Nonlinear Analysis of Transient Seepage by the Coupled Finite Element Method

Ahad Ouria, Ahmad Fahmi and Mohammad M. Toufigh

Abstract— In this paper, transient seepage beneath a dam is investigated considering the effect of the change of the permeability of the soil by the finite element method. Change of the permeability is considered as a function of the change of the void ratio. In order to calculate the change of the void ratio, the force equilibrium equations are taken in to account by coupling with the fluid continuity equation. Displacement method is used in the finite element formulation. Change of the soil permeability is calculated based on the soil deformation which is the primary unknown in the coupled models. Generalized Hook's law is used for stress-strain behavior of the soil. Galerkin residuals method is used in the finite element formulations. Coupled analyses showed the effect of the change of the permeability on the transient seepage in the coupled models is les than its effect in the uncoupled models for elastic soils.

Keywords— Seepage, Transient, Nonlinear, Coupled, Finite Element Method

I. Introduction

Nonlinear analysis of seepage flow requires calculating the change of the void ratio of soil. Nonlinear analysis of seepage flow requires calculating the change of the void ratio of soil. In routine analyses, change of the void ratio is calculated as a function of the total head. Also steady-state conditions commonly are considered ignoring the time required to reach the steady-state condition. But actually, change of the void ratio must be calculated based on the soil mass deformations. In the other hand, moving water trough soil voids is a time consuming procedure. Increase of the water table height in a dam reservoir is gradual and time dependent, so transient analysis is essential to yield reliable results.

In uncouple systems there is only one degree of freedom in the governing equation which is the hydraulic potential. So in order to determine deformations of the soil mass due to seepage forces it is essential to consider element equilibrium equations besides the fluid continuity equation. In this case a set of partial differential equations including the continuity and equilibrium equations must be solved at the same time.

In this method changes of the void ratio would be calculated based on the soil body deformation which is the primary unknown in the governing equations.

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II. Finite Element Formulation

Fluid flow through porous media is governed by hydrodynamic equations considering the interaction of the fluid in motion with the porous media ensuring the continuity of the fluid. Continuity is ensured by requiring that the net volume of water flowing per unit of time into or out of an element of soil be equal to change per unit of time of the volume of water in that element. Difference of the quantity of the water that leaves or inters to an element is equal to change of the element volume. Therefore the fluid continuity equation in the transient state would be [1]:

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=-\frac{\partial}{\partial t}(ndv) \tag{1}$$

Where; vx and vy are flow velocities in horizontal and vertical directions respectively and, n, is the porosity.

Biot formulated the theory of coupled solid-fluid interaction where the soil skeleton is treated as porous elastic solid and the laminar pore fluid is coupled to the solid by the conditions of compressibility and of continuity. Thus Biot's governing equations are combination of equation (1) and element equilibrium equations.

For two-dimensional equilibrium in the absence of body forces considering seepage forces, the gradient of effective stress must be augmented by the gradients of the fluid pressure as follows [2]:

$$\begin{aligned}&\frac{\partial \sigma'_x}{\partial x}+\frac{\partial \tau_{xz}}{\partial y}+\frac{\partial u_e}{\partial x}=0\\&\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial u_e}{\partial y}=0\end{aligned} \tag{2}$$

The stress-strain relations for plane strain condition can be written as follows [4]:

$$\begin{Bmatrix} \sigma'_x \\ \sigma'_y \\ \tau_{xy} \end{Bmatrix} = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)} \times \begin{bmatrix} 1 & \frac{\mu}{1-\mu} & 0 \\ \frac{\mu}{1-\mu} & 1 & 0 \\ 0 & 0 & \frac{1-2\mu}{2(1-\mu)} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (3)$$

Continuity equation can be obtained using equation (2) where the volume change of the element is written in terms of displacement components:

$$\frac{k_x}{\gamma_w} \cdot \frac{\partial^2 u_e}{\partial x^2} + \frac{k_y}{\gamma_w} \frac{\partial^2 u_e}{\partial y^2} = -\frac{d}{dt}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \quad (4)$$

Where ue is the pre-water pressure, u and v are the horizontal and vertical components of displacements.

As is usual in the displacement method in solid mechanics, stress and strain are replaced with displacement components so, final coupled variables are the pore water pressure, and horizontal and vertical displacements. Use of the displacements method is useful for nonlinear analysis with variable permeability because the change of the void ratio would be calculated based on the volumetric strain.

Using the linear shape functions with rectangular elements for the solid body and the pore fluid, the final form of the set of P.D.E which is combining equations (2) and (4) discretized by the F.E.M can be as follows [2]:

$$\begin{aligned} & KMr + Cu_e = 0 \\ & C^T \frac{dr}{dt} - KPu_e = 0 \end{aligned} \quad (5)$$

Where:, r is the displacement vector [u,v]; and also KP is the pre-fluid matrix, KM is solid body stiffness matrix and C is the coupling matrix which are as follows:

$$KP_{ij} = \iint \left(k_x \frac{\partial N_i}{\partial x} \cdot \frac{\partial N_j}{\partial x} + k_y \frac{\partial N_i}{\partial y} \cdot \frac{\partial N_j}{\partial y}\right) dx.dy \quad (6)$$

$$KM = \begin{bmatrix} KMxx & KMxy \\ KMxy^T & KMyy \end{bmatrix} \quad (7)$$

$$\begin{aligned} KMxx_{ij} &= \iint \left(R_{11} \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + R_{33} \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y}\right) dxdy \\ KMxy_{ij} &= \iint \left(R_{12} \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial y} + R_{33} \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial x}\right) dxdy \\ KMyy_{ij} &= \iint \left(R_{22} \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} + R_{33} \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x}\right) dxdy \end{aligned} \quad (8)$$

In the above equations, R is the stress strain relationship matrix according to equation (3), and;

$$C = \begin{bmatrix} Cx \\ Cy \end{bmatrix} \quad (9)$$

Where:

$$\begin{aligned} Cx_{ij} &= \iint N_i \frac{\partial N_j}{\partial x} dxdy \\ Cy_{ij} &= \iint N_i \frac{\partial N_j}{\partial y} dxdy \end{aligned} \quad (10)$$

To integrate the Equation (5) with respect to time, Crank-Nicholson method is implemented, therefore [5]:

$$\begin{aligned} & \theta KMr_1 + \theta Cu_{e1} = (\theta - 1)KMr_0 + (\theta - 1)Cu_{e0} \\ & \theta C^T r_1 - \theta^2 \Delta t KPu_{e1} = \theta C^T r_0 - \theta(\theta - 1)\Delta t KPu_{e0} \end{aligned} \quad (11)$$

In above equations, if $\theta = 1$, the system will be absolutely stable without any oscillatory results. Therefore the final form of the Equation (15) in fully implicit type of time-integration will be as follow [5]:

$$\begin{bmatrix} KM & C \\ C^T & -\frac{\Delta t}{2} KP \end{bmatrix} \begin{Bmatrix} r_1 \\ u_{e_1} \end{Bmatrix} = \begin{bmatrix} 0 & 0 \\ C^T & 0 \end{bmatrix} \begin{Bmatrix} r_0 \\ u_{e_0} \end{Bmatrix} \quad (12)$$

After finding governing matrix equations for a single element, the assembled matrices for total elements can be obtained and boundary conditions can be introduced.

Solving such equations at any time, horizontal and vertical deformations (u,v) and the fluid pressure at various nodal points can be found and strain values for each element can be calculated.

Base on the changes of the volumetric strains, the change of the permeability for each element would be calculated. The volumetric strain can be defined in term of the changes of the void ratio as follow:

$$\Delta e = \Delta \varepsilon_v (1 + e_0) \quad (13)$$

Where: e is the void ratio in the above equation.

The change of the soil permeability is a logarithmic function of the change of the void ratio [1]:

$$\Delta \log k = \frac{\Delta e}{c_k} \tag{14}$$

Where: the ck is a factor which is $\frac{1}{2}e_0$, for normally consolidated clays.

Therefore the change of the soil permeability at each time step would be:

$$k_{t+\Delta t} = 10^{\left[\log k_t + \frac{(\Delta \varepsilon_v)(1+e_t)}{c_k}\right]} \tag{15}$$

In uncoupled formulation, because of the absence of body deformations, an indirect approach is used to calculate the permeability variations. In this case, change in void ratio is related to change of mean effective stress that can be calculated by changes of the total head. Therefore equation (15) in uncoupled formulation will be as follow [6]:

$$k_{t+\Delta t} = 10^{[\log k_t + 0.03 \Delta \log p']} \tag{16}$$

Where; p' is vertical effective stress can be calculated based on the seepage follow gradients.

III. NUMERICAL RESULTS

For both linear (with constant permeability) and nonlinear (with variable permeability) formulations, finite element codes are developed. In order to compare the effect of the change of the permeability on the seepage calculations, the transient seepage beneath a concrete dam in investigated by both methods. Also the results of the coupled formulations compared with the results obtained from uncoupled formulations.

The geometry of the finite element model and material properties are illustrated in Figure (1).

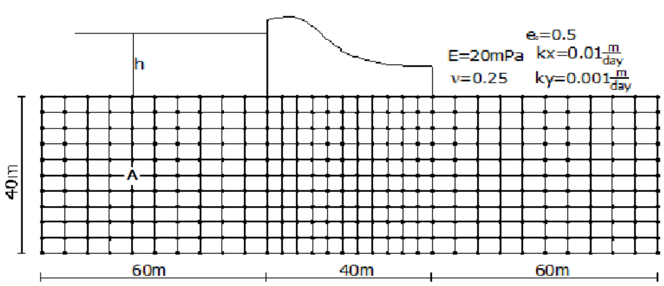

Fig. 1 Finite element mesh and material properties.

Water table in reservoir rises to 30 meter gradually during 90 days.

At first stage, effect of the change of the permeability on the water pressure distribution is investigated. Figure (2), shows the equipotential lines for analysis with variable and constant permeability at steady-state conditions resulted from coupled model. It should be noted that the steady-state conditions reached after about 600 days.

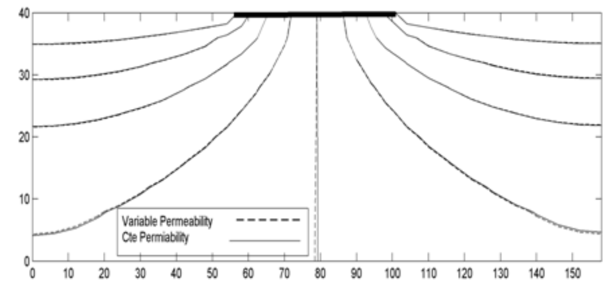

Fig.2 water pressure distribution resulted from coupled linear and nonlinear analysis

In this figure, dashed lines are the results of nonlinear analysis with variable permeability.

Figure (3) shows the outlet flow rate vs. time from down stream resulted from nonlinear analyses.

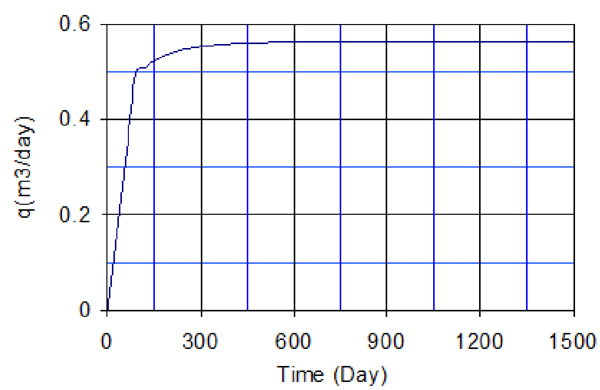

Fig. 3 outlet flow rate from coupled nonlinear analysis

The effect of the change of the permeability coefficient on the outlet flow rate is illustrated in the Figure (4).

In this figure, the vertical axis shows the differences between the results of the nonlinear analysis with variable permeability and linear analysis with constant permeability.

It can be seen in Figure (4) the effect of the change of the permeability on the outlet flow decreases as the time increases. At the steady-state conditions, the results of the analysis with variable and constant permeability became close together. Their differences is about 8% as the increment of the water table in the reservoir completed and its value is about 0.02% at steady-state conditions.

Figure (5) shows the change of the void ratio due to seepage forces in the dam foundation at steady-state conditions.

Change of the void ratio in the dam foundation is symmetrical about the dam centerline. It's because of the seepage force in the dam foundation which is compressive in the upstream and tensile in the down stream.

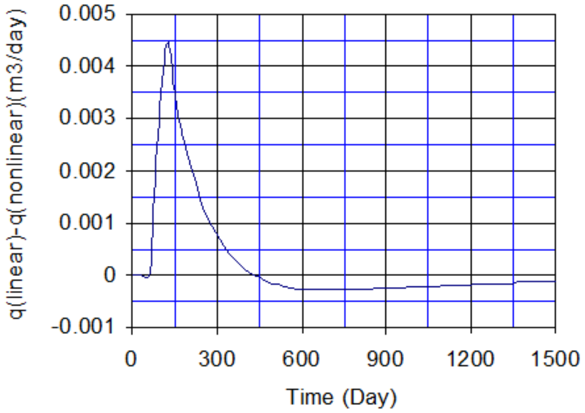

Fig. 4 effect of the change of the permeability on the outlet flow

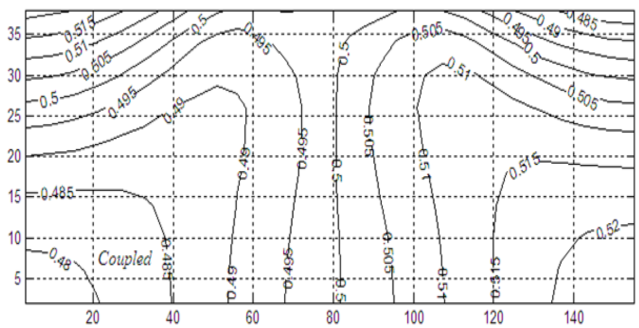

Fig. 5 changes of void ratio due to seepage forces

Change of the vertical coefficient of permeability in the dam foundation is shown in Figure 6. It can be seen that the maximum amount of the change of the vertical permeability is about 4%.

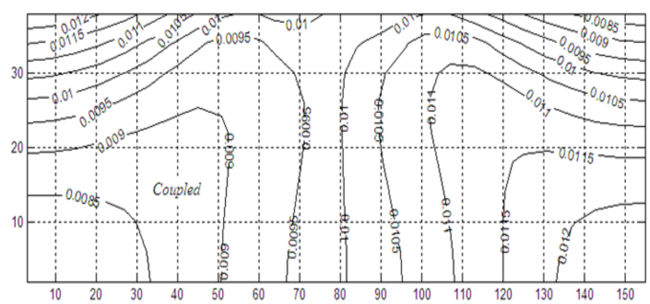

Fig. 6 changes of the permeability due to seepage forces

In Figure (7), the change of the void ratio at node (A) is illustrated. Change of the void ratio is used to determine the variation of the permeability. In this research change of the permeability is related to the change of the void ratio in the soil elements which is a function of the volumetric strain.

In this rsearch the anisotropic ratio of permeability is assumed to be constant as the change of the void ratio.

Stress conditions in the upstream and downstream are different. Principal stresses with different signs cause to reduce the change of the void ratio and consequently reduces the change of the permeability.

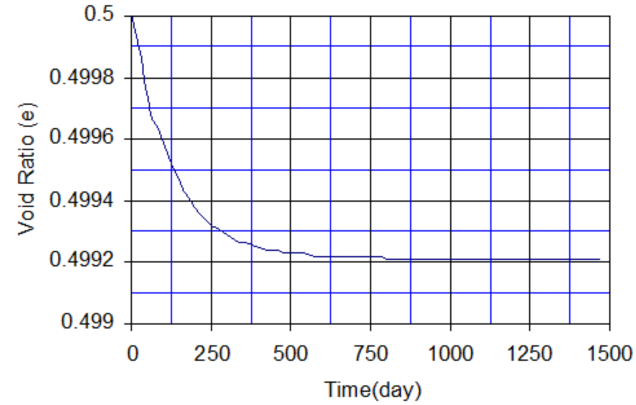

Fig. 7 variation of the void ratio at node A

Figure (8) shows the change of the vertical and horizontal effective stresses due to seepage forces at the steady-state conditions.

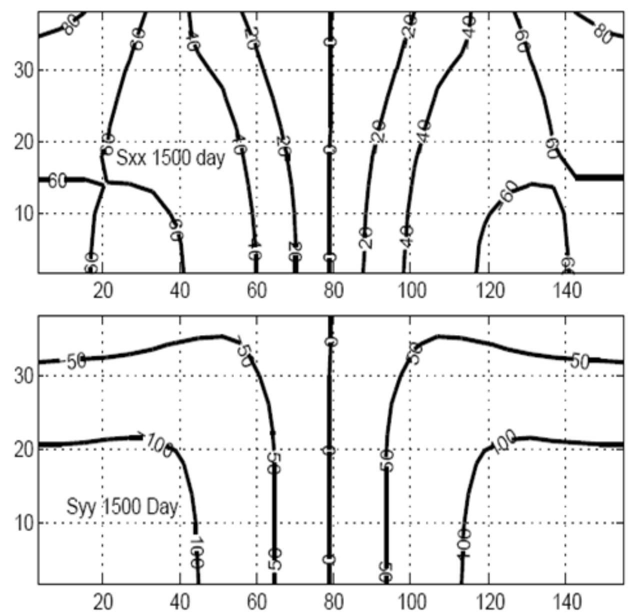

Fig. 8 the change of the horizontal and vertical effective stresses due to seepage force

In Figure (8), the positive and negative signs refer to tensile and compressive incremental stresses respectively.

Referring to figures (6) and (5), the change of the void ratio and permeability coefficient in the down stream and upstream are same as the change of the stress conditions.

Also the effect of permeability variation in uncoupled formulation is investigated.

Figure (9) shows outlet flow rate calculated by the coupled and uncoupled models with variable and constant permeability.

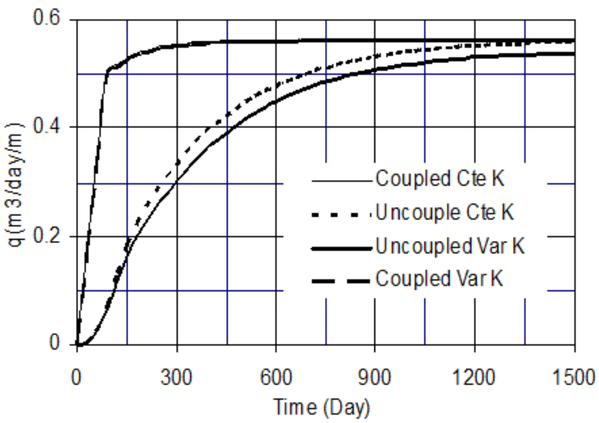

Figure (9): effect of the permeability change on outlet flow rate for coupled and uncoupled models.

It can be seen in figure (9) the change of the permeability coefficient has a large effect in the outlet follow in uncoupled model comparing to coupled model. For coupled model, effect of the permeability change is not considerable. It would be due to calculation of permeability change from volumetric strain in coupled model because compression of an element in horizontal direction consequents swelling in other direction. Therefore change of the volumetric strain due to seepage force is not considerable in coupled model.

Effect of the change of the permeability on the seepage results based on the coupled formulations would be less than its effect based on the uncoupled formulations. It's due to the effect of the 2-d or 3-d stress conditions in the coupled models, produced vertical effective stress in the coupled models are less than its value resulted from uncoupled models based on classical relationships [5].

IV. Conclusion

In this paper the effect of the changes of the soil permeability on the transient seepage is investigated. The transient state of seepage is due to the porous media compressibility which causes the problem to become as a consolidation problem. A numerical model based on the Biot's coupled consolidation theory is developed to investigate this phenomenon. The finite element technique is implemented using the Galerkin's method. The change of the void ratio and permeability are calculated based on the element displacement components which are the primary unknowns of the model. The results of the nonlinear analysis are compared with the linear analysis with constant permeability. The results showed the effect of the change of the permeability on the outlet flow is different depending on the coupled or uncoupled formulations.

Change of permeability has considerable effect on result of uncoupled formulation in comparison with coupled formulations.

In this paper, elastic behavior is presumed for solid body of the soil. Generalized Hook's law is used as the constitutive model for stress-strain relation ships. Elastic assumption is the shortcoming of this study. Also it should be noted that in the traditional methods based on the uncoupled formulations, effect of the inelastic behavior of the solid phase of the soil would not be applied in the calculations directly. The coupled model would be used with other advanced elastic-plastic constitutive models for the stress-strain behavior and failure of the soil body for implementation in the seepage models.

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