Flexure of Thick Beams using New Hyperbolic Shear Deformation Theory

Atteshamuddin S. Sayyad
Department of Civil Engineering, SRES's College of Engineering,
Kopargaon-423601, (M.S.), India,
E-mail: attu_sayyad@yahoo.co.in

Yuwaraj M. Ghugal
Department of Applied Mechanics, Government Engineering College,
Aurangabad-431005, (M.S.) India
E-mail: ghugal@rediffmail.com

Abstract- In this paper a new hyperbolic shear deformation theory is developed for the static flexure of thick isotropic beam, considering hyperbolic functions in terms of thickness co-ordinate associated with transverse shear deformation effect. Rotation of normal is taken as combined effect of shear slope and bending slope at the neutral axis. The most important feature of the theory is that the transverse shear stress can be obtained directly from the constitutive relations satisfying the shear stress free surface conditions on the top and bottom of the beam. Hence the theory obviates the need of shear correction factor. Governing equations and boundary conditions of the theory are obtained using the principle of virtual work. Results obtained for static flexure of simply supported isotropic beam subjected to single sine load are compared with those of other refined theories and exact solution.

Keywords: Shear deformation, thick beam, flexure, transverse shear stress.

I. INTRODUCTION

Classical Euler-Bernoulli theory of beam (ETB) bending is based on hypothesis that the plane section which is perpendicular to the neutral axis before bending remains plane and perpendicular to the neutral axis after bending. The theory should not applied to deep beams since it disregards the effect of shear deformation.

Timoshenko [1] has developed first order shear deformation theory (FSDT) which is based on hypothesis that the plane section which is perpendicular to the neutral axis before bending remains plane but not necessarily perpendicular to the neutral axis after bending. In this theory the transverse shear strain distribution over the cross-section of the beam is assumed to be constant through the thickness and thus require shear correction factor. Cowper [2] has given refined expression for the shear correction factor for different cross-section of the beam. To remove the discrepancies in the ETB and FSDT higher order theories were developed for the static and vibration analyses of beams. Soler [3] developed the higher order theory for thick isotropic rectangular elastic beams using Legendre polynomials and Tsai and Soler [4] extended it to orthotropic beams. Levinson [5], Bickford [6], and Krishna Murty [7] presented parabolic shear deformation theories assuming a higher order variation of inplane displacement in terms of thickness coordinates. Irretier [8] and Heyliger and Reddy [9] presented higher order shear deformation theories for the static and free vibration analysis of shear deformable rectangular beams. Stein [10] has developed refined shear deformation theories for thick beams including sinusoidal functions in terms of thickness coordinate in the displacement field. However, in this theory shear stress free boundary conditions are not satisfied at top and bottom surfaces of the beam. Comprehensive reviews of these theories have been given by Ghugal and Shimpi [11].

Ghugal and Sharma [12] have developed a variationally consistent refined hyperbolic shear deformation theory for flexure and free vibration of thick isotropic beam. Recently Ghugal and Nakhate [13] has developed trigonometric shear deformation theory for the static flexure of thick isotropic beam and obtained the general solution of thick isotropic beam with various support and loading conditions.

In this paper a variationally consistent new hyperbolic shear deformation theory for beam is developed. In this theory rotation of normal is taken as combined effect of shear slope and bending slope at the neutral axis. The theory is applied to simply supported isotropic beam of rectangular cross-section for static flexure analysis. A close form solution for simply supported beam subjected to single sine load is obtained. The results obtained are compared with those of elementary, refined and exact beam theories available in the literature.

Beam under Consideration

The beam under consideration occupies the region:

$$0 \le x \le L; \quad -\frac{b}{2} \le y \le \frac{b}{2}; \quad -\frac{h}{2} \le z \le \frac{h}{2} \tag{1}$$

where x, y, z are Cartesian coordinates, L is the length of beam, b is the width and h is the total depth of beam. The beam is subjected to transverse load of intensity $q(x)$ per unit length of the beam.

Assumptions Made in Theoretical Formulation

1. The in-plane displacement u in x direction consists of two parts:
   a) A displacement component analogous to displacement in elementary beam theory of bending;
   b) Displacement component due to shear deformation which is assumed to be hyperbolic in nature with respect to thickness coordinate.
2. The transverse displacement w in z direction is assumed to be a function of x coordinate.
3. One dimensional constitutive law is used.
4. The beam is subjected to lateral load only.

The Displacement Field

Based on the before mentioned assumptions, the displacement field of the present hyperbolic shear deformation theory is given as below:

$$u = -z\frac{dw}{dx} + f(z)\left(\phi + \frac{dw}{dx}\right) \qquad w = w(x) \tag{2}$$

where

$$f(z) = h\left[\frac{z}{h}\cosh\left(\frac{1}{2}\right) - \sinh\left(\frac{z}{h}\right)\right]$$

here u and w are the axial and transverse displacements of the beam centre line. The hyperbolic function is assigned according to the shearing stress distribution through the thickness of the beam. The ϕ represents the rotation of the cross-section of the beam at neutral axis which is unknown function to be determined. The normal and transverse shear strains are obtained from linear theory of elasticity.

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{3}$$

One dimensional law is used to obtained normal bending and transverse shear stresses.

$$\sigma_x = E\,\varepsilon_x, \quad \tau_{zx} = G\,\gamma_{zx} \tag{4}$$

Using the Eqns. (3) and (4) for strains, stresses and principle of virtual work, variationally consistent differential equations for the beam under consideration are obtained. The principle of virtual work when applied to the beam leads to:

$$\int_{-h/2}^{h/2}\int_0^L \left(\sigma_x\,\delta\varepsilon_x + \tau_{zx}\,\delta\gamma_{zx}\right)dx\,dz - \int_0^L q(x)\,\delta w\,dx = 0 \tag{5}$$

where the symbol δ denotes the variational operator . Integrating Eqn. (5) by parts and collecting the coefficients of δw and $\delta\phi$ the governing equations obtained are as follows:

$$K_1\frac{d^4w}{dx^4} - K_2\left(\frac{d^2w}{dx^2} + \frac{d\phi}{dx}\right) + K_3\frac{d^3\phi}{dx^3} = q \tag{6}$$

$$-K_3\frac{d^3w}{dx^3} + K_2\left(\frac{dw}{dx} + \phi\right) - K_4\frac{d^2\phi}{dx^2} = 0 \tag{7}$$

The associated boundary conditions obtained are of the following form at $x = 0$ and $x = L$:

$$K_1\frac{d^3w}{dx^3} - K_2\left(\frac{dw}{dx} + \phi\right) + K_3\frac{d^2\phi}{dx^2} = 0 \quad \text{or } w \text{ is prescribed} \tag{8}$$

$$K_1\frac{d^2w}{dx^2} + K_3\frac{d\phi}{dx} = 0 \quad \text{or } \frac{dw}{dx} \text{ is prescribed} \tag{9}$$

$$K_3\frac{d^2w}{dx^2} + K_4\frac{d\phi}{dx} = 0 \quad \text{or } \phi \text{ is prescribed} \tag{10}$$

Thus, the variationally consistent governing differential equations and boundary conditions are obtained. The constants appear in the governing equations and boundary conditions are the stiffness given in Appendix.

II. ILLUSTRATIVE EXAMPLES

A simply supported uniform beam of rectangular cross-section occupying the region given by expression (1) is considered for detailed numerical study.

Example 1: The beam is subjected to single sine load $q(x) = q_0\sin(\pi x/L)$ acting in the z –direction, where q_0 is the intensity of load.

Example 2: The beam is subjected to uniformly distributed load $q(x) = \sum_{m=1}^{m=\infty} q_m \sin(m\pi x/L)$ acting in the z –direction, where $q_m = (4q_0/m\pi)$ is the coefficient of single Fourier expansion of loads.

Example 3: The beam is subjected to linearly varying load $q(x) = \sum_{m=1}^{m=\infty} q_m \sin(m\pi x/L)$ acting in the z –direction, where $q_m = -(2q_0/m\pi)\cos(m\pi)$ is the coefficient of single Fourier expansion of loads.

The material properties of the beam are as follows:

$$E = 210 \ GPa \qquad \mu = 0.3$$

where E is the Young's modulus and μ is the Poisson's ratio of the beam material.

The Solution Scheme

The following is the solution form assumed for $w(x)$ and $\phi(x)$ which satisfies the boundary conditions exactly:

$$w(x) = w_m \sin\frac{m\pi x}{L}\ ; \quad \phi(x) = \phi_m \cos\frac{m\pi x}{L} \tag{11}$$

where w_m and ϕ_m are the unknown coefficients. For single sine load m = 1. Substituting this form of solution and the load $q(x)$ into governing equations, yields the two algebraic simultaneous equations from which the unknowns w_m and ϕ_m can be readily determined.

III. NUMERICAL RESULTS

The results obtained for displacements and stresses are presented in the following non-dimensional form

$$\bar{u} = \frac{E\,b\,u}{q_0\,h}; \qquad \bar{w} = \frac{E\,w10\,h^3}{q_0\,L^4}; \qquad \bar{\sigma}_x = \frac{b\,\sigma_x}{q_0};$$

$$\bar{\tau}_{zx} = \frac{b\,\tau_{zx}}{q_0}; \qquad S = L/h.$$

The percentage error in the results obtained by models of other researchers with respect to the corresponding results obtained by the theory of elasticity is calculated as follows:

$$\%\,\text{error} = \frac{\begin{array}{c}\text{value by a particular model} - \\ \text{value by exact solution}\end{array}}{\text{value by exact solution}} \times 100$$

Table 1 Comparison of axial displacement $\bar{u}$ at (x = 0, z = ± h / 2), for isotropic beam subjected to single sine load

S	Theory	Model	$\bar{u}$	% Error
4	Present	NHySDT	12.704	3.31
	Reddy [9]	HSDT	12.715	3.40
	Timoshenko [1]	FSDT	12.385	0.72
	Bernoulli-Euler	ETB	12.385	0.72
	Ghugal [14]	Exact	12.297	0.00
10	Present	NHySDT	194.31	0.70
	Reddy [9]	HSDT	194.34	0.72
	Timoshenko [1]	FSDT	193.51	0.29
	Bernoulli-Euler	ETB	193.51	0.29
	Ghugal [14]	Exact	192.95	0.00

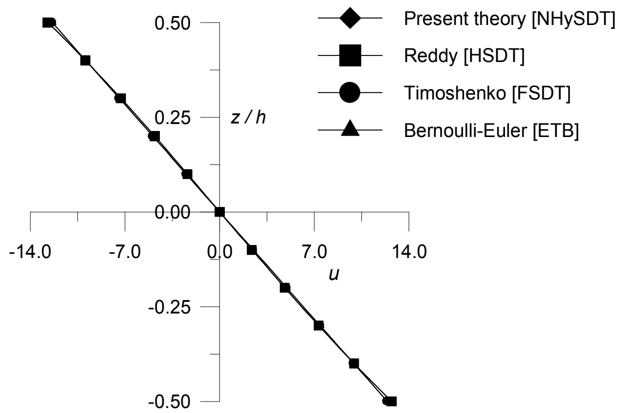

Fig. 1. Variation of axial displacement through the thickness of isotropic beam subjected to single sine load at (x = 0, z) for aspect ratio 4

Table 2 Comparison of transverse displacement $\bar{w}$ at (x = L / 2, z = 0), for isotropic beam subjected to single sine load

S	Theory	Model	$\bar{w}$	% Error
4	Present	NHySDT	1.427	1.13
	Reddy [9]	HSDT	1.429	1.28
	Timoshenko 1]	FSDT	1.430	1.35
	Bernoulli-Euler	ETB	1.232	-12.68
	Ghugal [14]	Exact	1.411	0.00
10	Present	NHySDT	1.263	0.15
	Reddy [9]	HSDT	1.264	0.24
	Timoshenko[1]	FSDT	1.264	0.24
	Bernoulli-Euler	ETB	1.232	-2.30
	Ghugal [14]	Exact	1.261	0.00

Table 3 Comparison of axial bending stress $\bar{\sigma}_x$ at ($x = L/2$, $z = \pm h/2$) for isotropic beam subjected to single sine load

S	Theory	Model	$\bar{\sigma}_x$	% Error
4	Present	NHySDT	9.977	0.19
	Reddy [9]	HSDT	9.986	0.28
	Timoshenko[1]	FSDT	9.727	-0.31
	Bernoulli-Euler	ETB	9.727	-0.31
	Ghugal [14]	Exact	9.958	0.00
10	Present	NHySDT	61.04	0.21
	Reddy [9]	HSDT	61.05	0.22
	Timoshenko[1]	FSDT	60.79	-0.20
	Bernoulli-Euler	ETB	60.79	-0.20
	Ghugal [14]	Exact	60.91	0.00

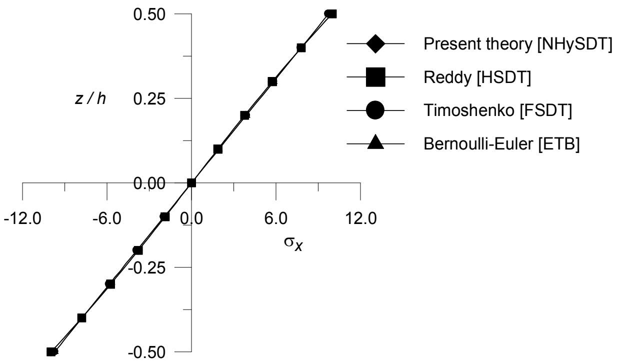

Fig. 2. Variation of axial stress through the thickness of isotropic beam subjected to single sine load at ($x = L/2$, z) for aspect ratio 4.

Table 4 Comparison transverse shear stress $\bar{\tau}_{zx}$ at ($x = 0$, $z = 0$) for isotropic beam subjected to single sine load via constitutive relation

S	Theory	Model	$\bar{\tau}_{zx}^{CR}$	% Error
4	Present	NHySDT	1.894	-0.32
	Reddy [9]	HSDT	1.906	0.32
	Timoshenko[1]	FSDT	1.270	-33.0
	Bernoulli-Euler	ETB	---	---
	Ghugal [14]	Exact	1.900	0.00
10	Present	NHySDT	4.745	-0.54
	Reddy [9]	HSDT	4.773	0.04
	Timoshenko[1]	FSDT	3.183	-33.3
	Bernoulli-Euler	ETB	---	---
	Ghugal [14]	Exact	4.771	0.00

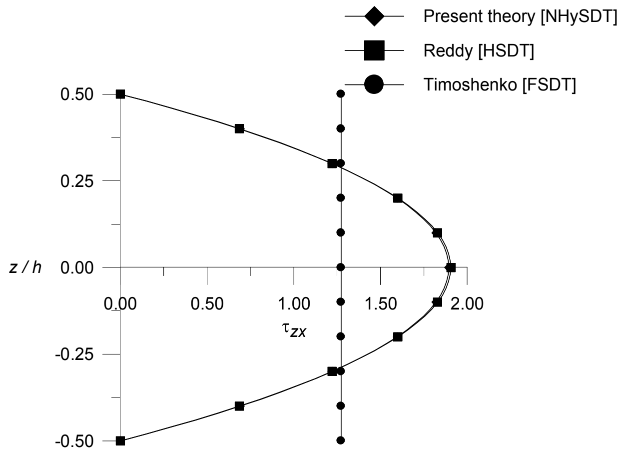

Fig. 3. Variation of transverse shear stress through the thickness of isotropic beam subjected to single sine load at ($x = 0$, z) for aspect ratio 4 *via* constitutive relation.

Table 5 Comparison transverse shear stress $\bar{\tau}_{zx}$ at ($x = 0$, $z = 0$) for isotropic beam subjected to single sine load via equation of equilibrium

S	Theory	Model	$\bar{\tau}_{zx}^{EE}$	% Error
4	Present	NHySDT	1.896	-0.21
	Reddy [9]	HSDT	1.895	-0.21
	Timoshenko[1]	FSDT	1.910	0.52
	Bernoulli-Euler	ETB	1.910	0.52
	Ghugal [14]	Exact	1.900	0.00
10	Present	NHySDT	4.769	-0.04
	Reddy [9]	HSDT	4.769	-0.04
	Timoshenko[1]	FSDT	4.769	-0.04
	Bernoulli-Euler	ETB	4.769	-0.04
	Ghugal [14]	Exact	4.771	0.00

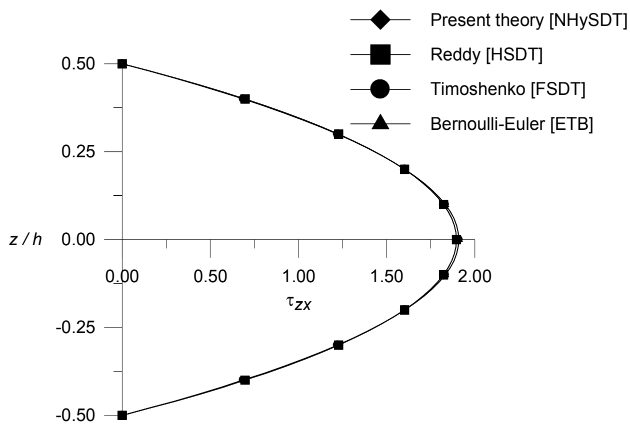

Fig. 4. Variation of transverse shear stress through the thickness of isotropic beam subjected to single sine load at ($x = 0$, z) for aspect ratio 4 *via* equation of equilibrium.

Table 6 Comparison of axial displacement $\bar{u}$ at (x = 0, z = ± h / 2), for isotropic beam subjected to uniformly distributed load

S	Theory	Model	$\bar{u}$	%Error
4	Present	NHySDT	16.486	4.341
	Reddy [9]	HSDT	16.504	4.455
	Timoshenko [1]	FSDT	16.000	1.265
	Bernoulli-Euler	ETB	16.000	1.265
	Timoshenko and Goodier [15]	Exact	15.800	0.000
10	Present	NHySDT	251.23	0.693
	Reddy [9]	HSDT	251.27	0.709
	Timoshenko [1]	FSDT	250.00	0.200
	Bernoulli-Euler	ETB	250.00	0.200
	Timoshenko and Goodier [15]	Exact	249.50	0.000

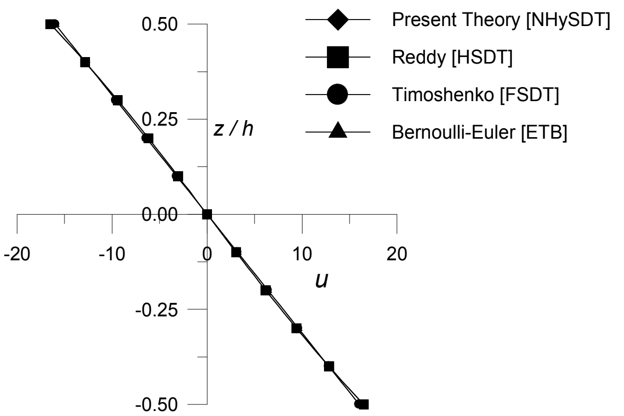

Fig. 5. Variation of axial displacement through the thickness of isotropic beam subjected to uniformly distributed load at (x = 0, z) for aspect ratio 4

Table 8 Comparison of axial bending stress $\bar{\sigma}_x$ at (x = L/2, z = ±h/2) for isotropic beam subjected to uniformly distributed load

S	Theory	Model	$\bar{\sigma}_x$	%Error
4	Present	NHySDT	12.254	0.442
	Reddy [9]	HSDT	12.263	0.516
	Timoshenko [1]	FSDT	12.000	-1.639
	Bernoulli-Euler	ETB	12.000	-1.639
	Timoshenko and Goodier [15]	Exact	12.200	0.00
10	Present	NHySDT	75.259	0.078
	Reddy [9]	HSDT	75.268	0.090
	Timoshenko [1]	FSDT	75.000	-0.265
	Bernoulli-Euler	ETB	75.000	-0.265
	Timoshenko and Goodier [15]	Exact	75.200	0.000

Table 7 Comparison of transverse displacement $\bar{w}$ at (x=L/2, z=0), for isotropic beam subjected to uniformly distributed load

S	Theory	Model	$\bar{w}$	%Error
4	Present	NHySDT	1.804	1.064
	Reddy [9]	HSDT	1.806	1.176
	Timoshenko [1]	FSDT	1.806	1.176
	Bernoulli-Euler	ETB	1.563	-12.4
	Timoshenko and Goodier [15]	Exact	1.785	0.000
10	Present	NHySDT	1.601	0.187
	Reddy [9]	HSDT	1.602	0.250
	Timoshenko [1]	FSDT	1.602	0.250
	Bernoulli-Euler	ETB	1.563	-2.19
	Timoshenko and Goodier [15]	Exact	1.598	0.000

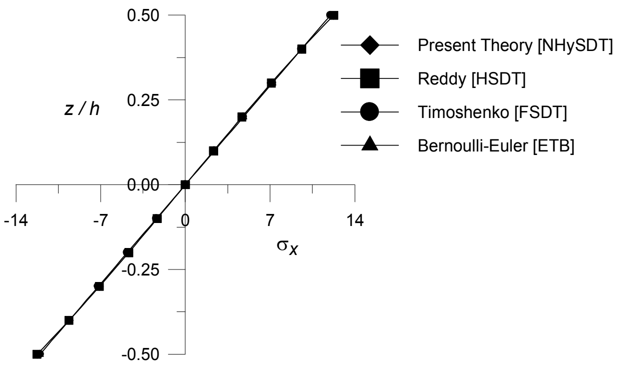

Fig. 6. Variation of axial stress through the thickness of isotropic beam subjected to uniformly distributed load at (x = L/2, z) for aspect ratio 4

Table 9 Comparison transverse shear stress $\bar{\tau}_{zx}$ at (x = 0, z = 0) for isotropic beam subjected to uniformly distributed load via constitutive relation

S	Theory	Model	$\bar{\tau}_{zx}^{CR}$	%Error
4	Present	NHySDT	2.882	-3.933
	Reddy [9]	HSDT	2.908	-3.066
	Timoshenko [1]	FSDT	1.969	-34.36
	Bernoulli-Euler	ETB	---	---
	Timoshenko and Goodier [15]	Exact	3.000	0.00
10	Present	NHySDT	7.312	-2.506
	Reddy [9]	HSDT	7.361	-1.853
	Timoshenko [1]	FSDT	4.922	-34.37
	Bernoulli-Euler	ETB	---	---
	Timoshenko and Goodier [15]	Exact	7.500	0.00

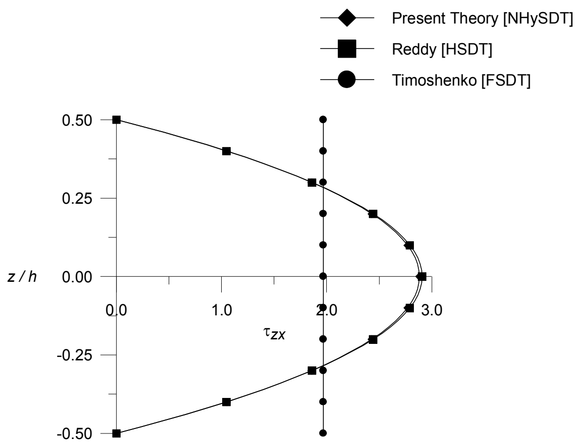

Fig. 7. Variation of transverse shear stress through the thickness of isotropic beam subjected to uniformly distributed load at (x = 0, z) for aspect ratio 4 *via* constitutive relation.

Table 10 Comparison transverse shear stress $\overline{\tau}_{zx}$ at (x = 0, z = 0) for isotropic beam subjected to uniformly distributed load via equation of equilibrium

S	Theory	Model	$\overline{\tau}_{zx}^{EE}$	%Error
4	Present	NHySDT	2.791	-6.966
	Reddy [9]	HSDT	2.795	-6.833
	Timoshenko [1]	FSDT	2.953	-1.566
	Bernoulli-Euler	ETB	2.953	-1.566
	Timoshenko and Goodier [15]	Exact	3.000	0.00
10	Present	NHySDT	7.299	-2.680
	Reddy [9]	HSDT	7.304	-2.613
	Timoshenko [1]	FSDT	7.383	-1.560
	Bernoulli-Euler	ETB	7.383	-1.560
	Timoshenko and Goodier [15]	Exact	7.500	0.00

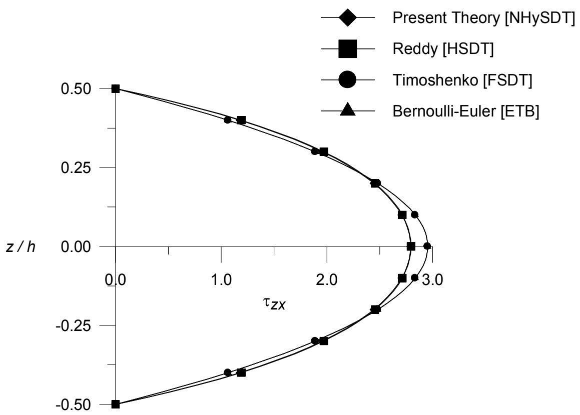

Fig. 8. Variation of transverse shear stress through the thickness of isotropic beam subjected to uniformly distributed load at (x = 0, z) for aspect ratio 4 *via* equation of equilibrium.

Table 11 Comparison of axial displacement $\overline{u}$ at (x=0, z=±h/2), for isotropic beam subjected to linearly varying load

S	Theory	Model	$\overline{u}$	%Error
4	Present	NHySDT	8.243	4.341
	Reddy [9]	HSDT	8.252	4.455
	Timoshenko [1]	FSDT	8.000	1.265
	Bernoulli-Euler	ETB	8.000	1.265
	Timoshenko and Goodier [15]	Exact	7.900	0.000
10	Present	NHySDT	125.61	0.693
	Reddy [9]	HSDT	125.63	0.709
	Timoshenko [1]	FSDT	125.00	0.200
	Bernoulli-Euler	ETB	125.00	0.200
	Timoshenko and Goodier [15]	Exact	124.75	0.000

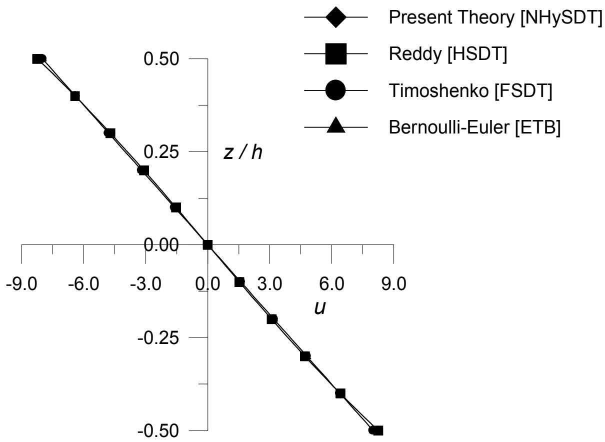

Fig. 9. Variation of axial displacement through the thickness of isotropic beam subjected to linearly varying load at (x=0, z) for aspect ratio 4

Table 12 Comparison of transverse displacement $\overline{w}$ at (x=L/2, z=0), for isotropic beam subjected to linearly varying load

S	Theory	Model	$\overline{w}$	%Error
4	Present	NHySDT	0.902	1.007
	Reddy [9]	HSDT	0.903	1.119
	Timoshenko [1]	FSDT	0.903	1.119
	Bernoulli-Euler	ETB	0.782	-12.4
	Timoshenko and Goodier [15]	Exact	0.893	0.000
10	Present	NHySDT	0.800	0.125
	Reddy [9]	HSDT	0.801	0.250
	Timoshenko [1]	FSDT	0.801	0.250
	Bernoulli-Euler	ETB	0.782	-2.19
	Timoshenko and Goodier [15]	Exact	0.799	0.000

Table 13 Comparison of axial bending stress $\bar{\sigma}_x$ at ($x = L/2$, $z = \pm h/2$) for isotropic beam subjected to linearly varying load

S	Theory	Model	$\bar{\sigma}_x$	%Error
4	Present	NHySDT	6.127	0.442
	Reddy [9]	HSDT	6.131	0.516
	Timoshenko [1]	FSDT	6.000	-1.639
	Bernoulli-Euler	ETB	6.000	-1.639
	Timoshenko and Goodier [15]	Exact	6.100	0.00
10	Present	NHySDT	37.630	0.078
	Reddy [9]	HSDT	37.634	0.090
	Timoshenko [1]	FSDT	37.500	-0.265
	Bernoulli-Euler	ETB	37.500	-0.265
	Timoshenko and Goodier [15]	Exact	37.600	0.000

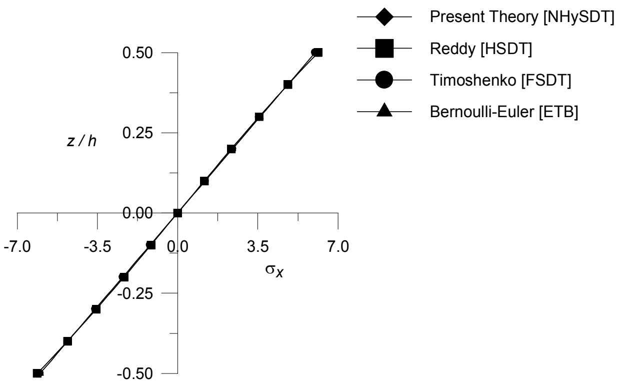

Fig. 10. Variation of axial stress through the thickness of isotropic beam subjected to linearly varying load at ($x = L/2$, z) for aspect ratio 4

Table 14 Comparison transverse shear stress $\bar{\tau}_{zx}$ at ($x = 0$, $z = 0$) for isotropic beam subjected to linearly varying load via constitutive relation

S	Theory	Model	$\bar{\tau}_{zx}^{CR}$	%Error
4	Present	NHySDT	1.441	-3.933
	Reddy [9]	HSDT	1.454	-3.066
	Timoshenko [1]	FSDT	0.985	-34.36
	Bernoulli-Euler	ETB	---	---
	Timoshenko and Goodier [15]	Exact	1.500	0.00
10	Present	NHySDT	3.656	-2.506
	Reddy [9]	HSDT	3.680	-1.853
	Timoshenko [1]	FSDT	2.461	-34.37
	Bernoulli-Euler	ETB	---	---
	Timoshenko and Goodier [15]	Exact	3.750	0.00

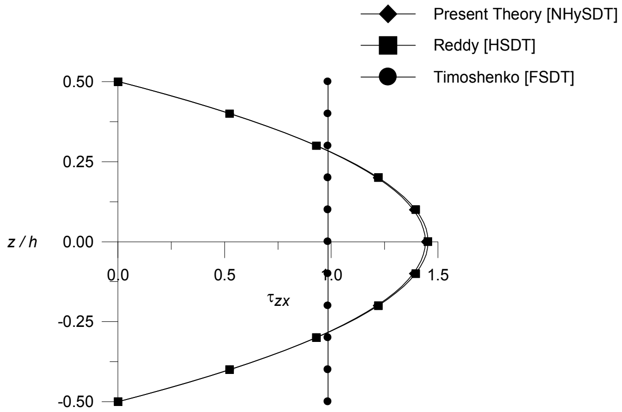

Fig. 11. Variation of transverse shear stress through the thickness of isotropic beam subjected to linearly varying load at ($x = 0$, z) for aspect ratio 4 *via* constitutive relation.

Table 15 Comparison transverse shear stress $\bar{\tau}_{zx}$ at ($x = 0$, $z = 0$) for isotropic beam subjected to linearly varying load via equation of equilibrium

S	Theory	Model	$\bar{\tau}_{zx}^{EE}$	%Error
4	Present	NHySDT	1.395	-6.966
	Reddy [9]	HSDT	1.397	-6.833
	Timoshenko [1]	FSDT	1.476	-1.566
	Bernoulli-Euler	ETB	1.476	-1.566
	Timoshenko and Goodier [15]	Exact	1.500	0.00
10	Present	NHySDT	3.649	-2.680
	Reddy [9]	HSDT	3.652	-2.613
	Timoshenko [1]	FSDT	3.691	-1.560
	Bernoulli-Euler	ETB	3.691	-1.560
	Timoshenko and Goodier [15]	Exact	3.750	0.00

Discussion of Results

The comparison of maximum non-dimensional axial displacement for various aspect ratios is presented in Table 1. The present theory overestimates the maximum value of axial displacement by 3.31 % and 0.70 % for aspect ratios 4 and 10 respectively as compared to that of exact solution. The theory of Reddy (HSDT) overestimates the value of axial displacement by 3.4 % and 0.72 % for aspect ratios 4 and 10 respectively. The theory of Timoshenko (FSDT) and ETB yields identical value for the axial displacement for all aspect ratios. The through thickness variation of axial displacement of isotropic beam subjected to single sine load is shown in Fig. 1.

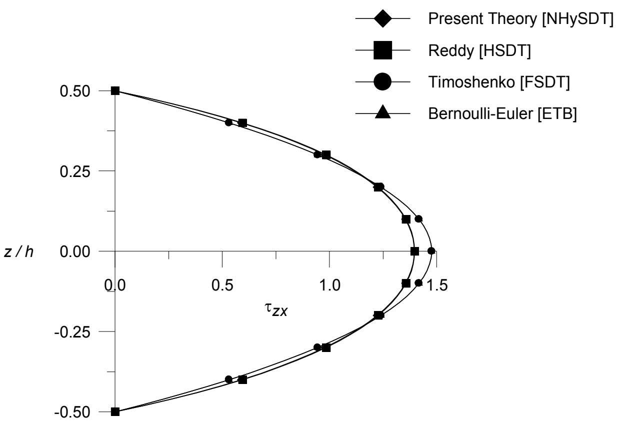

Fig. 12. Variation of transverse shear stress through the thickness of isotropic beam subjected to linearly varying load at (x = 0, z) for aspect ratio 4 *via* equation of equilibrium.

The comparison of maximum transverse displacement for the aspect ratios 4 and 10 is presented in Table 2. The maximum central deflection predicted by present theory is in excellent agreement with the exact solution for all aspect ratios. The HSDT and FSDT overestimate the value of maximum transverse deflection by 1.13 % and 1.28 % for aspect ratio 4 respectively. The ETB underestimates the value of maximum transverse displacement by 12.68 % and 2.30 % for aspect ratios 4 and 10 respectively due to neglect of transverse shear deformation.

Table 3 shows comparison of non-dimensional normal bending stress for aspect ratios 4 and 10. The examination of Table 3 reveals that, the value of maximum normal bending stress obtained by present theory is in tune with the exact solution for all aspect ratios. Theory of Reddy overestimates the normal bending stress by 0.28 % and 0.22 % for aspect ratios 4 and 10 respectively compared to those of exact values. The values of normal bending stress predicted by FSDT and ETB are identical for all aspect ratios. FSDT and ETB underestimate the value of normal bending stress by 0.31 % and 0.20 % for aspect ratios 4 and 10 respectively as compared to exact value. Variation of axial stress through the thickness of beam subjected to single sine load for aspect ratio 4 is shown in Fig. 2.

The comparison of maximum non-dimensional transverse shear stress for aspect ratios 4 and 10 obtained by constitutive relation is presented in Table 4. The transverse shear stress satisfies the stress free boundary conditions on the top and bottom surfaces of the beam when obtained by constitutive relation. It may be noted that τ_{zx} and $\bar{\tau}_{zx}$ obtained by constitutive relations are indicated by τ_{zx}^{CR} and $\bar{\tau}_{zx}^{CR}$. The maximum transverse shear obtained by present theory using constitutive relation is in excellent agreement with that of exact solution for the aspect ratios 4 and 10. The maximum transverse shear stress obtained using constitutive relation by theory of Reddy is in close agreement with that of exact value for all aspect ratios. The FSDT yields lower value of transverse shear stress when obtained using constitutive relation. Variation of transverse shear stress through the thickness of beam subjected to single sine load for aspect ratio 4 obtained using constitutive relation is shown in Fig. 3.

The comparison of maximum non-dimensional transverse shear stress for aspect ratios 4 and 10 obtained by equation of equilibrium is presented in Table 5. Further it may be noted that τ_{zx} and $\bar{\tau}_{zx}$ obtained by equation of equilibrium is indicated by τ_{zx}^{EE} and $\bar{\tau}_{zx}^{EE}$.

The maximum transverse shear obtained by present theory using equations of equilibrium is in excellent agreement with that of exact solution for the aspect ratios 4 and 10. The present theory, HSDT, FSDT and ETB gives identical values of this stress for aspect ratio 10 when obtained using equation of equilibrium. Variation of transverse shear stress through the thickness of beam subjected to single sine load for aspect ratio 4 obtained using equation of equilibrium is shown in Fig. 4.

Comparison of displacements and stresses for the isotropic beams subjected to uniformly distributed load are shown in Table 6 through 10 and found in excellent agreement with those of exact solution. Through thickness variation of displacement and stresses for the isotropic beam subjected to uniformly distributed load for aspect ratio 4 are shown in Figs. 5 through 8.

The comparison of axial displacement for isotropic beam subjected to linearly varying load is shown in Table 11. The examination of Table 11 reveals that the axial displacement predicted by present theory is in excellent agreement with that of exact solution for aspect ratio 10 whereas HSDT of Reddy overestimates the same by 0.709 %. The axial displacement predicted by FSDT and ETB are identical for both the aspect ratios. Through thickness variation of axial displacement for isotropic beam subjected to linearly varying load is shown in Fig. 9. Table 12 shows the comparison of transverse displacement for isotropic beam subjected to linearly varying load and found in good agreement when predicted by present theory. FSDT and ETB show the identical values for transverse displacement for both the aspect ratios. The comparison of axial bending stress for isotropic beam subjected to linearly varying load is shown in Table 13.

The axial bending stress predicted by present theory is in excellent agreement with that of exact solution whereas FSDT and ETB underestimate the same for both the aspect ratios. The through thickness variation of axial bending stress for the isotropic beam subjected to linearly varying load for aspect ratio 4 is shown in Fig. 10. The comparison of transverse shear stress for isotropic beam subjected to linearly varying load via constitutive relation and equation of equilibrium is represented in Tables 14 and 15 respectively. The examination of Tables 14 and 15 reveals that transverse shear stress predicted by present theory is in excellent agreement when obtained using constitutive relations. The through thickness variation of transverse shear stress for isotropic beam subjected to linearly varying load is shown in Figs 11 and 12.

IV. CONCLUSIONS

Following conclusions are drawn from this study.

1. The results of maximum transverse deflection obtained by present theory are in excellent agreement with the exact solution.
2. The transverse shear stress obtained from constitutive relation using present theory gives near to exact values and it is in excellent agreement when obtained using equation of equilibrium.
3. The present theory obviates need of shear correction factor.
4. The governing differential equation and boundary conditions are variationally consistent.

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Appendix

$$K_1 = \left(D - 2B + A\right),$$
$$K_2 = Ac\,,\ K_3 = \left(A - B\right),$$
$$K_4 = A.$$

where

$$D = E\int_{-h/2}^{h/2} z^2\,dz\;;$$
$$B = E\int_{-h/2}^{h/2} z\,f\left(z\right)\,dz\;;$$
$$A = E\int_{-h/2}^{h/2} \left[f\left(z\right)\right]^2 dz\;;$$
$$Ac = G\int_{-h/2}^{h/2} \left[\frac{d}{dz}f\left(z\right)\right]^2 dz;$$

BIOGRAPHIES

Sayyad A. S.

Sayyad A. S. is presently Assistant Professor in Civil Engineering Department, SRES's College of Engineering, Kopargaon-423601, (M.S.), India. His research interests are in the areas of theory of beams and plates, composite materials. He has submitted his Ph.D. thesis on "Assessment of refined shear deformation theories on laminated composites".

Ghugal Y. M.

Ghugal Y. M. Obtained the Ph.D. from I.I.T. Bombay, India. He is presently Professor in Department of Applied Mechanics, Government College of Engineering, Aurangabad-431005, (M.S.), India. His research interests are in the areas of theory of plates, composite materials. His research interests are in the areas of theory of beams and plates, composite materials, fibre reinforced concrete.