, SIAM Review 10, 422-437, 1968.

[27] Nualart D. Malliavin Calculus and Related Topics, 2nd edn. Springer- Verlag, 2006.

[28] Novikov A. and Valkeila E. On some maximal inequalities for fractional Brownian motions, Statistics & Probability Letters 44, 47-54, 1999.
[29] Øksendal B. Fractional Brownian motion in finance, Preprint-series in Oslo University, Pure Mathematics 13, 1-39, 2003.
[30] Pemantle R. Phase transition in reinforced random walk and RWRE on trees, Ann. Probab. 16, 1229-1241, 1998.
[31] Sebastian K. L. Path integral representation for fractional Brownian motion, J. Phys. A 28, 4305-4311, 1995.

Yu Sun (1982-), Glorious Sun School of Business &Management, Donghua University, 1882 West Yan’An Rd., Shanghai 200051, P. R. CHINA.

Education:
Ph.D. in management science and engineering, Donghua University, Shanghai, China, 2010;
M.S. in applied mathematics, Donghua University, Shanghai, China, 2007;
B.S. in mathematics, Qingdao University, Qingdao, China, 2004;
Publications are as following:
Y. Sun and C. Gao, Research on Competitive Intelligence System of Enterprise, New advances in simulation, modeling and optimization proceedings of the 7th WSEAS international conference on simulation, modeling and optimization, 381-386.
Y. Sun and L. Yan, On Fractional Bessel-type Processes, Toyama Math. J. 28(2006), 45-54
L. Yan, Y. Sun and Y. Lu, On The Line Fractional Self-Attracting Diffusion, Journal of Theoretical Probability, 10.1007/s10959-007-0113-y

Current research interests is stochastic processes and mathematics of finance.