The 2-Dimensional Probabilistic Bin Packing Problem : An average case analysis

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***Abstract*—In the probabilistic two-dimensional Bin Packing problem (2D-PBPP), one is asked to pack a random number of rectangular items, without overlap and any rotation, into the minimum number of identical square bins. In this paper we consider the re-optimization procedure used for solving probabilistic combinatorial optimization problems and an approximation of this strategy: the redistribution strategy according to an heuristic. According to computational results we show that the redistribution strategy according to some efficient heuristics generates results near those given by the re-optimization strategy which is impossible to be carried out.**

***Index Terms*—Probabilistic Combinatorial Optimization Problem; Two Dimensional Probabilistic Bin Packing Problem; Re-optimization strategy; Redistribution strategy; Approximation algorithms.**

I. INTRODUCTION

IN the Two-Dimensional Bin Packing Problem (2D-BPP), one is given a collection of rectangles specified by their widths and heights that need to be packed into larger square bins. The most interesting and well-studied version of this problem is the so-called orthogonal packing without rotation where each rectangle must be packed parallel to the edge of a bin and cannot be rotated. The goal is to find the feasible packing i.e. a packing where rectangles do not overlap using the smallest number of bins. The great interest of 2D-BPP is mainly due to the tremendous number of its real-world applications: From packing newspaper commercials to cutting-stock problems. Bin Packing and its multi-dimensional variations are classical problems in the Combinatorial Optimization Problem and have been studied extensively. Since the one-dimensional Bin Packing problem in which n items of lengths $\{h_i : 1 \leq i \leq n\}$ have to be packed in the minimum number of bins of capacity c so that the capacity of none is exceeded, is known to be *NP-hard* (Garey and Johnson (1979)), the same holds for the 2D-BPP.

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Let's consider the problem of the allocation of advertisements in newspapers. It can be seen as a 2D-BPP, where the advertisements are the items and the pages are the bins of the packing. But in reality, the number of items to be published varies from a day to another so in this packing problem the items appears as a random variable, so we have to solve several new problems which are a simple variation of the initial one. This new problem can be formulated as a 2D-BPP where the number of the items to be packed is a random variable, called Two-Dimensional Probabilistic Bin Packing Problem (2D-PBPP). The 2D-PBPP involves in the field of Probabilistic Combinatorial Optimization Problem (PCOP). The idea of Probabilistic Combinatorial Optimization Problem (PCOP) comes from Jaillet (Jaillet (1988) and Jaillet (1993)) who introduced it for the Travelling Salesman Problem (see also (Bertsimas (1988),Bertsimas et al (1990),Bertsimas and Hell (1988)). Among several motivations for the PCOPs studies, we are mostly interested in two: The first is the wish of formulating and analyzing models which are more appropriate for real-world problems where randomness is present, the second motivation is an attempt to study a perturbation of an initial problem, simulated by either the presence or the absence of subsets of the given list of items. The Probabilistic Bin Packing Problem was first studied in (Bellalouna (1993)) where we assume that a list L_n of n items is given, and that some items disappear from L_n and in (Bellalouna et al(2004)) an average case analysis is proposed for the one-dimensional probabilistic bin packing problem.

To solve such problem, we consider two strategies. The natural one is the re-optimization strategy (find optimal solution for each potential instance) but the problem is NP-hard then this strategy is not realistic. As an alternative we propose a near optimal strategy : find solution according to efficient heuristic A for each potential instance; we call it redistribution strategy according to A. An exact analysis of these strategies seems to be quite difficult, then we propose in this paper an asymptotic analysis. For an estimation of the performance of the redistribution strategy, we have first to know the asymptotic average behavior of re-optimization strategy. Under assumption, we show that redistribution strategy according to

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FBS is asymptotic to the re-optimization strategy.

II. Definitions for 2D-PBPP

The 2D-PBPP is essentially a 2D-BPP in which the number of items to be packed is a random variable: On any given instance of the problem, only a subset I of L_n is present with a probability $P(I)$. We denote N the number of present items ($0 \leq N \leq n$). In this paper, we will concentrate on the particular case where each item has a probability p of being present independently of the others. Hence, N is a binomial random variable with parameters p and n. Under these assumptions, we propose to study re-optimization strategy to solve 2D-PBPP. These strategies will be detailed in the following paragraphs.

A. Re-optimization strategy

The re-optimization strategy is the most natural one. It consists in finding the optimal solution for each potential instance. The smallest number of bins needed to pack the present items of the list L_n is a random variable. We note it $\sum_{opt}(L_n)$, its expected value is defined by :

$$\mathbb{E}(\Sigma_{Opt}(L_n)) = \sum_{I \subseteq L_n} P(I) Opt(I), \quad (1)$$

where $Opt(I)$ is the smallest number of bin need to pack instance I.
But the 2D-BPP is *NP-hard*, so we have to solve an exponential number of $NP-hard$ problems, this is impossible to perform in the practice. As an alternative, we propose a near-optimal procedure: Find solution according to an efficient heuristic A for each potential instance; we call it *redistribution* strategy according to A.

B. Redistribution strategy

For the redistribution strategy, according to an heuristic A,we consider a list of items $L_n = \{(w_i, h_i)\}$ where (w_i, h_i) is the size of the i^{th} item, we suppose that we have a probability distribution $\mathcal{P}$ on all the possible subset of L_n i.e every potential subset $I \subseteq L_n$ has probability $\mathcal{P}(I)$ of occurring.
the total number of bins needed to pack the present items of the list L_n by the heuristic A is a random variable, we denoted it $\sum_{redist}^{H}(L_n)$, its expected value is defined by the expression :

$$\mathbb{E}(\Sigma_{Redist}^{A}(L_n)) = \sum_{I \subseteq L_n} \mathcal{P}(I) A(I), \quad (2)$$

where $A(I)$ is the total number of bins needed to pack I by the heuristic A.

C. Approximation algorithms

In this section, we survey some algorithms devoted to the Two-dimensional Bin Packing Problem (For a larger survey see [10], [12], [11]). Up to the mid-nineties, almost all result in the literature concerned heuristic algorithms. Many approximation heuristics have been proposed and studied (See [6] for an exhaustive survey).
Most of the approaches are shelf algorithms, i.e., the packing is obtained by placing the item, from left to right, in rows forming levels (shelves). The first shelf is the bottom of the bin, and subsequent shelves are produced by the horizontal line coinciding with the top of the tallest item packed on the shelf below.
In what follows, we present the heuristics used in our experimental study in section III.

1) The Next Fit Level algorithm (NFL): NFL packs the current item on the current level of the current bin if it fits. If not, a new level is created in the current bin if the height of the item is less than or equal to the height of the bin minus the hight of the current level, the current level is closed off, and the item is packed in the new level created. When the next item to be packed cannot be packed on a new level in the current bin, a new bin is opened.

2) The First Fit Level algorithm (FFL): FFL packs the current item in the lowest shelf on the first bin where it fits. If no shelf can accommodate it, a new shelf is created either in the first suitable bin, or by initializing a new bin (If no bin have enough vertical space available).

3) The Finite First-Fit algorithm (FFF): First the items are stored by non increasing heights, the *FFF* packs the current item in the lowest level of the first bin where it fits; if no level can accommodate it, a new level is created in the first bin having sufficient vertical space, otherwise, the new level is created in a new bin.

4) The Finite Best Strip algorithm (FBS): FBS sorts initially the items by decreasing heights and consists of two phases. First, items are packed into an infinite height strip using a Best Fit algorithm, the resulting strip packing is made up of ”shelves” each corresponding to a different level, having equal width and different heights. In the second phase, the shelves are packed into finite bins using the well known Best Fit (BF)[1] heuristic for the 1D-BPP.

[1] Best Fit maintains a list of current bins, ordered by sizes, and upon arrival of item x, puts it in the current fullest bin in which it fits, opening a new bin for x if this fails

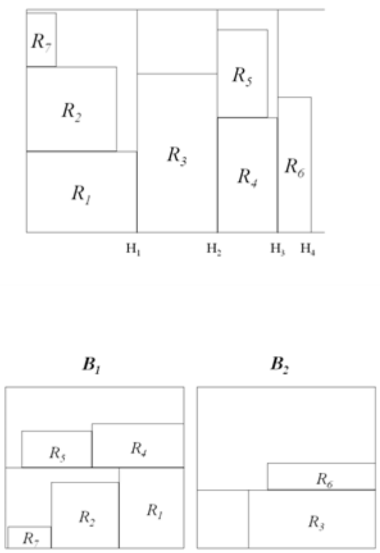

Fig. 1. First and second phase of algorithm FBS

5) Matching (KLM): The last heuristic *KLM* is considered as an extension of matching to multi-dimensions. We recall that in One Dimension Bin Packing Problem, the matching consists, first, in sorting the pieces according the decreasing heights, and then putting the current object with the biggest that fits with, if it exists. Otherwise, it is kept in a single bin. In the two dimensional case, the algorithm first divides the items into four sets S_1, S_2, S_3 and S_4 containing items with dimensions in the quadrants of the unit square in the order $\left[0, \frac{1}{2}\right]^2, \left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, 1\right], \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]^2$, respectively. The algorithm then attempts to pack as many bins as possible with four items, one from each set. This is done by producing 3 one-dimensional (pairwise) matchings, M_{12}, M_{34} and M_{24}, as follows. Let $J_i = (h_i, w_i)$ denote the size of the j^{th} item. M_{12} provides a maximum matching of items in S_1 with those from S_2 such that for all $(J_i, J_j) \in M_{12}$ with $J_i \in S_1$ and $I_j \in S_2$ we have $w_i + w_j \leq 1$ and $h_i \leq h_j$. Similarly, M_{34} is a maximum matching of items in S_3 and S_4 such that if $(I_i, I_j) \in M_{34}$ with $I_i \in S_3$ and $I_j \in S_4$ we have $w_i + w_j \leq 1$ and $h_i \leq h_j$. Finally, M_{24} is a maximum matching of items in S_2 et S_4 such that if $(J_i, J_j) \in M_{24}$ with $J_i \in S_2$ and $J_j \in S_4$ then $h_i + h_j \leq 1$.

III. EXPERIMENTAL STUDY

In this section we attempt to do a comparison between the re-optimization strategy and the re-distribution strategies through an experimental study, in which we have implemented the redistribution strategy according to the *FBS* heuristic.

A. Asymptotic behavior of the re-optimization strategy

We aim here to show that under assumption, the redistribution strategy according to *FBS* is asymptotic to the re-optimization strategy.

So for an estimate of the performance of a distribution strategy, we have first to know the asymptotic behavior of the re-optimization strategy. In fact, it can be easily proved that the expected number of bins obtained by the re-optimization strategy, when items are independently and identically distributed on the the unit square $[0, 1]^2$, and when each of them have the same probability of being present p, divided by np is $\frac{1}{4}$

$$\lim_{n \to +\infty} \frac{\mathbb{E}u(\sum_{Reopt}(L_n))}{np} = \frac{1}{4} \qquad (3)$$

B. Asymptotic behavior of the redistribution strategy

According to the result (3) , we compute the following ratio:

$$R^A_{Redist} = \frac{Eu(\sum^A_{Redist}(L_n))}{np}. \qquad (4)$$

where $\mathbb{E}u(\sum^A_{Redist}(L_n))$ is the expected total number of bin needed to pack the present items of the random list $L_n = \{(W_1, H_1); (W_2, H_2); ...; (W_n, H_n)\}$ by the heuristic A.

The strategy of redistribution according to FBS will be all the more interesting as this ratio will be close to $\frac{1}{4}$.

C. Experimental results

The experience consists in generating a list $L_n = \{(W_1, H_1); (W_2, H_2); ...; (W_n, H_n)\}$ of n items uniformly and independently distributed on $[0, 1]^2$. We attribute the same probability p of being present for each item. So we generate the subset I of present items.

We implemented the redistribution strategy according to an heuristic A, where A will be in turn NFL, FFL, FFF, FBS and KLM, as an approximation to the re-optimization strategy which is impossible to carry out since the problem is *NP-Hard*. The runs were executed with values of n ranging from 100 to 1000, with a step of 100 and with different values of $p \in \{0.1, 0.3, 0.5, 0.7, 0.9, 0.9999\}$. Fig.2 presents the variation of R^{FBS}_A

We notice that:

- The redistribution strategy converges for the different heuristics
- For all tested values of p, and for any heuristic A, the ratio R_A^{FBS} is almost constant for $n \geq 700$, so we can conclude that this ratio is asymptotically independant from n.
- The FBS heuristic is the most performant one in fact:

$$R_{Redist}^{FBS} < R_{Redist}^{FFF} < R_{Redist}^{FFL} < R_{Redist}^{KLM} < R_{Redist}^{NFL}$$

Since the redistribution strategy according to FBS seems to be the most efficient one, we present in the following figure 3 the variation of the ratio R_{Redist}^{FBS} when p varies and when n takes a large value, here $n = 1000$.

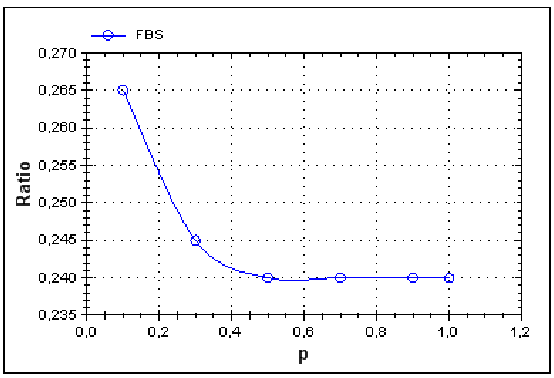

Fig. 3. Variation of R_{Redist}^{FBS} for $n = 1000$

The figure 3 also shows that this same ratio computed for the redistribution strategy according to FBS converges toward a constant and this independently from the values of n and p. This implies that for any probability p for being present, the packing obtained by redistribution strategy is asymptotically equivalent to $O(np)$

We notice that the ratio so calculated, for the heuristic **FBS** diverges at most about 6% from the re-optimization strategy.

Therefore, we can conclude that under assumption, the redistribution strategy according to FBS is asymptotic and the re-optimization strategy are asymptotically equivalent.

IV. Conclusion

This paper has suggested an average case behavior of the redistribution strategy for the probabilistic bin packing problem where to sorts of randomness was considered : the size of the items to be packed and their presence. We tried to evaluate how well the redistribution strategy can approximate the re-optimization strategy, and through an experimental study we can conclude that the redistribution strategy according to some heuristics is asymptotic to the re-optimization strategy which is impossible to carried out in the practice.

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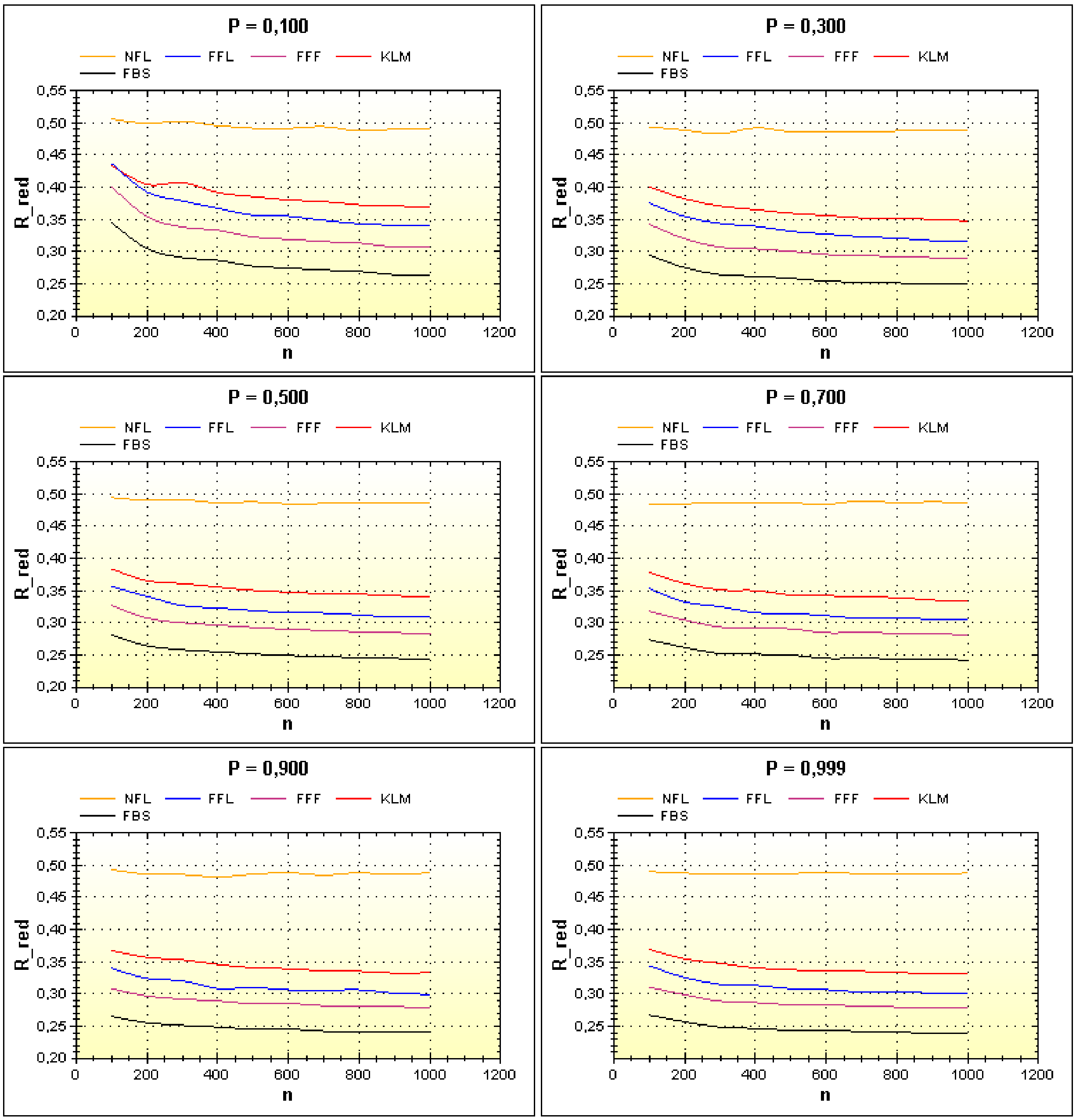

Fig. 2. Redistribution Ratio