The Statistical Properties of Fluctuations of Interfaces for Voter Model

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***Abstract*—**We consider the limiting statistical properties of fluctuations of the voter model. The voter model is one of interacting particle systems, and it is a continuous Markov process on the lattice. Applying the theory of the voter model, an interface model defined from the voter model is defined and studied in the present paper. The interface model is analyzed and estimated by the voter model and the theory of stochastic analysis, including the stopping time method. Further, we show that the probability distributions of the fluctuations, under some conditions, converge to the corresponding distribution of a geometric Brownian motion.

***Keywords*—**Voter model, statistical properties, fluctuation, interface, geometric Brownian motion.

I. Introduction

In this paper, we consider the statistical properties of the fluctuations of interfaces that defined from the voter model, see [1-7]. The problem of description of shapes of interfaces is a well-known problem of statistical mechanics (see [8-10]), in details that, the aim of the research is to investigate the asymptotical behavior of the corresponding sequence of probability measure describing the statistical properties of these interfaces. The voter model is one of statistical physics models, as the name might suggest that the voter model can model political systems, but rather the fact that the voter model is exactly the class of spin systems which duality can be applied most completely and successfully, the voter model is a continuous Markov process on $\{0,1\}^{Z^d}$, see [8]. This work originates in an attempt to describe the fluctuations of interfaces of the voter model, and study the convergence of the corresponding probability distributions. Recently, some research work has been done to study the statistical properties of the random interfaces for some statistical physics models, for example the two-dimensional Widom-Rowlinson model, see Refs. [2-6]. The research work of interfaces fluctuations focus on the Ising model, Widom-Rowlinson model and S.O.S. model, but there is no research work on the voter model, in fact that there is no definition of interfaces for the voter model until now. And their research work heavy depends on the theory of the cluster expansion and the partition functions expansion. But in the present paper, we give a definition of interfaces for the voter model, and the stopping time methods will be used to study the fluctuations of the model. In the last section of this paper, we extend the interfaces voter model and extend the result of Theorem 1, the theory of compound Poisson process is applied to study the properties of interfaces model, and we obtain Theorem 2.

First, we give the brief definitions and properties of the voter model, for details see [8]. One interpretation for the voter model is, for a collection of individuals, each of which has one of two possible positions on a political issue, at independent exponential times, an individual reassesses his view by choosing a neighbor at random with certain probabilities and then adopting his position. Specifically, the voter model is one of the statistical physics models, we think of the sites of the d -dimensional integer lattice as being occupied by persons who either in favor of or opposed to some issue. To write this as a set--valued process, we let $\{\xi(s), s \geq 0\}$ the set of voters in favor, we can also think of the sites in $\xi(s)$ as being occupied by cancer cells, and the other sites as being occupied by healthy cells. We can formulate the dynamics as follows: (i) An occupied site becomes vacant at a rate equal to the number of the vacant neighbors; (ii) An vacant site becomes occupied at a rate equal to λ times the number of the occupied neighbors, where λ is a intensity which is called the ``carcinogenic advantage" in voter model. When $\lambda = 1$, the model is called the voter model, and when $\lambda > 1$, the model is called the biased voter model.

Next we introduce the graphical representation of the model, since the graphical representation is necessary for us to give a good description of the model. For simplicity, we give the construction of graphical representation for 1 -dimensional voter model ($\lambda = 1$), for more general cases, see [8]. Thinking of 1 -dimensional integer points as being laid out on a horizontal axis, with the time lines being placed vertically, above that axis. Define independent Poisson processes with rate 1 for each time lines, at each event time (x, s) , we choose one of its two neighbors with probability $1/2$, and draw an arrow from that neighbor point to (x, s) , and write a δ at (x, s) . To construct the process from this ``graphical representation", we imagine fluid entering the bottom at the

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points in $\xi(0)$ and flowing up the structure. The δ 's are the dams and the arrows are pipes which allow the fluid to flow in the indicated direction. To make this definition mathematical, we say that there is a path from $(x,0)$ to (y,t) if there is a sequence of times $s_0 = 0 < s_1 < s_2 < ... < s_n < s_{n+1} = 1$ and spatial locations $x = x_0, x_1, ..., x_n = y$ so that:

(i) for $i = 1, 2, ..., n$ there is an arrow from x_{i-1} to x_i at time s_i, and

(ii) the vertical segments $\{x_i\} \times (s_i, s_{i+1})$, $i = 0, 1, ..., n$ do not contain any δ 's.

When there is a path from $(x,0)$ to (y,t), it follows that the individual at y at time t has the same opinion as individual at time 0. Since every individual has the same opinion as some individual at time 0, it follows that

$$\xi^A(s) = \{ y: \text{ for some } x \in A \text{ there is a path from } (x,0) \text{ to } (y,t) \}.$$

Let $\xi^A(s)$ ($s \in I$) denote the state at time s with the initial state $\xi^A(0) = A$, when $A = \{0\}$, recorded as $\{\xi^{\{0\}}(s),\ s \ge 0\}$. Then from [8], the voter model $\xi^A(s)$ approaches total consensus in $d = 1$ and $d = 2$. But in higher dimensions $d \ge 3$, the differences of opinion may persist. For more generally, we consider the initial distribution as ν_θ, the product measure with density θ, that is, each site is independently occupied with probability θ and let $\xi^\theta(s)$ denote the voter model with initial distribution ν_θ. For the biased voter model ($\lambda > 1$), there is a ``critical value'' for the process which is defined as following

$$\lambda_c = \inf \left\{ \lambda: \ P(|\xi^{\{0\}}(s)| > 0, \forall s \ge 0) > 0 \right\}.$$

It can be shown that $\lambda_c = 1$ for the voter model. This means that, on d -dimensional lattice, if $\lambda < \lambda_c$, the process dies out (becomes vacant) exponentially fast, if $\lambda > \lambda_c$, the process survives with positive probability.

II. Notations and Definitions

First we define the interfaces by the voter model on discrete time $k \in \{1, \cdots, n\}$, then take the scaling limit of discrete time model, we will obtain a continuous time process, this is the main work of the present paper. Although a lot of research work has been done for studying the voter model in the past twenty years, the interfaces model defined in the following (1) is the first time, so we think that this kind of work is important for us to further understand the statistical properties of the voter model, especially understand the fluctuation properties of the voter model.

Suppose that the interfaces fluctuate at each time k. For each $k \in \{1, \cdots, n\}$, let ω_k be random variable such that

$$P(\omega_k = +1) = p_k,\ P(\omega_k = -1) = q_k,\ P(\omega_k = 0) = r_k,$$

where $p_k + q_k + r_k = 1$, and $\{\omega_1, \omega_2, \cdots, \omega_n\}$ is an independent random sequence. Let l_n denote a positive integer, and let $\Lambda_{l_n} = [-l_n, l_n]$ be a subset of 1-dimensional lattice Z. For a parameter $\beta_k > 0$, we define a function of interfaces by the voter model at time k by

$$A(\sigma_k) = \beta_k \omega_k \frac{|\xi_k^{\{0\}}(s)|}{|\Lambda_{l_n}|}$$

where $|\Lambda_{l_n}|, |\xi_k^{\{0\}}(s)|$ are the cardinality of Λ_{l_n} and $\xi_k^{\{0\}}(s)$, $s \in I$. Now we define the interfaces of the model by

$$G(k) = G(k-1) \exp\{A(\sigma_k)\}.$$

Then for $k \in \{1, \cdots, n\}$,

$$G(k) = G(0) \exp\left\{ \sum_{l=1}^{k} A(\sigma_l) \right\}$$
$$= G(0) \exp\left\{ \sum_{l=1}^{k} \beta_l \omega_l \frac{|\xi_l^{\{0\}}(s)|}{|\Lambda_{l_n}|} \right\} \qquad (1)$$

where $G(0)$ is an initial state at time 0, and let

$$A_k = \sum_{l=1}^{k} A(\sigma_l).$$

The interpretation for the interfaces model of (1) is, for example, for a collection of individuals, each of individuals has one of two possible positions on a political issue, and $G(k)$ is the number that reflects this political issue at time k. From (1), if $\omega_k = +1$, then $A(\sigma_k) = \beta_k \omega_k |\xi_k^{\{0\}}(s)| / |\Lambda_{l_n}| > 0$, this means that the number of voters in favor at time k is more than that of voters in favor at time $k-1$, so it implies that $G(k)$ increases; On the contrary, if $\omega_k = -1$, $A(\sigma_k) = \beta_k \omega_k |\xi_k^{\{0\}}(s)| / |\Lambda_{l_n}| < 0$, then $G(k)$ decreases.

III. The Stopping Time for the Model

In this section, first we introduce some results of the voter model (see [8]), then we define the stopping times for the interfaces model, at last we show the main results of the present paper. From [8], we have the following Lemma 1, here we omit the proof of Lemma 1. According to above Section 2 and Lemma 1, we can show the following Corollary 1. Corollary 1 is important for us to estimate the main results of the present paper.

Lemma 1 (a) If $\lambda < \lambda_c$, there is a $\rho > 0$ such that

$$P\left(\xi^{\{0\}}(s) \ne \varnothing\right) \le e^{-\rho s}$$

then the process dies out exponentially fast;

(b) If $\lambda > \lambda_c$, then on $\{\xi^{\{0\}}(s) \ne \varnothing,\ \text{for all } s \ge 0\}$,

$$\frac{|\xi^{\{0\}}(s)|}{s} \to 2(\lambda - 1), \quad \text{a.s.,} \quad \text{as } s \to \infty.$$

Corollary 1 For any $\varepsilon > 0$ and l_n large enough,

(a) If $\lambda < \lambda_c$, for any fixed k,

$$E\left(\frac{|\xi_k^{\{0\}}(s)|}{|\Lambda_{l_n}|}\right)<\varepsilon\,,\quad \text{as}\quad s\to\infty\,.$$

(b) If $\lambda>\lambda_c$, for any fixed k, there is a $\rho>0$ such that, as $s\to\infty$,

$$E\left(\frac{|\xi_k^{\{0\}}(s)|}{|\Lambda_{l_n}|}\right)\geq\rho\,,\qquad E\left(\frac{|\xi_k^{\{0\}}(s)|}{|\Lambda_{l_n}|}\right)^2\geq\rho\,.$$

Next we define the stopping time for the interfaces model. Let $\tau_1,\tau_2,\cdots,\tau_m,\cdots$, denote the stopping times defined as followings

$$\tau_1=\min\left\{k\geq 1;\ n^{-\frac{1}{3}}\sum_{l=1}^{k}A(\sigma_l)\geq 1\right\},$$

$$\tau_2=\min\left\{k\geq 1;\ n^{-\frac{1}{3}}\sum_{l=\tau_1+1}^{\tau_1+k}A(\sigma_l)\geq 1\right\},\cdots$$

$$\tau_m=\min\left\{k\geq 1;\ n^{-\frac{1}{3}}\sum_{l=\tau_{m-1}}^{\tau_{m-1}+k}A(\sigma_l)\geq 1\right\},\ \cdots\,. \tag{2}$$

For every stopping time intervals $[\tau_{m-1}+1,\tau_m]$, define a $\lambda_m>0$ on this time interval, such that for some $0<\alpha<1$, if $m\leq n^\alpha/2$ then $\lambda_m<\lambda_c$, if $m>n^\alpha/2$ then $\lambda_m>\lambda_c$. Then we have the following results. For any fixed k,

$$E\left[A(\sigma_k)\right]=\beta_k(p_k-q_k)E\left(\frac{|\xi_k^{\{0\}}(s)|}{|\Lambda_{l_n}|}\right) \tag{3}$$

$$E\left[A(\sigma_k)\right]^2=\beta_k^2(p_k+q_k)E\left(\frac{|\xi_k^{\{0\}}(s)|}{|\Lambda_{l_n}|}\right)^2 \tag{4}$$

By Lemma 1, Corollary 1, (3) and (4), If $\lambda<\lambda_c$ (or $m\leq n^\alpha/2$), we can properly choose β_k,p_k,q_k, where k belongs to some time interval $[\tau_{m-1}+1,\tau_m]$, such that

$$E\left[A(\sigma_k)\right]=E\left[A(\sigma_k)\right]^2=\frac{1}{\sqrt{n}}\,. \tag{5}$$

If $\lambda>\lambda_c$ (or $m>n^\alpha/2$), by Lemma 1 and Corollary 1, (3) and (4), we properly choose β_k,p_k,q_k, where k belongs to some time interval $[\tau_{m-1}+1,\tau_m]$, such that

$$E\left[A(\sigma_k)\right]=E\left[A(\sigma_k)\right]^2=c \tag{6}$$

where c is a positive constant.

Taking the scaling limit of discrete time model of (1), we will obtain a continuous time process—the continuous time interfaces model, and we discuss the probability distribution of this continuous time model. Let

$$0<v<1\,,\qquad [nv]\in\left[1+\tau_1+\cdots+\tau_{m-1},\tau_1+\cdots+\tau_m\right]$$

where $[nv]$ is the integer part of nv. Then m can be expressed by $m=m(n,v)$, let

$$A_v^n=\frac{1}{\sqrt{n}}\sum_{k=1}^{\tau_1+\cdots+\tau_{m(n,v)}}A(\sigma_k)\,,\quad 0<v<1\,. \tag{7}$$

Now we define the interfaces of the model in terms of above (7) by (see (1))

$$G(n,v)=G(0)\exp\left\{A_v^n\right\},\quad 0<v<1\,. \tag{8}$$

Theorem 1 Suppose that the interfaces model follows (8), when $n\to\infty$, the probability distribution of the process $G(n,v)$ convergence to the corresponding distribution of

$$G(0)\exp\left\{\int_0^v\mu(v)dv+\int_0^v\sigma(v)B(v)dv\right\},\qquad 0<v<1$$

where $B(v)$ is the one dimensional standard Brownian motion, $G(0)$ is an initial state at time 0, $\mu(v)$ is the trend function and $\sigma(v)$ is the volatility function.

IV. Convergence of the Fluctuations of Interfaces

In this section, we consider the convergence of the probability distribution of the process $G(n,v)$ which is introduced in Section 3, and we show the proof of the main results in this paper. The proof of Theorem 1 is divided into four parts, and the supercritical case ($\lambda>\lambda_c$) and subcritical case ($\lambda<\lambda_c$) are considered in the proof.

Proof of Theorem 1 In order to show the convergence of the distribution, we consider the convergence of the characteristic function of A_v^n, i.e.,

$$\varphi_v^n(z)=E\left[\exp\{izA_v^n\}\right],\qquad \text{as}\quad n\to\infty$$

where $i=\sqrt{-1}$. $\varphi_v^n(z)$ is divided into two terms as follows

$$\varphi_v^n(z)=E[\exp\{izA_v^n\};\ |\tau_m-n^{5/6}|\leq n^{2/3+\varepsilon},\ \text{for all}\ m=1,\cdots,\frac{n^\alpha}{2}]$$
$$+\ E[\exp\{izA_v^n\};\ |\tau_m-n^{5/6}|>n^{2/3+\varepsilon},\ \text{for some}\ m=1,\cdots,\frac{n^\alpha}{2}]\,. \tag{9}$$

Next we define the conditional expectations,

$$K_1=E[\exp\{izA_v^n\}\big|\ |\tau_m-n^{5/6}|\leq n^{2/3+\varepsilon},\ \text{for all}\ m=1,\cdots,\frac{n^\alpha}{2}]\,, \tag{10}$$

$$K_2=E[\exp\{izA_v^n\}\big|\ |\tau_m-n^{5/6}|>n^{2/3+\varepsilon},\ \text{for some}\ m=1,\cdots,\frac{n^\alpha}{2}]\,. \tag{11}$$

(I) Now we estimate the second term K_2. Let $1/12<\varepsilon<1/6$ and $A_k=\sum_{l=1}^{k}A(\sigma_l)$, then

$$P\left(|\tau_m-n^{5/6}|>n^{2/3+\varepsilon}\right)$$
$$=P\left(\tau_m>n^{2/3+\varepsilon}+n^{5/6}\right)+P\left(\tau_m<n^{5/6}-n^{2/3+\varepsilon}\right)$$
$$=P\left(A_{n^{2/3+\varepsilon}+n^{5/6}}\leq n^{1/3}\right)+P\left(A_{n^{5/6}-n^{2/3+\varepsilon}}\geq n^{1/3}\right)$$
$$=P\left((A_{n^{2/3+\varepsilon}+n^{5/6}}-E[A_{n^{2/3+\varepsilon}+n^{5/6}}]\leq -n^{1/6+\varepsilon}\right)$$

$$+P\left((A_{n^{5/6}-n^{2/3+\varepsilon}}-E[A_{n^{5/6}-n^{2/3+\varepsilon}}]\geq n^{1/6+\varepsilon}\right)$$
$$\leq\frac{1}{n^{1/3+2\varepsilon}}\left(n^{1/3}-n^{1/6+\varepsilon}\right)+\frac{1}{n^{1/3+2\varepsilon}}\left(n^{1/3}+n^{1/6+\varepsilon}\right)$$
$$=\frac{2}{n^{2\varepsilon}}\,. \qquad (12)$$

By (11) and (12), when $\alpha=1/6$, then we have $K_2\leq 2n^{\alpha}/n^{2\varepsilon}$, so that

$$K_2\to 0\,,\qquad \text{as}\quad n\to\infty\,.$$

This implies that the length of stopping time τ_m is about $n^{5/6}$.

(II) From above discussion, for $\alpha=1/6$, $(n^{\alpha}/2)\times n^{5/6}=n/2$. So, when $k\leq n/2$, $\lambda<\lambda_c$; when $k>n/2$, $\lambda>\lambda_c$.

(a) If $m\leq n^{\alpha}/2$, and $k\leq n/2$, by (5) we have

$$E[\exp\{\frac{iz}{\sqrt{n}}A(\sigma_k)\}]$$
$$=1+\frac{iz}{\sqrt{n}}E[A(\sigma_k)]-\frac{z^2}{2n}E[A(\sigma_k)]^2+o(\frac{1}{n\sqrt{n}})$$
$$=1+\frac{iz}{n}+o(\frac{1}{n\sqrt{n}}) \qquad (13)$$

(b) If $m>n^{\alpha}/2$, and $k>n/2$, by (6) we have

$$E[\exp\{\frac{iz}{\sqrt{n}}A(\sigma_k)\}]$$
$$=1+\frac{iz}{\sqrt{n}}E[A(\sigma_k)]-\frac{z^2}{2n}E[A(\sigma_k)]^2+o(\frac{1}{n\sqrt{n}})$$
$$=1+\frac{iz-z^2c/2}{n}+o(\frac{1}{n\sqrt{n}})\,. \qquad (14)$$

(III) We estimate the first term K_1 in two parts.

(a) If $0<v<1/2$ ($k\leq n/2$), so $m(n,v)=[n^{1/6}v]$, by (13) we have

$$K_1=\sum_{\substack{|r_m-n^{5/6}|\leq n^{2/3+\varepsilon}\\ m=1,\cdots,n^{\alpha}/2}}E[\exp\{izA_v^n\}\big|\ \tau_m=r_m\,,$$
$$\text{for all } m=1,\cdots,\frac{n^{\alpha}}{2}]\,,$$
$$=\sum_{\substack{|r_m-n^{5/6}|\leq n^{2/3+\varepsilon}\\ m=1,\cdots,n^{\alpha}/2}}\prod_{m=1}^{[n^{1/6}v]}E[\exp\{\frac{iz}{\sqrt{n}}A(\sigma_k)\}]^{r_m}$$
$$=\sum_{\substack{|r_m-n^{5/6}|\leq n^{2/3+\varepsilon}\\ m=1,\cdots,n^{\alpha}/2}}\prod_{m=1}^{[n^{1/6}v]}(1+\frac{iz}{n}+o(\frac{1}{n\sqrt{n}}))^{r_m}$$
$$=\sum_{\substack{|r_m-n^{5/6}|\leq n^{2/3+\varepsilon}\\ m=1,\cdots,n^{\alpha}/2}}\exp[\sum_{m=1}^{[n^{1/6}v]}r_m\ln(1+\frac{iz}{n}+o(\frac{1}{n\sqrt{n}}))]$$
$$=\sum_{\substack{|r_m-n^{5/6}|\leq n^{2/3+\varepsilon}\\ m=1,\cdots,n^{\alpha}/2}}\exp[izv+o(\frac{1}{n^{1/6-\varepsilon}})]\,. \qquad (15)$$

On the other hand, by (12) we have

$$P(|\tau_m-n^{5/6}|\leq n^{2/3+\varepsilon},m=1,\cdots,\frac{n^{\alpha}}{2})$$
$$=\prod_{m=1}^{n^{\alpha}/2}P(|\tau_m-n^{5/6}|\leq n^{2/3+\varepsilon})$$
$$=\prod_{m=1}^{n^{\alpha}/2}(1-n^{-2\varepsilon})=(1-n^{-2\varepsilon})^{n^{\alpha}/2}\,. \qquad (16)$$

So for $\alpha=1/6$ and $1/12<\varepsilon<1/6$, then $2\varepsilon>\alpha$, so that, as $n\to\infty$,

$$\ln(1-n^{-2\varepsilon})^{n^{\alpha}/2}\approx\frac{1}{2}n^{\alpha}\frac{1}{n^{2\varepsilon}}=\frac{n^{\alpha}}{2n^{2\varepsilon}}\to 0\,.$$

Then we have

$$\lim_{n\to\infty}P(|\tau_m-n^{5/6}|\leq n^{2/3+\varepsilon},m=1,\cdots,\frac{n^{\alpha}}{2})=1\,. \qquad (17)$$

Combining (9)-(11) and (15)-(17), if $0<v<1/2$, then we obtain

$$\lim_{n\to\infty}\varphi_v^n(z)=\lim_{n\to\infty}\exp\left\{izv+o(\frac{1}{n^{1/6-\varepsilon}})\right\}$$
$$=\exp\{izv\} \qquad (18)$$

(b) If $1/2\leq v<1$ ($k>n/2$), following the similar procedure of above (a), and by (13)(14), we have

$$E[\exp\{izA_v^n\}\big|\ \tau_m=r_m\,,\ \text{for all } m=1,\cdots,\frac{n^{\alpha}}{2}]$$
$$=\prod_{m=1}^{[n^{\alpha}/2]}E[\exp\{\frac{iz}{\sqrt{n}}A(\sigma_u)\}]^{r_m}\qquad (u<\frac{n}{2})$$
$$\times\ E[\exp\{\frac{iz}{\sqrt{n}}A(\sigma_k)\}]^{[nv]-(r_1+\cdots+r_{n^{\alpha}/2})}$$
$$=(1+\frac{iz}{\sqrt{n}}+o(\frac{1}{n\sqrt{n}}))^{[nv]-(r_1+\cdots+r_{n^{\alpha}/2})}$$
$$\times\ (1+\frac{iz-z^2c/2}{\sqrt{n}}+o(\frac{1}{n\sqrt{n}}))^{[nv]-(r_1+\cdots+r_{n^{\alpha}/2})}$$
$$=\exp[izv-\frac{1}{2}(v-\frac{1}{2})z^2c+o(\frac{1}{\sqrt{n}})]\,.$$

Then we have, for $1/2\leq v<1$

$$\lim_{n\to\infty}\varphi_v^n(z)=\lim_{n\to\infty}\exp\left\{izv-\frac{1}{2}(v-\frac{1}{2})z^2c+o(\frac{1}{\sqrt{n}})\right\}$$
$$=\exp\left\{izv-\frac{1}{2}(v-\frac{1}{2})z^2c\right\}\,. \qquad (19)$$

(IV) Combining (18) and (19), for $0<v<1$, we have

$$\lim_{n\to\infty}\varphi_v^n(z)=\lim_{n\to\infty}E\left[\exp\{izA_v^n\}\right]$$
$$=\exp\left\{izv\mu(v)-\frac{1}{2}\sigma^2(v)(v-\frac{1}{2})z^2\right\} \qquad (20)$$

where

$$\mu(v)=1\,,\ \sigma^2(v)=0\,,\ \text{if}\ 0<v<\frac{1}{2}\,,$$
$$\sigma^2(v)=c\,,\qquad \text{if}\ \frac{1}{2}\leq v<1\,.$$

By Refs. [11], above (20) shows that the probability distributions of interfaces model of the present paper converge to the corresponding distributions of a geometric Brownian motion, this completes the proof of Theorem 1.

Remark 1 The proof of Theorem 1 can be extended to more complicated interfaces model. For example, the time s in the voter model $\xi^A(s)$ ($s \in I$) , can be defined to be an independent random sequence $\{s_k(\sigma)\}(k=1,\cdots,n)$, then following the similarly proof methods, we can obtain the different trend function $\mu(v)$ and the different volatility function $\sigma(v)$.

V. The Extension of the Model and Theorem 1

In this Section, we will extend the interfaces model defined in (1) of Section 2 in this paper. For each $k \in \{1,\cdots,n\}$, we consider J as a random variable that follows Poisson distribution with parameter γ. Let weight parameters π_j, $j \in \{1,\cdots,J\}$ such that $\pi_1+\cdots+\pi_j=1$. For a fixed $k \in \{1,\cdots,n\}$, let

$$A(\sigma_k^j)=\pi_j\beta_k\omega_k\frac{|\xi_k^{\{0\}}(s)|}{|\Lambda_{l_n}|}, \qquad j \in \{1,\cdots,J\}.$$

Now we define the interfaces of the model by

$$G(k)=G(k-1)\exp\left\{\sum_{j=1}^{J}A(\sigma_k^j)\right\}.$$

Then for $k \in \{1,\cdots,n\}$,

$$G(k)=G(0)\exp\left\{\sum_{l=1}^{k}\sum_{j=1}^{J}A(\sigma_k^j)\right\}$$

$$=G(0)\exp\left\{\sum_{l=1}^{k}\sum_{j=1}^{J}\pi_j\beta_l\omega_l\frac{|\xi_l^{\{0\}}(s)|}{|\Lambda_{l_n}|}\right\} \qquad (21)$$

where $G(0)$ is an initial state at time 0. Similarly as Section 3, let

$$0<v<1, \qquad [nv]\in\left[1+\tau_1+\cdots+\tau_{m-1},\tau_1+\cdots+\tau_m\right]$$

where $[nv]$ is the integer part of nv. Then m can be expressed by $m=m(n,v)$, let

$$A_v^n=\frac{1}{\sqrt{n}}\sum_{k=1}^{\tau_1+\cdots+\tau_{m(n,v)}}\sum_{j=1}^{J}A(\sigma_k^j), \qquad 0<v<1. \qquad (22)$$

Now we define the interfaces of the model in terms of above (22) by

$$G(n,v)=G(0)\exp\left\{A_v^n\right\}, \qquad 0<v<1. \qquad (23)$$

Then we give the following Theorem 2.

Theorem 2 Suppose that the interfaces model follows (23), when $n\to\infty$, the probability distribution of the process $G(n,v)$ convergence to the corresponding distribution of

$$G(0)\exp\left\{\int_0^v\mu(v)dv+\int_0^v\sigma(v)B(v)dv\right\}, \qquad 0<v<1$$

where $B(v)$ is the one dimensional standard Brownian motion, $G(0)$ is an initial state at time 0, the trend function $\mu(v)$ and the volatility function $\sigma(v)$ are given by

$$\mu(v)=\gamma,\ \sigma^2(v)=0,\ \text{if } 0<v<\frac{1}{2},$$

$$\sigma^2(v)=\gamma c, \qquad \text{if } \frac{1}{2}\le v<1.$$

The proof of Theorem 2 can follow the proving procedure of Theorem1 in Section 4 of the present paper. Note that the theory of compound Poisson process is applied in the proof of Theorem 2, see [12].

VI. Conclusion

In this paper, we studied the statistical properties of the fluctuations of interfaces model given by the voter model and stopping times. Theorem 1 and Theorem 2 show that the probability distributions of the fluctuations of interfaces model converge to the corresponding distributions of a geometric Brownian motion. This work models an interface of the voter model, and is useful for us to understanding the fluctuations of the voter model.

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