The analysis of the transient dynamic response of elastic thin-walled beams of open section via the ray method *

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Abstract – The problem on the normal impact of an elastic rod with a rounded end upon an elastic Timoshenko arbitrary cross section thin-walled beam of open section is considered. The process of impact is accompanied by the dynamic flexure and torsion of the beam, resulting in the propagation of plane flexural-warping and torsional-shear waves of strong discontinuity along the beam axis. Behind the wave fronts upto the boundaries of the contact region, the solution is constructed in terms of one-term ray expansions. During the impact the rod moves under the action of the contact force which is determined due to the Hertz's theory, while the contact region moves under the attraction of the contact force, as well as the twisting and bending-torsional moments and transverse forces, which are applied to the lateral surfaces of the contact region. The procedure proposed allows one to obtain rather simple relationship for estimating the maximal magnitude of the contact force, which can be very useful in engineering applications.

Keywords – Normal impact, ray method, thin-walled beam of open section, transient waves.

1 Introduction

It seems likely that Crook [1] pioneered the application of the wave approach in the theory of impact when considering the longitudinal impact of an elastic sphere against the end of a thin elastic bar. As this takes place, the deformation of the bar's material in the contact region was considered through the use of the Hertz's contact theory; but in the vicinity of the contact region, it was taken into account using one-term ray expansions constructed behind the longitudinal wave front. The problem was reduced to the solution of the nonlinear integro-differential equation in the contact force, whose numerical integration allowed the author to determine the time dependence of the contact force and the dynamic stress in the bar.

The same approach was used by Rossikhin and Shitikova [2,3] for investigating the transverse impact of an elastic bar and sphere upon an Uflyand-Mindlin plate [4,5]. The material local bearing dependence of the force has been defined on a basis of quasi-static analysis; however, in this problem, a major portion of energy transformed into energy of the nonstationary transverse shear wave, behind the front of which, upto the boundary of the contact region, the values to be found were constructed in terms of one-term or multiple-term ray expansions. The ray expansions employed allowed to consider reflected waves as well, if these latter had had time to return at the point of the impact prior to the completion of the colliding process. The conditions of matching of the desired values in the contact region and its vicinity, which were to be fulfilled on the boundary of the contact region, permitted to obtain the closed system of equations for determining all characteristics of the shock interaction.

The problem of the response of rods, beams, plates and shells to low velocity impact with the emphasis on the wave theories of shock interaction has been reviewed by Rossikhin and Shitikova in [6]. These theories are based on the fact that at the moment of impact transient waves (surfaces of strong discontinuity) are generated within the contact domain, which further propagate along the thin bodies and thereby influence the process of the shock interaction. The desired functions behind the strong discontinuity surfaces are found in terms of one-term, two-term or multiple-term ray expansions, the coefficients of which are determined with an accuracy of arbitrary functions from a set of equations describing the dynamic behavior of the thin body. On the contact domain boundary, the ray expansions for the desired functions go over into the truncated power series with respect to time and are matched further with the desired functions within the contact region that are represented by the truncated power series with respect to time with uncertain coefficients. As a result of such a procedure, it has been possible to determine all characteristics of shock interaction and, among these, to find the time dependence of the contact force and the displacements of the contact region.

The procedure proposed in [3] for investigating the transverse impact upon a plate has been generalized to the case of the shock interaction of an elastic Timoshenko thin-walled beam of open section with an elastic sphere [7]. It has been revealed that the impact upon a thin-walled beam has its own special features. First, the transverse deformation in the contact region of colliding bodies may be so large that can result in the origination of longitudinal shock

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waves. Second, the deflection of the beam in the place of contact may be so large that one is led to consider the projection of the membrane contractive (tensile) forces onto the normal to the beam's median surface in the place of contact.

In the present paper, this approach is generalized for the analysis of the thin-walled open section beam response to the impact by a thin long elastic rod with a rounded end.

2 The Engineering Theories of Thin-Walled Rods of Open Section

Thin-walled beams of open section are extensively used as structural components in different structures in civil, mechanical and aeronautical engineering fields. These structures have to resist dynamic loads such as wind, traffic and earthquake loadings, so that the understanding of the dynamic behavior of the structures becomes increasingly important. Ship hulls are also can be modelled as thin-walled girders during investigation of hydroelastic response of large container ships in waves.

The classical engineering theory of thin-walled uniform open cross-section straight beams as well as horizontally curved ones was developed by Vlasov [8] in the early 60-s without due account for rotational inertia and transverse shear deformations [9]. The Vlasov theory is the generalization of the Bernoulli-Navier law to the thin-walled open section beams by including the sectorial warping of the section into account by the law of sectorial ares, providing that the first derivative of the torsion angle with respect to the longitudinal axis serves as a measure of the warping of the section. Thus, this theory results in the four differential equations of free vibrations of a thin-walled beam with an open inflexible section contour of arbitrary shape. For the case of a straight beam, the first second-order equation determines, independently of the other three and together with the initial and boundary conditions, the longitudinal vibrations of the beam. The remaining three fourth-order differential equations form a symmetrical system which, together with the initial and boundary conditions determines the transverse flexural-torsional vibrations of the beam (see page 388 in [8]). In the case of a curved beam, all four equations are coupled. However, as it will be shown later on, Vlasov's equations are inappropriate for use in the problems dealing with the transient wave propagation.

Many researchers have tried to modify the Vlasov theory for dynamic analysis of elastic isotropic thin-walled beams with uniform cross-section by including into consideration the rotary inertia and/or transverse shear deformations [10]–[25].

It is well known that Timoshenko [26] in order to generalize the Bernoulli-Euler beam model has introduced two distinct functions, namely: the deflection of the centroid of the cross-section and the rotation of the normal to the cross-section through the centroid, i.e., he considered the transverse shear angle to be the independent variable. This starting point was the basis for the derivation of a set of two hyperbolic differential equations describing the dynamic behavior of a beam, resulting in the fact that two transient waves propagate in the Timoshenko beam with finite velocities: the longitudinal wave with the velocity equal to $G_L = \sqrt{E/\rho}$, and the wave of transverse shear with the velocity equal to $G_T = \sqrt{K\mu/\rho}$, where E and μ are the elastic moduli, ρ is the density, and K is the shear coefficient which is weakly dependent on the geometry of the beam [27].

Many of the up-to-date technical articles involve the derivation of the equations which, from the authors viewpoint, should describe the dynamic behavior of *thin-walled beams of the Timoshenko type* [16]–[25]. Moreover, practically in each such paper it is written that such equations are novel, and no analogs were available previously in scientific literature [17], [20], [22]–[24].

All papers in the field can be divided into three groups. The papers, wherein the governing set of equations is both hyperbolic and correct from the viewpoint of the physically admissible magnitudes of the velocities of the transient waves resulting from these equations, fall into the first category, i.e., the velocity of the longitudinal wave is $G_L = \sqrt{E/\rho}$, while the velocities of the three transverse shear waves, in the general case of arbitrary cross sections of thin-walled beams with open profile, depend essentially of the geometry of the open section beam [13], [15], [17]. There are seven independent unknowns in the displacement field in the general case if only primary warping is included into consideration [13], or with additional three generalized displacements describing the variation of the secondary warping due to non-uniform bending and torsion [17], or with additional three variables describing a "complete homogeneous deformation of the microstructure" [15]. As this takes place, different authors obtain different magnitudes for the velocities of transverse shear waves.

The second category involves the articles presenting hyperbolic but incorrect equations from the above mentioned viewpoint, i.e., resulting in incorrect magnitudes of the transient waves. This concerns, first of all, the velocity of the longitudinal waves which should not deviate from $G_L = \sqrt{E/\rho}$, nevertheless, there are some examples [11] where such a situation takes place. Secondly, in some papers one can find equations looking like hyperbolic ones [14], [18], [24], [25] but from which it is impossible to obtain the velocity, at least, of one transient wave at all. In such papers, usually six generalized displacements are independent (for monosymmetric cross sections they are four, and two in the case of bisymmetric profiles) while warping is assumed to be dependent on the derivative of the torsional rotation with respect to the beam axial coordinate [24], [25] or is neglected in the analysis [14], [18]. In other words, there is a hybrid of two approaches: Timoshenko's beam theory [26] and Vlasov's thin-walled beam theory [8], some times resulting to a set of equations wherein some of them are hyperbolic, while others are not. Thirdly, not all inertia terms are included into consideration.

The papers providing the governing system of equations which are not hyperbolic belong to the third group [20], [21], [22]. In such papers, the waves of transverse shear belong to the diffusion waves possessing infinitely large velocities, and therefore, from our point of view, the dynamic equations presented in [20], [21], and [22] cannot be named as the Timoshenko type equations.

Checking for the category, within which this or that paper falls in, is carried out rather easily if one uses the following reasoning.

Suppose that the given governing set of equations is the hyperbolic one. Then as a result of non-stationary excitations on a beam, transient waves in the form of surfaces of strong or weak discontinuity are generated in this beam. We shall interpret the wave surface as a limiting layer with the thickness h, inside of which the desired field Z changes monotonically and continuously from the magnitude Z_+ to the magnitude Z_-. Now we can differentiate the set of equations n times with respect to time t, then rewrite it inside the layer, and change all time-derivatives by the derivatives with respect to the axial coordinate z using the one-dimensional condition of compatibility (see Appendix A)

$$\begin{aligned}(-1)^n Z_{,(n)} &= G^n \frac{\partial^n Z}{\partial z^n} \\ &+ \sum_{m=0}^{n-1} (-1)^{m+1} \frac{n!}{m!(n-m)!} \frac{\delta^{n-m} Z_{,(m)}}{\delta t^{n-m}},\end{aligned} \tag{1}$$

where G is the normal velocity of the limiting layer, $\delta/\delta t$ is the Thomas δ -derivative [28], and $Z_{,(k)} = \partial^k Z/\partial t^k$.

Integrating the resulting equations n times with respect to z, where n is the order of the highest z-derivative, writing the net equations at $z = -h/2$ and $z = h/2$, and taking their difference, we are led at $h \to 0$ to the relationships which involve the discontinuities in the desired field $[Z] = Z_+ - Z_-$ and which are used for determining the velocities of the transient waves, i.e., the magnitude of G, what allows one to clarify the type of the given equations.

If the values entering in the governing equations could not experience the discontinuity during transition through the wave surface, generalized displacements as an example, then in this case the governing set of equations should be differentiated one time with respect to time in order to substitute the generalized displacements by their velocities. Thus, after the procedure described above, the governing equations will involve not the discontinuities in the desired values Z but the discontinuities in their time-derivatives, i.e., $[\dot{Z}] = (\partial Z/\partial t)_+ - (\partial Z/\partial t)_-$.

Using the procedure described above, it can be shown that the correct hyperbolic set of equations taking shear deformation due to bending and coupled bending torsion was suggested by Aggarwal and Cranch [12], but their theory is strictly applied only to a channel-section beam.

It seems likely that for a straight elastic thin-walled beam with a generic open section this problem was pioneered in 1974 by Korbut and Lazarev [13], who generalized the Vlasov theory by adopting the assumptions proposed in 1949 by Gol'denveizer [29] that the angles of in-plane rotation do not coincide with the first derivatives of the lateral displacement components and, analogously, warping does not coincide with the first derivative of the torsional rotation. It should be emphasized that it was precisely Gol'denveizer [29] who pioneered in combining Timoshenko's beam theory [26] and Vlasov thin-walled beam theory [8] (note that the first edition of Vlasov's book was published in Moscow in 1940) and who suggested to characterize the displacements of the thin-walled beam's cross-section by seven generalized displacements. It is interesting to note that the approach proposed by Gol'denveizer [29] for solving static problems (which has being widely used by Russian researchers and engineers since 1949) was re-discovered approximately 50 years later by Back and Will [30], who have inserted it in finite element codes.

The set of seven second-order differential equations with due account for rotational inertia and transverse shear deformations derived in [13] using the Reissner's variational principle really describes the dynamic behavior of a straight beam of the Timoshenko type and has the following form:
the equations of motion

$$\begin{aligned}\rho I_x \dot{\mathrm{B}}_x - M_{x,z} + Q_{y\omega} &= 0,\\ \rho I_y \dot{\mathrm{B}}_y - M_{y,z} - Q_{x\omega} &= 0,\\ \rho I_\omega \dot{\Psi} - \mathrm{B}_{,z} - Q_{xy} &= 0,\\ \rho F \dot{v}_z - N_{,z} &= 0,\end{aligned} \tag{2}$$

$$\begin{aligned}\rho F \dot{v}_x &+ \rho a_y F \dot{\Phi} - Q_{x\omega,z} = 0,\\ \rho F \dot{v}_y &- \rho a_x F \dot{\Phi} - Q_{y\omega,z} = 0,\\ \rho I_p \dot{\Phi} &+ \rho a_y F \dot{v}_x - \rho a_x F \dot{v}_y - (Q_{xy} + H)_{,z} = 0;\end{aligned} \tag{3}$$

the generalized Hook's law

$$\begin{aligned}\dot{M}_x &= EI_x \mathrm{B}_{x,z}\,, \quad \dot{M}_y = EI_y \mathrm{B}_{y,z},\\ \dot{\mathrm{B}} &= EI_\omega \Psi_{,z}\,, \quad \dot{N} = EF v_{z,z},\end{aligned} \tag{4}$$

$$\begin{aligned}\mu(v_{x,z} - \mathrm{B}_y) &= k_y \dot{Q}_{x\omega} + k_{xy} \dot{Q}_{y\omega} + k_{y\omega} \dot{Q}_{xy},\\ \mu(v_{y,z} + \mathrm{B}_x) &= k_{xy} \dot{Q}_{x\omega} + k_x \dot{Q}_{y\omega} + k_{x\omega} \dot{Q}_{xy},\\ \mu(\Phi_{,z} - \Psi) &= k_{y\omega} \dot{Q}_{x\omega} + k_{x\omega} \dot{Q}_{y\omega} + k_\omega \dot{Q}_{xy},\\ \dot{H} &= \mu I_k \Phi_{,z}\,,\end{aligned} \tag{5}$$

where ρ is the beam's material density, F is the cross-section area, ω is the sectorial coordinate, I_x and I_y are centroidal moments of inertia, I_ω is the sectorial moment of inertia, I_p is the polar moment of inertia about the flexure center A, I_k is the moment of inertia due to pure torsion, a_x and a_y are the coordinates of the flexural center, E and μ are the Young's and shear moduli, respectively, $\mathrm{B}_x = \dot{\beta}_x$, $\mathrm{B}_y = \dot{\beta}_y$, $\Phi = \dot{\varphi}$, β_x, β_y and φ are the angles of rotation of the cross section about x-, y- and z-axes, respectively, $\Psi = \dot{\psi}$, ψ is the warping function, v_x, v_y, v_z are the velocities of displacements of the flexural center, u, v, and w, along the central principal axes x and y and the longitudinal z-axis, respectively, M_x and M_y are the bending moments, B is the bimoment, N is the longitudinal (membrane) force, $Q_{x\omega}$ and $Q_{y\omega}$ are the transverse forces, H is the moment of pure torsion, Q_{xy} is the bending-torsional moment from the axial shear forces acting at a tangent to the contour of the cross section about the flexural center, overdots denote the time derivatives, and the index z after a point defines the derivative with respect to the z-coordinate.

In (2)-(5), k_x, k_y, k_ω, $k_{x\omega}$, $k_{y\omega}$, and k_{xy} are the cross-sectional geometrical characteristics which take shears into consideration:

$$\begin{aligned}
k_x &= \frac{1}{I_x^2}\int_F \frac{S_x^2}{\delta_s^2}dF, \\
k_y &= \frac{1}{I_y^2}\int_F \frac{S_y^2}{\delta_s^2}dF, \\
k_\omega &= \frac{1}{I_\omega^2}\int_F \frac{S_\omega^2}{\delta_s^2}dF, \\
k_{x\omega} &= \frac{1}{I_x I_\omega}\int_F \frac{S_x S_\omega}{\delta_s^2}dF, \\
k_{y\omega} &= \frac{1}{I_y I_\omega}\int_F \frac{S_y S_\omega}{\delta_s^2}dF, \\
k_{xy} &= \frac{1}{I_x I_y}\int_F \frac{S_x S_y}{\delta_s^2}dF,
\end{aligned} \tag{6}$$

where S_x, S_y, and S_ω are the axial and sectorial static moments of the intercepted part of the cross section, and δ_s is the width of the web of the beam.

Note that 25 years later the shear coefficients (6) were re-derived by means of the Reissner principle in [21].

2.1 Velocities of the transient waves propagating in the thin-walled beam of open section due to the Korbut–Lazarev theory and its generalizations

To show that the set of equations (2)-(5) governs three transient shear waves which propagate with the fnite velocities depending on the geometrical characteristics of the thin-walled beam (6), we can use the approach suggested above. If we write (2)-(5) inside the layer and apply the condition of compatibility (1) at $n = 1$, as a result, we find [7]

$$\begin{aligned}
-\rho I_x G[\mathrm{B}_x] &- [M_x] = 0, \\
-\rho I_y G[\mathrm{B}_y] &- [M_y] = 0, \\
-\rho I_\omega G[\Psi] &- [\mathrm{B}] = 0, \\
-\rho F G[v_z] &- [N] = 0,
\end{aligned} \tag{7}$$

$$\begin{aligned}
-\rho F G[v_x] &- \rho F G a_y[\Phi] - [Q_{x\omega}] = 0, \\
-\rho F G[v_y] &+ \rho F G a_x[\Phi] - [Q_{y\omega}] = 0, \\
-\rho I_p G[\Phi] &- \rho F G a_y[v_x] + \rho F G a_x[v_y], \\
&- [Q_{xy}] - [H] = 0,
\end{aligned} \tag{8}$$

$$\begin{aligned}
-G[M_x] &= E I_x[\mathrm{B}_x], \\
-G[M_y] &= E I_y[\mathrm{B}_y], \\
-G[\mathrm{B}] &= E I_\omega[\Psi], \\
-G[N] &= E F[v_z],
\end{aligned} \tag{9}$$

$$\begin{aligned}
\mu[v_x] &= -G k_y[Q_{x\omega}] - G k_{xy}[Q_{y\omega}] - G k_{y\omega}[Q_{xy}], \\
\mu[v_y] &= -G k_{xy}[Q_{x\omega}] - G k_x[Q_{y\omega}] - G k_{x\omega}[Q_{xy}], \\
\mu[\Phi] &= -G k_{y\omega}[Q_{x\omega}] - G k_{x\omega}[Q_{y\omega}] - G k_\omega[Q_{xy}], \\
-G[H] &= \mu I_k[\Phi].
\end{aligned} \tag{10}$$

Eliminating the values $[M_x]$, $[M_y]$, [B] and $[N]$ from (7) and (9), we obtain the velocity of the longitudinal-flexural-warping wave

$$G_4 = \sqrt{E\rho^{-1}}\,, \tag{11}$$

on which $[\mathrm{B}_x] \neq 0$, $[\mathrm{B}_y] \neq 0$, $[\Psi] \neq 0$, and $[v_z] \neq 0$, while $[v_x] = [v_y] = [\Phi] = 0$.

Eliminating the values $[Q_{x\omega}]$, $[Q_{y\omega}]$, $[Q_{xy}]$, and $[H]$ from (8) and (10), we arrive at the system of three linear homogeneous equations:

$$\sum_{j=1}^{3} a_{ij}[v_j] = 0 \quad (i, j = 1, 2, 3)\,, \tag{12}$$

where $[v_1] = [v_x]$, $[v_2] = [v_y]$, $[v_3] = [\Phi]$,

$$\begin{aligned}
a_{11} &= \rho F G^2(k_y + a_y k_{y\omega}) - \mu, \\
a_{12} &= \rho F G^2(k_{xy} - a_x k_{y\omega}), \\
a_{13} &= \rho F G^2(a_y k_y - a_x k_{xy}) + k_{y\omega}(\rho I_p G^2 - \mu I_k), \\
a_{21} &= \rho F G^2(k_{xy} + a_y k_{x\omega}), \\
a_{22} &= \rho F G^2(k_x - a_x k_{x\omega}) - \mu, \\
a_{23} &= \rho F G^2(a_y k_{xy} - a_x k_x) + k_{x\omega}(\rho I_p G^2 - \mu I_k), \\
a_{31} &= \rho F G^2(k_{y\omega} + a_y k_\omega), \\
a_{32} &= \rho F G^2(k_{x\omega} - a_x k_\omega), \\
a_{33} &= \rho F G^2(a_y k_{y\omega} - a_x k_{x\omega}) \\
&+ k_\omega(\rho I_p G^2 - \mu I_k) - \mu.
\end{aligned}$$

Setting determinant of the set of equations (12) equal to zero

$$|a_{ij}| = 0\,, \tag{13}$$

we are led to the cubic equation governing the velocities G_1, G_2, and G_3 of three twisting-shear waves, on which only the values $[v_x]$, $[v_y]$ and $[\Phi]$ are nonzero such that

$$[v_x] = \gamma[\Phi], \quad [v_y] = \delta[\Phi]\,, \tag{14}$$

where

$$\gamma = \frac{a_{23}a_{12} - a_{13}a_{22}}{a_{11}a_{22} - a_{12}a_{21}}, \quad \delta = \frac{a_{13}a_{21} - a_{23}a_{11}}{a_{11}a_{22} - a_{12}a_{21}}.$$

For the bisymmetrical beam, the values a_x, a_y, k_{xy}, $k_{x\omega}$, and $k_{y\omega}$ vanish. In this case, the set of (13) becomes the three independent equations defining the velocities of two shear waves [7]

$$G_1 = \sqrt{\frac{\mu}{\rho F k_y}}, \quad G_2 = \sqrt{\frac{\mu}{\rho F k_x}}, \tag{15}$$

and one twisting wave

$$G_3 = \sqrt{\frac{\mu(1 + k_\omega I_k)}{\rho I_p k_\omega}}, \tag{16}$$

on which $[v_x]$, $[v_y]$, and $[\Phi]$ are nonzero, respectively.

It is strange to the authors of this paper that the Korbut and Lazarev theory [13] appeared in 1974 is absolutely unaware to the international mechanics community, in spite of the fact that it was published in the Soviet academic journal which is available in English due to translation made by Springer.

The Korbut–Lazarev theory [13], which provides the physically admissible velocities of propagation of transient waves, was generalized in [7] taking the extension of the thin-walled beam's middle surface into account.

Nine years later after the appearance of [13], Muller [15] suggested the theory (which generalized the Korbut–Lazarev approach [13]), wherein the additional deformations of two lateral contractions and the so-called effect of distortion shear were taken into consideration. This allowed the author to receive correctly the velocity of the longitudinal-flexural-warping wave (11), three velocities of the transverse shear waves due to coupled flexural translational-torsional motions, which strongly depend of the geometry of the beam's cross section as in the case of (13) defined by the Korbut-Lazarev theory [13], and the wave of pure shear due to lateral distortion deformation, which propagates with the velocity $G_T = \sqrt{\mu/\rho}$.

One more example of the correct generalization of the Timoshenko beam model to an open section thin-walled beam is the approach proposed in [16] and [17] in the early 90-s. Once again it is the generalization of the Korbut-Lazarev theory [13], since three additional deformations describing the secondary warping due to nonuniform bending and torsion are taken into account. The hyperbolic set of ten equations presented in [16] allows one to obtain the velocity of longitudinal-flexural-warping wave (11), and three velocities of the transverse shear waves due to coupled flexural translational-torsional motions similar to (13). As this takes place, the found shear constants (see relationships (45) in [17]) coincide completely with those of (6).

The presence of three [7], [13], [17], or four [15] transverse shear waves, which propagate with different velocities dependent strongly on geometric characteristics of the thin-walled beam, severely limits the application of such theories in solving engineering problems. As for the experimental verification of the existence of the three shear waves in thin-walled beams of open section, then it appears to be hampered by the fact that the velocities of these waves depend on the choice of the beam's cross section.

2.2 Velocities of the transient waves propagating in the thin-walled beam of open section due to the Vlasov theory and its modifications

Note that only inclusion into consideration of three factors, namely: shear deformations, rotary inertia, and warping deformations as the independent field - could lead to the correct system of hyperbolic equations of the Timoshenko type for describing the dynamic behaviour of thin bodies. Ignoring one of the factors or its incomplete account immediately results in an incorrect set of governing equations.

Let us consider, as an example, the dynamic equations suggested by Vlasov (see (1.8) in page 388 in [8]) to describe the behaviour of thin-walled straight beams of open profile:

$$\begin{aligned}
&EF\frac{\partial^2\zeta}{\partial z^2} - \rho F\frac{\partial^2\zeta}{\partial t^2} = 0, \qquad (17)\\
&EI_y\frac{\partial^4\xi}{\partial z^4} - \rho I_y\frac{\partial^4\xi}{\partial z^2\partial t^2} + \rho F\frac{\partial^2\xi}{\partial t^2} + a_y\rho F\frac{\partial^2\theta}{\partial t^2} = 0,\\
&EI_x\frac{\partial^4\eta}{\partial z^4} - \rho I_x\frac{\partial^4\eta}{\partial z^2\partial t^2} + \rho F\frac{\partial^2\eta}{\partial t^2} - a_x\rho F\frac{\partial^2\theta}{\partial t^2} = 0,\\
&EI_\omega\frac{\partial^4\theta}{\partial z^4} - \mu I_k\frac{\partial^2\theta}{\partial z^2} - \rho I_\omega\frac{\partial^4\theta}{\partial z^2\partial t^2} + \rho I_p\frac{\partial^2\theta}{\partial t^2}\\
&+a_y\rho F\frac{\partial^2\xi}{\partial t^2} - a_x\rho F\frac{\partial^2\eta}{\partial t^2} = 0,
\end{aligned}$$

which was obtained with due account for the rotary inertia but neglecting the shear deformations, where z is the beam's longitudinal axis.

If we differentiate all equations in (17) one time with respect to time, and then apply to them the suggested above procedure, as a result we obtain

$$\begin{aligned}
(\rho G^2 - E)[\dot\zeta] &= 0,\\
(\rho G^2 - E)[\dot\xi] &= 0,\\
(\rho G^2 - E)[\dot\eta] &= 0,\\
(\rho G^2 - E)[\dot\theta] &= 0.
\end{aligned} \tag{18}$$

Reference to (18) shows that on the transient longitudinal wave of strong discontinuity propagating with the velocity $G_L = \sqrt{E/\rho}$, not only the velocity of longitudinal displacement ζ experiences discontinuity but the velocities of transverse displacements ξ and η as well, what is characteristic for the transient transverse shear wave of strong discontinuity. Therefore, the set of equations (17) could not be considered as a correct hyperbolic set of equations. In other words, the values connected with the phenomenon of shear propagate with the velocity G_L, what falls into contradiction with the physical sense, and thus the Vlasov theory is applicable only for the static problems.

Note that for a rod of a massive cross-section the account only for the rotary inertia was made for the first time by Lord Rayleigh in his *Theory of Sound* in the form of a mixed derivative of the displacement with respect to time and coordinate.

If we exclude from (17) the terms responsible for the rotary inertia, i.e., $\rho I_y \frac{\partial^4 \xi}{\partial z^2 \partial t^2}$, $\rho I_x \frac{\partial^4 \eta}{\partial z^2 \partial t^2}$, and $\rho I_\omega \frac{\partial^4 \theta}{\partial z^2 \partial t^2}$, then we obtain the equations describing the dynamic behaviour of the Bernoulli-Euler beams. In such beams, the velocity of the propagation of the transient transverse shear wave of strong discontinuity is equal to infinity.

The second example is not mere expressive. Let us consider the set of equations suggested by Meshcherjakov [11] for describing the straight thin-walled beam of open bisymmetric profile

$$\begin{aligned} EI_y \frac{\partial^4 \beta_y}{\partial z^4} \quad &- \quad \rho I_y \frac{\partial^4 \beta_y}{\partial z^2 \partial t^2} + \rho F \frac{\partial^2 \beta_y}{\partial t^2} \\ &+ \quad 2(1+\nu) \frac{S_{xx}}{I_y} \rho F \frac{\partial^4 \beta_y}{\partial z^2 \partial t^2} = 0, \\ EI_x \frac{\partial^4 \beta_x}{\partial z^4} \quad &- \quad \rho I_x \frac{\partial^4 \beta_x}{\partial z^2 \partial t^2} + \rho F \frac{\partial^2 \beta_x}{\partial t^2} \\ &+ \quad 2(1+\nu) \frac{S_{yy}}{I_x} \rho F \frac{\partial^4 \beta_x}{\partial z^2 \partial t^2} = 0, \\ EI_\omega \frac{\partial^4 \psi}{\partial z^4} \quad &- \quad \rho I_\omega \frac{\partial^4 \beta_x}{\partial z^2 \partial t^2} - \rho I_k \frac{\partial^2 \psi}{\partial z^2} \\ &+ \quad \rho I_p \left[\frac{\partial^2 \psi}{\partial t^2} + 2(1+\nu) \frac{S_{\omega\omega}}{I_\omega} \frac{\partial^4 \psi}{\partial z^2 \partial t^2} \right] = 0, \end{aligned} \tag{19}$$

where S_{xx}, S_{yy}, and $S_{\omega\omega}$ are shear coefficients [11], and ν is the Poisson's ratio.

If we differentiate all equations from (19) one time with respect to time t and then apply to them the procedure described above, as a result we obtain

$$\begin{aligned} I_y \left\{ E - \rho G^2 \left[1 - 2(1+\nu) \frac{S_{xx}}{I_y^2} F \right] \right\} [\dot{\beta}_y] &= 0, \\ I_x \left\{ E - \rho G^2 \left[1 - 2(1+\nu) \frac{S_{yy}}{I_x^2} F \right] \right\} [\dot{\beta}_x] &= 0, \\ I_\omega \left\{ E - \rho G^2 \left[1 - 2(1+\nu) \frac{S_{\omega\omega}}{I_\omega^2} I_p \right] \right\} [\dot{\psi}] &= 0. \end{aligned} \tag{20}$$

Reference to (20) shows that absolutely absurd velocities of three transient longitudinal waves of strong discontinuity

$$\begin{aligned} G_1 &= \sqrt{E\rho^{-1} \left[1 - 2(1+\nu) \frac{S_{xx}}{I_y^2} F \right]^{-1}}, \\ G_2 &= \sqrt{E\rho^{-1} \left[1 - 2(1+\nu) \frac{S_{yy}}{I_x^2} F \right]^{-1}}, \\ G_3 &= \sqrt{E\rho^{-1} \left[1 - 2(1+\nu) \frac{S_{\omega\omega}}{I_\omega^2} I_p \right]^{-1}} \end{aligned} \tag{21}$$

are obtained.

If the author of [11] considered sequentially the rotary inertia, as it was done by S.P. Timoshenko in *Vibration Problems in Engineering* [26], then the additional terms

$$\begin{aligned} &-2(1+\nu) \frac{S_{xx}}{I_y} \rho F \, \rho E^{-1} \frac{\partial^4 \beta_y}{\partial t^4}, \\ &-2(1+\nu) \frac{S_{yy}}{I_x} \rho F \, \rho E^{-1} \frac{\partial^4 \beta_x}{\partial t^4}, \\ &-2(1+\nu) \frac{S_{\omega\omega}}{I_\omega} \rho I_p \, \rho E^{-1} \frac{\partial^4 \psi}{\partial t^4}, \end{aligned} \tag{22}$$

will enter in (19), which could remedy all velocities of transient longitudinal waves, since the procedure suggested by the authors of the given paper transforms the additional terms (22) to the form

$$\begin{aligned} &-2(1+\nu) \frac{S_{xx}}{I_y} \rho F \, \rho E^{-1} G^4 [\dot{\beta}_y], \\ &-2(1+\nu) \frac{S_{yy}}{I_x} \rho F \, \rho E^{-1} G^4 [\dot{\beta}_x], \\ &-2(1+\nu) \frac{S_{\omega\omega}}{I_\omega} \rho I_p \, \rho E^{-1} G^4 [\dot{\psi}]. \end{aligned} \tag{23}$$

Relationships (23) will be added, respectively, in (20), and will transform them, in their turn, to the form

$$\begin{aligned} (E - \rho G^2) \left[1 + 2(1+\nu) \frac{S_{xx}}{I_y^2} EF\rho G^2 \right] [\dot{\beta}_y] = 0, \\ (E - \rho G^2) \left[1 + 2(1+\nu) \frac{S_{yy}}{I_x^2} EF\rho G^2 \right] [\dot{\beta}_x] = 0, \\ (E - \rho G^2) \left[1 + 2(1+\nu) \frac{S_{\omega\omega}}{I_\omega^2} EI_p\rho G^2 \right] [\dot{\psi}] = 0, \end{aligned} \tag{24}$$

whence it follows that the velocity of the longitudinal wave of strong discontinuity is equal to $G_L = \sqrt{E/\rho}$, what matches to the reality.

Moreover, a reader could find such papers in the field which are apparently false. Thus, the following set of equations is presented in [24] (it is written below in the notation adopted in this paper for convenience):

$$\begin{aligned} &EFw'' - \rho F\ddot{w} = 0, \\ &k_x \mu F(u'' - \beta_y') - \rho F\ddot{u} - \rho a_y F\ddot{\varphi} = 0, \\ &k_y \mu F(v'' + \beta_x') - \rho F\ddot{v} + \rho a_x F\ddot{\varphi} = 0, \end{aligned}$$

$$EI_x\beta_x'' - k_y\mu F(v' + \beta_x) - \rho I_x\ddot{\beta}_x = 0, \quad (25)$$
$$EI_y\beta_y'' + k_x\mu F(u' - \beta_y) - \rho I_y\ddot{\beta}_y = 0,$$
$$EI_\omega\varphi'''' - \mu I_k\varphi'' - \rho I_\omega\ddot{\varphi}'' + \rho a_y F\ddot{u}$$
$$-\rho a_x F\ddot{v} + \rho I_p\ddot{\varphi} = 0,$$

where primes denote derivatives with respect to the coordinate z, and k_x and k_y are the shear correction factors in principal planes [24].

The author of [24] has declared that the system of equations (25) is responsible for describing the transverse shear deformations and rotary inertia in a thin-walled beam of open profile, that is to describe the dynamic response of a Timoshenko-like beam.

But this set of equations is not even correct one, and thus it could not describe the dynamic behaviour of the thin-walled Timoshenko-like beam. Really, applying the procedure proposed above, we can rewrite (25) in terms of discontinuities

$$EF[v_z] - \rho G^2 F[v_z] = 0,$$
$$k_x\mu F[v_x] - \rho G^2 F[v_x] - \rho G^2 a_y F[\Phi] = 0,$$
$$k_y\mu F[v_y] - \rho G^2 F[v_y] + \rho G^2 a_x F[\Phi] = 0,$$
$$EI_x[B_x] - \rho G^2 I_x[B_x] = 0, \quad (26)$$
$$EI_y[B_y] - \rho G^2 I_y[B_y] = 0,$$
$$EI_\omega[\Phi] - \rho G^2 I_\omega[\Phi] = 0.$$

From (26) it follows that when $[v_z] \neq 0$, $[B_x] \neq 0$, and $[B_y] \neq 0$, i.e., on the longitudinal wave, the velocity G is equal to the velocity of the longitudinal wave $\sqrt{E/\rho}$. Furthermore, on the longitudinal wave, the discontinuity $[\Phi]$ is also distinct from zero, while the value $[\Phi]$ should be nonzero only on the transverse wave. Moreover, the velocity of the transverse shear wave could not be obtained from the second and third equations of (26) at all.

The contradiction obtained points to the fact that (25) is the incorrect system of equations, and nobody, including the author of [24], knows what phenomenon is described by these equations.

3 The Response of a Thin-Walled Beam of Open Section to the Normal Impact of a Rod

Based on the aforesaid it can be deduced that the Korbut-Lazarev theory [13] is the most acceptable for engineering applications from the physical viewpoint, since it gives the physically admissible velocities of propagation of transient waves. Below we shall use this theory for analyzing the impact response of a thin-walled straight beam of open profile. This boundary-value problem has been chosen for consideration in the given paper by no means accidentally.

The matter is fact that during the past two decades foreign object impact damage to structures has received a great deal of attention, since thin-walled structures are known to be susceptible to damage resulting from accidental impact by foreign objects. Impact on aircraft structures or civil engineering structures, for instance, from dropped tools, hail, and debris thrown up from the runway, poses a problem of great concern to designers. Since the impact response is not purely a function of materials properties and depends also on the dynamic structural behavior of a target, it is important to have a basic understanding of the structural response and how it is affected by different parameters [6]. From this point of view, analytical models are useful as they allow systematic parametric investigation and provide a foundation for prediction of impact damage.

It should be noted that except paper [7], these authors have found in literature only one paper by Taiwanese researchers Lin et al. [31] suggesting a numerical approach to determining the transient response of nonrectangular bars subjected to transverse elastic impact. To our great surprise, this paper is free from any formulas, although it is devoted to ‘transverse impact response’ of straight thin-walled beams with channel and tee profiles. The results obtained in [31] via finite element method (but it is impossible to understand what theory was adopted during solution, as well as what numerical algorithms were implemented) were compared graphically via numerous figures with experimental data obtained by the same authors themselves. As this takes place, only longitudinal waves were taken into account. But numerous data on impact analysis of structures [6] shows that during transverse impact the transverse forces and, thus, the shear waves predominate in the wave phenomena. That is why, despite the fact that the authors of the cited paper [31] declared the good agreement between their numerical and experimental investigations, it is hard to believe in such perfect matching.

Thus, let us consider the normal impact of an elastic thin rod of circular cross section upon a lateral surface of a thin-walled elastic beam of open section (Fig. 1), the dynamic behavior of which is described by system (2)–(5). At the moment of impact, the velocity of the impacting rod is equal to V_0, and the longitudinal shock wave begins to propagate along the rod with the velocity $G_0 = \sqrt{E_0\rho_0^{-1}}$, where E_0 is its elastic modulus, and ρ_0 is its density. Behind the wave front the stress σ^- and velocity v^- fields can be represented using the ray series [32]

$$\sigma^- = -\sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k\sigma}{\partial t^k}\right]\left(t - \frac{n}{G_0}\right)^k, \quad (27)$$

$$v^- = V_0 - \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k v}{\partial t^k}\right]\left(t - \frac{n}{G_0}\right)^k, \quad (28)$$

where n is the coordinate directed along the rod's axis with the origin in the place of contact (Fig. 1).

Considering that the discontinuities in the elastic rod remain constant during the process of the wave propagation

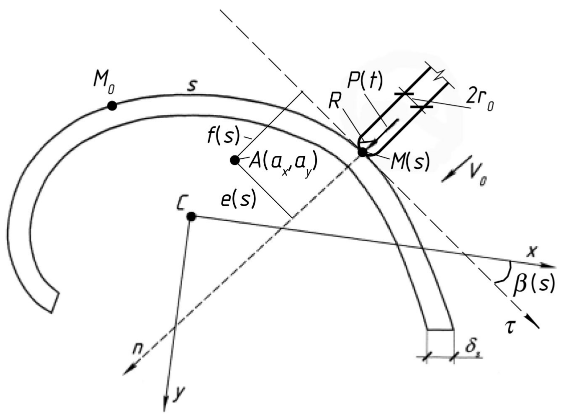


Figure 1: Scheme of shock interaction

and utilizing the condition of compatibility, we have

$$\left[\frac{\partial^{k+1}u}{\partial n \partial t^k}\right] = -G_0^{-1}\left[\frac{\partial^{k+1}u}{\partial t^{k+1}}\right] = -G_0^{-1}\left[\frac{\partial^k v}{\partial t^k}\right], \quad (29)$$

where u is the displacement.

With due account of (29) the Hook's law on the wave surface can be rewritten as

$$\left[\frac{\partial^k \sigma}{\partial t^k}\right] = -\rho_0 G_0 \left[\frac{\partial^k v}{\partial t^k}\right]. \quad (30)$$

Substituting (30) in (27) yields

$$\sigma^- = \rho_0 G_0 \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k v}{\partial t^k}\right]\left(t - \frac{n}{G_0}\right)^k. \quad (31)$$

Comparison of relationships (31) and (28) gives

$$\sigma^- = \rho G_0 \left(V_0 - v^-\right). \quad (32)$$

When $n = 0$, expression (32) takes the form

$$\sigma_{cont} = \rho G_0 \left(V_0 - v_\nu\right), \quad (33)$$

where $\sigma_{cont} = \sigma^- |_{n=0}$ is the contact stress, and $v_\nu = v^- |_{n=0}$ is the normal velocity of the beam's points within the contact domain.

Formula (33) allows one to find the contact force

$$P = \pi r_0^2 \rho_0 G_0 \left(V_0 - v_\nu\right), \quad (34)$$

where r_0 is the radius of the rod's cross section.

However, the contact force can be determined not only via (34), but using the Hertz's law as well

$$P = k\alpha^{3/2}, \quad (35)$$

where α is the value governing the local bearing of the target's material during the process of its contact interaction with the impactor.

If we suppose that the end of the rod is rounded with the radius of R, while the lateral surface of the thin-walled beam is flat in the place of contact, then $k = 4\sqrt{R}\Big/3\pi\left(k_1 + k_2\right)$, $k_1 = \left(1 - \nu_0^2\right)\big/\pi E_0$, $k_2 = \left(1 - \nu^2\right)\big/\pi E$, where ν_0 is the Poisson's ratio of the impactor.

Eliminating the force P from (34) and (35), we are led to the equation for determining the value $\alpha\left(t\right)$

$$v_\nu + \frac{k}{\pi r_0^2 \rho_0 G_0}\alpha^{3/2} = V_0. \quad (36)$$

In order to express the velocity v_ν in terms of α, let us analyze the wave processes occurring in the thin-walled beam of open section. At the moment of impact, three plane shock shear waves propagating with the velocities G_1, G_2, and G_3, which are found from (13) in the general case or from (15) and (16) in the case of bisymmetrical cross-section, are generated in the beam, as well as the longitudinal wave of acceleration.

Since the contours of the beam's cross sections remain rigid during the process of impact, then all sections involving by the contact domain form a layer which moves as rigid whole. Let us name it as a contact layer. If we neglect the inertia forces due to the smallness of this layer, then the equations describing its motion take the form

$$2Q_{x\omega} + P\sin\beta\left(s\right) = 0, \quad (37)$$

$$2Q_{y\omega} + P\cos\beta\left(s\right) = 0, \quad (38)$$

$$2\left(Q_{xy} + H\right) + Pe(s) = 0, \quad (39)$$

where $\beta(s)$ is the angle between the $x-$axis and the tangent to the contour at the point M with the $s-$coordinate, and $e(s)$ is the length of the perpendicular erected from the flexural center to the rod's axis.

The values $Q_{x\omega}$, $Q_{y\omega}$, and $Q_{xy} + H$ entering in (37)–(39) are calculated as follows: behind the wave fronts of three plane shear waves upto the boundary planes of the contact layer, the ray series can be constructed [32]. If we restrict ourselves only by the first terms, then it is possible to find them from (10). Considering (14), we obtain the following relationships for the values $Q_{x\omega}$, $Q_{y\omega}$, and $Q_{xy} + H$:

$$2Q_{x\omega} = -\sum_{i=1}^{3} L_i \Phi_i, \quad (40)$$

$$2Q_{y\omega} = -\sum_{i=1}^{3} M_i \Phi_i, \quad (41)$$

$$2\left(Q_{xy} + H\right) = -\sum_{i=1}^{3} d_i \Phi_i, \quad (42)$$

where $L_i = 2\rho F G_i(\gamma_i + a_y)$, $M_i = 2\rho F G_i(\delta_i - a_x)$, and $d_i = 2\rho F G_i(\gamma_i a_y - \delta_i a_x) + 2\rho I_p G_i$. From hereafter the sign [...] indicating the discontinuity in the corresponding value is omitted for the ease of presentation.

Substituting (40)–(42) and (35) in (37)–(39), we have

$$\sum_{i=1}^{3} L_i \Phi_i = k\alpha^{3/2} \sin\beta, \tag{43}$$

$$\sum_{i=1}^{3} M_i \Phi_i = k\alpha^{3/2} \cos\beta, \tag{44}$$

$$\sum_{i=1}^{n} d_i \Phi_i = k\alpha^{3/2} e. \tag{45}$$

Solving (43)–(45), we find

$$\Phi_i = k\alpha^{3/2} \Delta_i \Delta^{-1}, \tag{46}$$

where

$$\Delta = \begin{vmatrix} L_1 & L_2 & L_3 \\ M_1 & M_2 & M_3 \\ d_1 & d_2 & d_3 \end{vmatrix}, \ \Delta_1 = \begin{vmatrix} \sin\beta & L_2 & L_3 \\ \cos\beta & M_2 & M_3 \\ e & d_2 & d_3 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} L_1 & \sin\beta & L_3 \\ M_1 & \cos\beta & M_3 \\ d_1 & e & d_3 \end{vmatrix}, \ \Delta_3 = \begin{vmatrix} L_1 & L_2 & \sin\beta \\ M_1 & M_2 & \cos\beta \\ d_1 & d_2 & e \end{vmatrix}$$

Let us rewrite the relationship for v_ν

$$v_\nu = \dot{\alpha} - v_x \sin\beta(s) + v_y \cos\beta(s) + e(s)\Phi \tag{47}$$

with due account for (14)

$$v_\nu = \dot{\alpha} + \sum_{i=1}^{3} l_i \Phi_i, \tag{48}$$

and then consider (46) in (48)

$$v_\nu = \dot{\alpha} + k\Delta^{-1}\alpha^{3/2} \sum_{i=1}^{3} l_i \Delta_i\,, \tag{49}$$

where $l_i = \delta_i \cos\beta - \gamma_i \sin\beta + e$.

Substituting (49) in (36), we obtain the equation for defining α

$$\dot{\alpha} + \kappa\alpha^{3/2} = V_0\,, \tag{50}$$

where

$$\kappa = k\left(\frac{1}{\pi r_0^2 \rho_0 G_0} + \frac{1}{\Delta}\sum_{i=1}^{3} l_i \Delta_i\right).$$

The maximum deformation $\alpha_{\max}$ is reached at $\dot{\alpha} = 0$ and, due to (50), is equal to

$$\alpha_{\max} = \left(\frac{V_0}{\kappa}\right)^{2/3}. \tag{51}$$

Substitution of (51) in (35) gives us the maximal contact force

$$P_{\max} = kV_0\kappa^{-1}. \tag{52}$$

3.1 Numerical example

As an example, let us consider the impact of a steel thin cylindrical rod of radius $r_0 = 0.5$ cm with one rounded end of $R = 1$ cm upon steel thin-walled beams of open profile with different cross-section: I-beam (Fig. 2a), Z-shape beam (Fig. 2b), and channel beam (Fig. 2c), but with the equal cross-section area and with the following dimensions: $d = 20$ cm, and $\delta_s = \delta = 2$ cm.

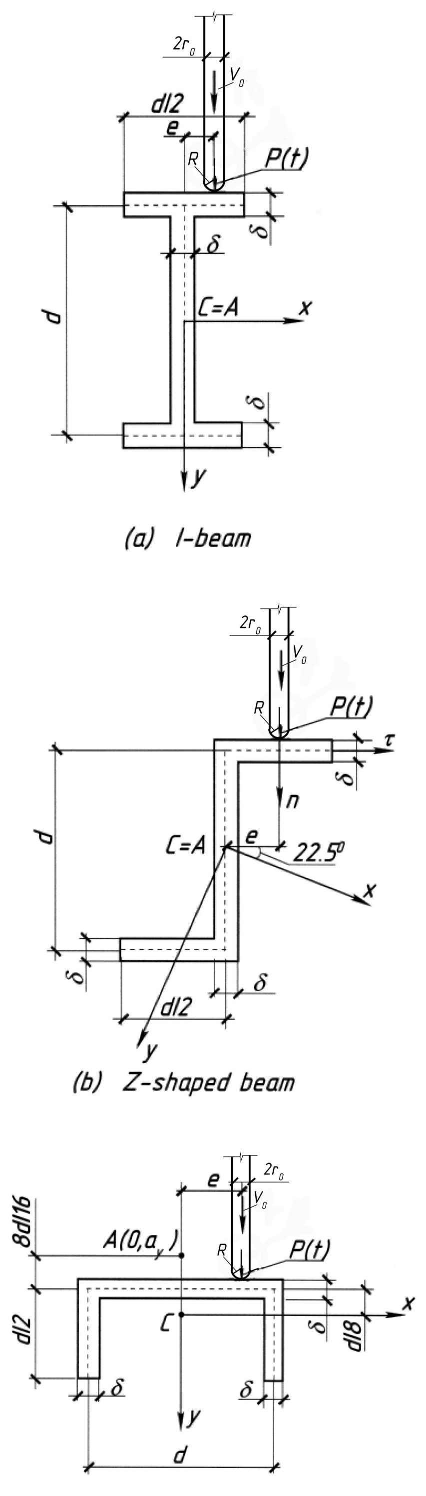
The following characteristics of the material have been adopted: $\rho = 7950$ kg/m^3, $E = 210$ GPa, $\mu = E/2.6$, and $\sigma = 0.3$. The impact occurs at the distance $e = 4$ cm from the flexural center of the thin-walled beam with different initial velocities.

The procedure of determining the geometrical characteristics of the beam cross section with the cross-section area $F = 2d\delta = 0.008\mathrm{m}^2$ is described in detail in [7]. The magnitudes of the shear coefficients calculated by formulas (6) and the wave speed data obtained according to (13) for the beams under consideration are presented in Table 1.

Table 1: Geometrical characteristics and wave velocities

geometrical characteristics and wave velocities	the type of the thin-walled beam cross section		
	I-beam	Z-shape beam	channel
F, m^2	0.008	0.008	0.008
a_x, m	0	0	0
a_y, m	0	0	-0.0665
I_x, m^4	5.33×10^{-5}	6.16×10^{-5}	8.33×10^{-6}
I_y, m^4	3.33×10^{-6}	5.06×10^{-6}	5.33×10^{-5}
I_p, m^4	5.667×10^{-5}	6.667×10^{-5}	9.292×10^{-5}
I_ω, m^6	3.33×10^{-8}	8.33×10^{-8}	5.833×10^{-8}
I_k, m^4	1.067×10^{-6}	1.067×10^{-6}	1.067×10^{-6}
k_x, m^{-2}	265.0	263.0	408.0
k_y, m^{-2}	300.0	257.5	300.0
k_ω, m^{-4}	3.0×10^4	4.08×10^4	3.184×10^4
$k_{x\omega}$, m^{-3}	0	0	0
$k_{y\omega}$, m^{-3}	0	0	964.25
k_{xy}, m^{-2}	0	196.925	0
G_1, m/s	2559.23	1974.24	1873.13
G_2, m/s	2057.48	4478.94	2674.24
G_3, m/s	2189.14	1666.66	1764.27
G_4, m/s	5139.56	5139.56	5139.56

The curves describing the initial velocity of impact V_0



(c)

Figure 2: The scheme of the shock interaction of a thin rod with a thin-walled beam of open profile: (a) I-beam, (b) Z-shape beam, and (c) channel beam.

dependence of the contact duration are given in Fig. 3. Reference to Fig. 3 shows that the duration of contact decreases with increase in the initial velocity of impact. As it takes place, the duration of contact for the I-beam is greater than that for the Z-shaped beam, but the latter, in its turn, is greater than that for the channel beam at common magnitudes of the initial velocity of impact.

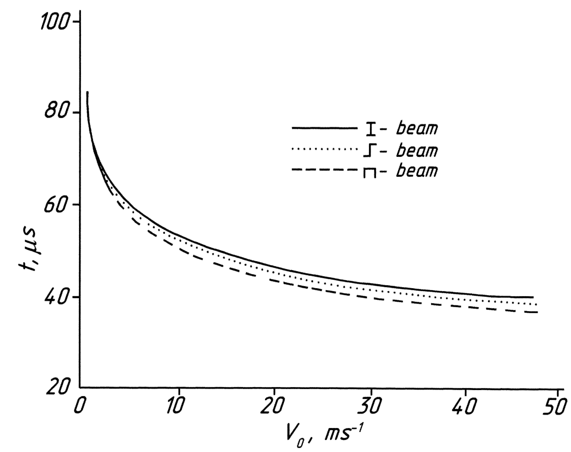


Figure 3: The initial velocity dependence of the contact duration.

Since the impact occurs with an eccentricity with respect to the flexural center in all considered cases, then the twisting motions dominate for the sections contacting with a striker. The inertia of area at the twisting motions is determined by the polar moment of inertia, which magnitudes for the three types of thin-walled beams are presented in Table 1. Reference to Table 1 shows that the channel beam and the I-beam have the largest and the smallest magnitudes of the polar moment of inertia, respectively, and the Z-shaped beam is sandwiched between them. It is obvious that during the impact of a sphere upon the channel beam the duration of contact will be the smallest, since this type of the section possesses the largest inertia under twisting, but the duration of contact of the striker with the I-beam will be the largest, since the I-beam has the smallest moment of inertia. In other words, the greater the magnitude of polar moment of inertia, the smaller the duration of contact at the same magnitude of the initial velocity of impact. However, the magnitude of the contact duration may not exceed the value calculated by the Hertz's contact theory for a semi-infinite medium at the same initial velocity of impact. Such a conclusion is supported by the experimental investigations reported in [33] and [34] for beams of continuous cross section. When $V_0 < 5$ m/s, the duration of contact practically coincides for all three thin-walled systems, since for small velocities the duration of contact is governed by the quasistatic process, which is common for all thin-walled systems under consideration.

4 Conclusion

The analytical review of the existing dynamic technical theories of thin-walled beams of open profile carried out in the given papers has shown that all papers in the field can be divided into three groups.

The papers, wherein the governing set of equations is both hyperbolic and correct from the viewpoint of the physically admissible magnitudes of the velocities of the transient waves resulting from these equations, fall into the first category, i.e., the velocity of the longitudinal wave is $G_L = \sqrt{E/\rho}$, while the velocities of the three (or four) transverse shear waves, in the general case of arbitrary cross sections of thin-walled beams with open profile, depend essentially of the geometry of the open section beam. Such theories describe the dynamic behavior of *thin-walled beams of the Timoshenko type*.

The second category involves the articles presenting hyperbolic but incorrect equations from the above mentioned viewpoint, i.e., resulting in incorrect magnitudes of the transient waves. In such papers, usually six generalized displacements are independent while warping is assumed to be dependent on the derivative of the torsional rotation with respect to the beam axial coordinate or is neglected in the analysis. In other words, there is a hybrid of two approaches: Timoshenko's beam theory and Vlasov's thin-walled beam theory, some times resulting to a set of equations wherein some of them are hyperbolic, while others are not.

The papers providing the governing system of equations which are not hyperbolic fall into the third group. In such papers, the waves of transverse shear belong to the diffusion waves possessing infinitely large velocities, and therefore, from our point of view, the dynamic equations due to such theories cannot be named as the Timoshenko type equations.

The simple but effective procedure for checking for the category, within which this or that paper falls in, has been proposed and illustrated by several examples. It has been shown that only the theories of the first group, such as the Korbut-Lazarev theory, could be used for solving the problems dealing with transient wave propagation, while the theories belonging to the second and third group could be adopted for static problems only.

The problem on the normal impact of an elastic thin rod with a rounded end upon an elastic Timoshenko arbitrary cross section thin-walled beam of open profile has been considered as an illustrative example for employing the Korbut-Lazarev theory for engineering applications. The process of impact is accompanied by the dynamic flexure and torsion of the beam, resulting in the propagation of plane flexural-warping and torsional-shear waves of strong discontinuity along the beam axis. Behind the wave fronts upto the boundaries of the contact region (the beam part with the contact spot), the solution is constructed in terms of one-term ray expansions. During the impact the rod moves under the action of the contact force which is determined due to the Hertz's theory, while the contact region moves under the attraction of the contact force, as well as the twisting and bending-torsional moments and transverse forces, which are applied to the lateral surfaces of the contact region.

The procedure proposed allows one to obtain rather simple relationships for estimating the maximal magnitude of the contact force and the contact duration, which can be very useful in engineering applications.

Appendix A

Let us prove the validity of formula (1) by the method of mathematical induction. At $n = 1$, the known formula is obtained, which is the basis for the definition of the Thomas $\delta-$derivative [28],

$$G\,\frac{\partial Z}{\partial z} = -Z_{,(1)} + \frac{\delta Z}{\delta t}. \qquad (A1)$$

Now we suppose that formula (1) is valid for $n-1$, i.e.,

$$G^{n-1}\frac{\partial^{n-1} Z}{\partial z^{n-1}} = \sum_{m=0}^{n-1}(-1)^m \frac{(n-1)!}{m!(n-1-m)!}\,\frac{\delta^{n-1-m} Z_{,(m)}}{\delta t^{n-1-m}}. \qquad (A2)$$

To prove the validity of (1), let us multiply (A2) by G, differentiate over z, and apply formula (A1). As a result we obtain

$$G^{n}\frac{\partial^{n} Z}{\partial z^{n}} = \sum_{m=0}^{n-1}(-1)^{m+1} \frac{(n-1)!}{m!(n-1-m)!}\,\frac{\delta^{n-1-m} Z_{,(m+1)}}{\delta t^{n-1-m}}$$
$$+\sum_{m=0}^{n-1}(-1)^m \frac{(n-1)!}{m!(n-1-m)!}\,\frac{\delta^{n-m} Z_{,(m)}}{\delta t^{n-m}}. \qquad (A3)$$

In the first sum of (A3), we substitute $m+1$ by m, in so doing its low limit becomes equal to unit, while the upper limit is equal to n.

Let us separate out the term at $m = n$ in the newly obtained sum and the term at $m = 0$ in the second sum of (A3), and add together all remained sums. As a result, we obtain

$$G^{n}\frac{\partial^{n} Z}{\partial z^{n}} = (-1)^n Z_{,(n)} + \frac{\delta^n Z}{\delta t^n} + \sum_{m=1}^{n-1}(-1)^m\left[\frac{(n-1)!}{(n-m)!(m-1)!}\right.$$
$$\left.+\frac{(n-1)!}{(n-1-m)!m!}\right]\frac{\delta^{n-m} Z_{,(m)}}{\delta t^{n-m}},$$

or

$$G^{n}\frac{\partial^{n} Z}{\partial z^{n}} = (-1)^n Z_{,(n)} + \frac{\delta^n Z}{\delta t^n}$$
$$+\sum_{m=1}^{n-1}(-1)^m \frac{n!}{m!(n-m)!}\,\frac{\delta^{n-m} Z_{,(m)}}{\delta t^{n-m}}. \qquad (A4)$$

If we include the second term standing in the right-hand side of (A4) into the sum, and express the value $(-1)^n Z_{,(n)}$, then we are led to relationship (1).

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