Energy balance equation, energetic theorems and variation equation for the general theory of micropolar elastic isotropic thin shells

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***Abstract*—**In the present paper main equations of three-dimensional theory of micropolar elasticity, energetic relations and general variation equation are introduced. Assumptions are accepted which have asymptotic justification and on the basis of these assumptions general applied theory of micropolar thin shells is constructed. Equation of energetic balance and general variation equation are constructed for micropolar elastic thin shells, theorems of uniqueness, existence, Betty's and other energetic theorems are proved.

***Keywords*—** Elastic, energetic balance, functional, isotropic, micropolar, thin shell, variation equation.

I. Introduction

MICROPOLAR theory of elasticity is one of the basic mathematical models for studying elastic bodies with internal structure. On importance of micropolar theory of elasticity is spoken in Yu. N. Rabatnov's monograph [1]. Variation principles, universally recognized in the classical theory of elasticity and structural mechanics [2]-[4], are essential also in the micropolar theory of elasticity [5], particularly, in the applied theories of micropolar elastic thin bars, plates and shells.

In the present paper energy balance equation, functional and general variation equation of micropolar elastic isotropic thin shells (bars, plates) with free fields of displacements and rotations are obtained, corresponding energetic theorems are proved.

II. The Energy Balance Equation, Functional and Variation Equation of Three-Dimensional Micropolar Theory of Elasticity

A shell of constant thickness $2h$ is considered as a three-dimensional elastic isotropic body. Tensor equations of the static problem of asymmetric (micropolar, momental) theory of elasticity with free fields of displacements and rotations are the followings [3],[5]:

Equilibrium equations:

$$\nabla_m\sigma^{mn}=0,\qquad \nabla_m\mu^{mn}+e^{nmk}\sigma_{mk}=0. \tag{1}$$

Elasticity relations:

$$\begin{aligned}\sigma_{mn}&=(\mu+\alpha)\gamma_{mn}+(\mu-\alpha)\gamma_{nm}+\lambda\gamma_{kk}\delta_{nm},\\ \mu_{mn}&=(\gamma+\varepsilon)\chi_{mn}+(\gamma-\varepsilon)\chi_{nm}+\beta\chi_{kk}\delta_{nm}.\end{aligned} \tag{2}$$

Geometrical relations:

$$\gamma_{mn}=\nabla_m V_n-e_{kmn}\omega^k,\qquad \chi_{mn}=\nabla_m\omega_n. \tag{3}$$

Here σ^{nm},μ^{nm} are contravariant components of the force and moment stresses tensor; γ_{mn},κ_{mn} are covariant components of the deformation and bending-torsion tensors; V^n are contravariant components of the displacement vector; ω^n are contravariant components of the free rotation; $\lambda,\mu,\alpha,\beta,\gamma,\varepsilon$ are the physical constants of micropolar material of the shell; indices m,n,k take values 1, 2, 3.

It should be noted that when $\alpha=0$, equations of the classical theory of elasticity will be separated from the equations (1)-(3).

We'll bring the shell to three orthogonal system of coordinates α_n $\left(H_i=A_i(1+\alpha_3/R_i),\quad H_3=1,\ i=1,2\right)$, accepted in the shell theory [6], and we'll pass to physical components for the above mentioned tensors and vectors, but with the same notations.

On the front surfaces of the shell boundary conditions of the first boundary-value problem are accepted, which can be written down as follows:

$$\sigma_{3n}=\pm p_n^{\pm},\ \mu_{3n}=\pm m_n^{\pm}\ \text{ on }\ \alpha_3=\pm h. \tag{4}$$

On the surface $\Sigma=\Sigma_1\cup\Sigma_2$ of the shell boundary conditions of the mixed boundary-value problem are accepted

$$\begin{aligned}&\sigma_{mn}n_m=p_n^*,\ \mu_{mn}n_m=m_n^*\ \text{ on }\Sigma_1,\\ &V_n=V_n^\bullet,\ \ \omega_n=\omega_n^\bullet\ \text{ on }\Sigma_2,\end{aligned} \tag{5}$$

where p_n^*,m_n^* are the components of external forces and moments on Σ_1; $V_n^\bullet$, $\omega_n^\bullet$ are the components of displacements and free rotation on Σ_2.

Multiplying equilibrium equations (1) on $V_1H_1H_2d\alpha_1d\alpha_2d\alpha_3$, $V_2H_1H_2d\alpha_1d\alpha_2d\alpha_3$, $V_3H_1H_2d\alpha_1d\alpha_2d\alpha_3$, $\omega_1H_1H_2d\alpha_1d\alpha_2d\alpha_3$, $\omega_2H_1H_2d\alpha_1d\alpha_2d\alpha_3$, $\omega_3H_1H_2d\alpha_1d\alpha_2d\alpha_3$, summarizing them and taking the integral by the shell volume, after some transformations with the help of formulas (2), (3), following equation of energetic balance will be obtained [5]:

$$\iint_S\int_{-h}^{h}WH_1H_2d\alpha_1d\alpha_2d\alpha_3=A, \tag{6}$$

where

$$
\begin{aligned}
W = \frac{1}{2}(&\sigma_{11}\gamma_{11} + \sigma_{22}\gamma_{22} + \sigma_{23}\gamma_{33} + \sigma_{12}\gamma_{12} + \sigma_{21}\gamma_{21} + \sigma_{13}\gamma_{13} + \\
&+ \sigma_{23}\gamma_{23} + \sigma_{32}\gamma_{32} + + \mu_{11}\chi_{11} + \mu_{22}\chi_{22} + \mu_{33}\chi_{33} + \\
&+ \mu_{12}\chi_{12} + \mu_{13}\chi_{13} + \mu_{31}\chi_{31} + \mu_{23}\chi_{23} + \mu_{32}\chi_{32},
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
A = \frac{1}{2}\Bigg\{ &\int_{-h}^{h} d\alpha_3 \int_{l_1} \left(\sigma_{21}^0 V_1 + \sigma_{22}^0 V_2 + \sigma_{23}^0 V_3 + \mu_{21}^0\omega_1 + \mu_{22}^0\omega_2 + \right. \\
&\left. + \mu_{23}^0\omega_3\right) H_1 d\alpha_1 + \int_{-h}^{h} d\alpha_3 \int_{l_2} \left(\sigma_{11}^0 V_1 + \sigma_{12}^0 V_2 + \sigma_{13}^0 V_3 + \right. \\
&\left. + \mu_{11}^0\omega_1 + \mu_{12}^0\omega_2 + \mu_{13}^0\omega_3\right) H_2 d\alpha_2 + \\
&+ \iint_{S^+} \left(q_1^+ V_1 + q_2^+ V_2 + q_3^+ V_3 + m_1^+\omega_1 + m_2^+\omega_2 + m_3^+\omega_3\right) H_1 H_2 d\alpha_1 d\alpha_2 + \\
&+ \iint_{S^-} \left(q_1^- V_1 + q_2^- V_2 + q_3^- V_3 + m_1^-\omega_1 + m_2^-\omega_2 + m_3^-\omega_3\right) H_1 H_2 d\alpha_1 d\alpha_2 \Bigg\}.
\end{aligned} \tag{8}
$$

Here W is the potential energy of deformation of the shell, enclosed in a unit of volume; A is the work of external surface forces and moments; S is the region, $l = l_1 \cup l_2$ is the shell middle surface contour.

The equation (6), as the energy balance equation, can be interpreted as Clapeyron theorem for micropolar body.

If force and moment stresses in formula (7) are replaced with deformations and bending-torsions, using physical relations (2), following formula will be obtained for elastic potential:

$$
\begin{aligned}
W = \frac{1}{2}\Big\{ &2\mu\left(\gamma_{11}^2 + \gamma_{22}^2 + \gamma_{33}^2\right) + \lambda\left(\gamma_{11} + \gamma_{22} + \gamma_{33}\right)^2 + \\
&+ (\mu+\alpha)\left(\gamma_{12}^2 + \gamma_{21}^2 + \gamma_{13}^2 + \gamma_{31}^2 + \gamma_{23}^2 + \gamma_{32}^2\right) + \\
&+ 2(\mu-\alpha)\left(\gamma_{12}\gamma_{21} + \gamma_{13}\gamma_{31} + \gamma_{23}\gamma_{32}\right) + \\
&+ 2\gamma\left(\chi_{11}^2 + \chi_{22}^2 + \chi_{33}^2\right) + \beta\left(\chi_{11} + \chi_{22} + \chi_{33}\right)^2 + \\
&+ (\gamma+\varepsilon)\left(\chi_{12}^2 + \chi_{21}^2 + \chi_{13}^2 + \chi_{31}^2 + \chi_{23}^2 + \chi_{32}^2\right) + \\
&+ 2(\gamma-\varepsilon)\left(\chi_{12}\chi_{21} + \chi_{13}\chi_{31} + \chi_{23}\chi_{32}\right)\Big\}.
\end{aligned} \tag{9}
$$

Analogically, the elastic potential can be expressed through force and moment stresses, using feedback physical relations.

As we'll see [7], the elastic potential (9) is a positively definite quadratic form. Thus, as in the classical theory [8], uniqueness theorem [5], existence theorem, Ritz and Galyorkin variation principles are available in the micropolar theory.

Formulas of Green type also take place in the micropolar theory:

$$
\sigma_{11} = \frac{\partial W}{\partial \gamma_{11}}, \ldots, \quad \sigma_{32} = \frac{\partial W}{\partial \gamma_{32}}, \quad \mu_{11} = \frac{\partial W}{\partial \chi_{11}}, \quad \ldots, \quad \mu_{32} = \frac{\partial W}{\partial \chi_{32}}.
$$

If we use feedback physical relations in formula (7), we'll obtain the elastic potential as a positively definite quadratic function, expressed through the components of force and moment stresses. Then it will be easy also to obtain formulas of Castigliano type [5]

$$
\gamma_{11} = \frac{\partial W}{\partial \sigma_{11}}, \ldots, \quad \gamma_{32} = \frac{\partial W}{\partial \sigma_{32}}, \quad \chi_{11} = \frac{\partial W}{\partial \mu_{11}}, \quad \ldots, \quad \chi_{32} = \frac{\partial W}{\partial \mu_{32}}.
$$

It should be noted that Betti's reciprocity theorem also true in the micropolar theory of elasticity [5]:

$$
\begin{aligned}
&\iint_{S^+} \left(q_1^+ V_1' + q_2^+ V_2' + q_3^+ V_3' + m_1^+\omega_1' + m_2^+\omega_2' + m_3^+\omega_3'\right) H_1 H_2 d\alpha_1 d\alpha_2 + \\
&+ \iint_{S^-} \left(q_1^- V_1' + q_2^- V_2' + q_3^- V_3' + m_1^-\omega_1' + m_2^-\omega_2' + m_3^-\omega_3'\right) H_1 H_2 d\alpha_1 d\alpha_2 + \\
&+ \int_{-h}^{h} d\alpha_3 \int_{l_1} \left(\sigma_{21}^0 V_1' + \sigma_{22}^0 V_2' + \sigma_{23}^0 V_3' + \mu_{21}^0\omega_1' + \mu_{22}^0\omega_2' + \mu_{23}^0\omega_3'\right) H_1 d\alpha_1 + \\
&+ \int_{-h}^{h} d\alpha_3 \int_{l_2} \left(\sigma_{11}^0 V_1' + \sigma_{12}^0 V_2' + \sigma_{13}^0 V_3' + \mu_{11}^0\omega_1' + \mu_{12}^0\omega_2' + \mu_{13}^0\omega_3'\right) H_2 d\alpha_1 = \\
&= \iint_{S^+} \left(q_1'^+ V_1 + q_2'^+ V_2 + q_3'^+ V_3 + m_1'^+\omega_1 + m_2'^+\omega_2 + m_3'^+\omega_3\right) \cdot \\
&\quad \cdot H_1 H_2 d\alpha_1 d\alpha_2 + \\
&+ \iint_{S^-} \left(q_1'^- V_1 + q_2'^- V_2 + q_3'^- V_3 + m_1'^-\omega_1' + m_2'^-\omega_2 + m_3'^-\omega_3'\right) \cdot \\
&\quad \cdot H_1 H_2 d\alpha_1 d\alpha_2 + \\
&+ \int_{-h}^{h} d\alpha_3 \int_{l_1} \left(\sigma_{21}'^0 V_1 + \sigma_{22}'^0 V_2 + \sigma_{23}'^0 V_3 + \mu_{21}'^0\omega_1 + \mu_{22}'^0\omega_2 + \mu_{23}'^0\omega_3\right) H_1 d\alpha_1 + \\
&+ \int_{-h}^{h} d\alpha_3 \int_{l_2} \left(\sigma_{11}'^0 V_1 + \sigma_{12}'^0 V_2 + \sigma_{13}'^0 V_3 + \mu_{11}'^0\omega_1 + \mu_{12}'^0\omega_2 + \mu_{13}'^0\omega_3\right) H_2 d\alpha_1.
\end{aligned}
$$

Both in the classical theory of elasticity and micropolar theory of elasticity variation principles are called to replace the task of integration of boundary-value problem (1)-(5) with the task of determination of extremum of corresponding functional.

Now we'll formulate general variation principle of the three-dimensional boundary-value problem of micropolar theory of elasticity with free fields of displacements and rotations and we'll show that all relations and boundary conditions (1)-(5) are the consequences of corresponding variation equation.

Following functional is studied:

$$
\begin{aligned}
I = \int_{-h}^{h}\iint_{S} \Bigg\langle W - \Bigg\{ &\sigma_{11}\left[\gamma_{11} - \left(\frac{1}{H_1}\frac{\partial V_1}{\partial \alpha_1} + \frac{1}{H_1 H_2}\frac{\partial H_1}{\partial \alpha_2} V_2 + \frac{1}{H_1}\frac{\partial H_1}{\partial \alpha_3} V_3\right)\right] + \\
&+ \sigma_{22}\left[\gamma_{22} - \left(\frac{1}{H_2}\frac{\partial V_2}{\partial \alpha_2} + \frac{1}{H_1 H_2}\frac{\partial H_2}{\partial \alpha_1} V_1 + \frac{1}{H_2}\frac{\partial H_2}{\partial \alpha_3} V_3\right)\right] + \\
&+ \sigma_{33}\left[\gamma_{33} - \frac{\partial V_3}{\partial \alpha_3}\right] + \\
&+ \sigma_{12}\left[\gamma_{12} - \left(\frac{1}{H_1}\frac{\partial V_2}{\partial \alpha_1} - \frac{1}{H_1 H_2}\frac{\partial H_1}{\partial \alpha_2} V_1 - \omega_3\right)\right] + \\
&+ \sigma_{21}\left[\gamma_{21} - \left(\frac{1}{H_2}\frac{\partial V_1}{\partial \alpha_2} - \frac{1}{H_1 H_2}\frac{\partial H_2}{\partial \alpha_1} V_2 + \omega_3\right)\right] +
\end{aligned}
$$

$$+\sigma_{31}\left[\gamma_{31}-\left(\frac{\partial V_1}{\partial\alpha_3}-\omega_2\right)\right]+$$
$$+\sigma_{13}\left[\gamma_{13}-\left(\frac{1}{H_1}\frac{\partial V_3}{\partial\alpha_1}-\frac{1}{H_1}\frac{\partial H_1}{\partial\alpha_3}V_1+\omega_2\right)\right]+$$
$$+\sigma_{32}\left[\gamma_{32}-\left(\frac{\partial V_2}{\partial\alpha_3}+\omega_1\right)\right]+$$
$$+\sigma_{23}\left[\gamma_{23}-\left(\frac{1}{H_2}\frac{\partial V_3}{\partial\alpha_2}-\frac{1}{H_2}\frac{\partial H_2}{\partial\alpha_3}V_2-\omega_1\right)\right]+$$
$$+\mu_{11}\left[\chi_{11}-\left(\frac{1}{H_1}\frac{\partial\omega_1}{\partial\alpha_1}+\frac{1}{H_1H_2}\frac{\partial H_1}{\partial\alpha_2}\omega_2+\frac{1}{H_1}\frac{\partial H_1}{\partial\alpha_3}\omega_3\right)\right]+$$
$$+\mu_{22}\left[\chi_{22}-\left(\frac{1}{H_2}\frac{\partial\omega_2}{\partial\alpha_2}+\frac{1}{H_1H_2}\frac{\partial H_2}{\partial\alpha_1}\omega_1+\frac{1}{H_2}\frac{\partial H_2}{\partial\alpha_3}\omega_3\right)\right]+$$
$$+\mu_{33}\left[\chi_{33}-\frac{\partial\omega_3}{\partial\alpha_3}\right]+$$
$$+\mu_{12}\left[\chi_{12}-\left(\frac{1}{H_1}\frac{\partial\omega_2}{\partial\alpha_1}-\frac{1}{H_1H_2}\frac{\partial H_1}{\partial\alpha_2}\omega_1\right)\right]+$$
$$+\mu_{21}\left[\chi_{21}-\left(\frac{1}{H_2}\frac{\partial\omega_1}{\partial\alpha_2}-\frac{1}{H_1H_2}\frac{\partial H_2}{\partial\alpha_1}\omega_2\right)\right]+$$
$$+\mu_{13}\left[\chi_{13}-\left(\frac{1}{H_1}\frac{\partial\omega_3}{\partial\alpha_1}-\frac{1}{H_1}\frac{\partial H_1}{\partial\alpha_3}\omega_1\right)\right]+\mu_{31}\left[\chi_{31}-\frac{\partial\omega_1}{\partial\alpha_3}\right]+$$
$$+\mu_{23}\left[\chi_{23}-\left(\frac{1}{H_2}\frac{\partial\omega_3}{\partial\alpha_2}-\frac{1}{H_2}\frac{\partial H_2}{\partial\alpha_3}\omega_2\right)\right]+$$
$$+\mu_{32}\left[\chi_{32}-\frac{\partial\omega_2}{\partial\alpha_3}\right]\Bigg\}\Bigg\rangle H_1H_2\,d\alpha_1 d\alpha_2 d\alpha_3-$$
$$-\iint_{S^+}\left[q_1^+V_1+q_2^+V_2+q_3^+V_3+m_1^+\omega_1+m_2^+\omega_2+m_3^+\omega_3\right]_{\alpha_3=h}\cdot$$
$$\cdot H_1H_2\,d\alpha_1 d\alpha_2+$$
$$+\iint_{S^-}\left[q_1^-V_1+q_2^-V_2+q_3^-V_3+m_1^-\omega_1+m_2^-\omega_2+m_3^-\omega_3\right]_{\alpha_3=-h}\cdot$$
$$\cdot H_1H_2\,d\alpha_1 d\alpha_2+ \tag{10}$$
$$+\int_{-h}^{+h}d\alpha_3\int_{l_1'}\left(\sigma_{21}^0V_1+\sigma_{22}^0V_2+\sigma_{23}^0V_3+\mu_{21}^0\omega_1+\mu_{22}^0\omega_2+\mu_{23}^0\omega_3\right)\cdot$$
$$\cdot H_1 d\alpha_1+$$
$$+\int_{-h}^{+h}d\alpha_3\int_{l_1''}\left[\sigma_{21}\left(V_1-V_1^0\right)+\sigma_{22}\left(V_2-V_2^0\right)+\sigma_{23}\left(V_3-V_3^0\right)+\right.$$
$$+\mu_{21}\left(\omega_1-\omega_1^0\right)+$$
$$\left.+\mu_{22}\left(\omega_2-\omega_2^0\right)+\mu_{23}\left(\omega_3-\omega_3^0\right)\right]H_1 d\alpha_1+$$
$$+\int_{-h}^{+h}d\alpha_3\int_{l_2'}\left(\sigma_{11}^0V_1+\sigma_{12}^0V_2+\sigma_{13}^0V_3+\mu_{11}^0\omega_1+\mu_{12}^0\omega_2+\mu_{13}^0\omega_3\right)\cdot$$
$$\cdot H_2 d\alpha_2+$$
$$+\int_{-h}^{+h}d\alpha_3\int_{l_2''}\left[\sigma_{11}\left(V_1-V_1^0\right)+\sigma_{12}\left(V_2-V_2^0\right)+\sigma_{13}\left(V_3-V_3^0\right)+\right.$$
$$\left.+\mu_{11}\left(\omega_1-\omega_1^0\right)+\mu_{12}\left(\omega_2-\omega_2^0\right)+\mu_{13}\left(\omega_3-\omega_3^0\right)\right]H_2 d\alpha_2,$$

where the integral is taken along the whole volume of the shell; surface integrals are taken on front surfaces $S^+,S^-(\alpha_3=\pm h)$ of the shell and on Σ, where external forces-moments and displacements-rotations are given; W is the density of potential energy of deformation ((7) or (9)); quantities with superscript 0 are the external forces and moments, given on some part of the shell side surface or displacements and rotations, given on the other part of the shell side surface; $l_1=l_1'\cup l_1''$, $l_2=l_2'\cup l_2''$ are corresponding parts of the shell middle surface contour. We call functional (10) full functional of three-dimensional micropolar theory of elasticity. Variation equation $(\delta I=0)$ can be obtained on the basis of this full functional, from where all main equations ((1)-(3)) and boundary conditions ((4), (5)) of micropolar theory are obtained.

III. Mathematical Model of Micropolar Elastic Thin Shells with Free Fields of Displacements and Rotations

Taking into consideration the qualitative results of the asymptotic solution of the system of equations (1) - (3) with above mentioned boundary conditions (4), (5) and on the basis of asymptotic integration process of this problem [9], [10] the following rather general assumptions (hypotheses) are put in the basis of the proposed theory of micropolar elastic thin shells with free fields of displacements and rotations:

1. During the deformation initially straight and normal to the middle surface fibers rotate freely in space at an angle as a whole rigid body, without changing their length and without remaining perpendicular to the deformed middle surface.

The formulated hypothesis is mathematically written as follows: tangential displacements and normal free rotation are changed by a linear law along the shell thickness:

$$V_i=u_i(\alpha_1,\alpha_2)+\alpha_3\psi_i(\alpha_1,\alpha_2),\quad \omega_3=\Omega_3(\alpha_1,\alpha_2)+\alpha_3\iota(\alpha_1,\alpha_2). \tag{11}$$

Normal displacement and tangential rotations do not depend on coordinate α_3.

$$V_3=w(\alpha_1,\alpha_2),\quad \omega_i=\Omega_i(\alpha_1,\alpha_2),\ i=1,2. \tag{12}$$

It should be noted that from the point of view of displacements the accepted hypothesis, in essence, is Timoshenko's kinematic hypothesis in the classical theory of elastic shells [4]. Here hypothesis (11), (12) in full we shall call Timoshenko's generalized kinematic hypothesis in the micropolar theory of shells.

2. In the generalized Hook's law (2) for γ_{ii}, force stress σ_{33} can be neglected in relation to the force

stresses σ_{ii}. Analogically, in the generalized Hook's law (2) for χ_{i3}, moment stresses μ_{3i} can be neglected in relation to $\mu_{i3}(i=1,2)$.

3. During the determination of the deformations, bending-torsions, force and moment stresses, first for the force stresses σ_{3i} and moment stress μ_{33} we'll take:

$$\sigma_{3i}=\overset{0}{\sigma}_{3i}(\alpha_1,\alpha_2),\quad \mu_{33}=\overset{0}{\mu}_{33}(\alpha_1,\alpha_2). \tag{13}$$

After determination of the above mentioned quantities, values of σ_{3i} and μ_{33} will be finally defined by the addition to the corresponding values (13) summed up, obtained by integration of the first two and the sixth equilibrium equations of (1), for which the condition will be required, that quantities, averaged along the shell thickness, are equal to zero.

4. Quantities $\dfrac{\alpha_3}{R_i}$ can be neglected in relation to 1.

During the construction of theory of micropolar elastic thin shells with the help of the assumptions 1-4 transverse shears and related deformations are completely taken into account.

On the basis of Timoshenko's kinematic hypothesis ((11), (12)) following formulas for components of deformations and bending-torsions will be obtained from relations (3):

$$\begin{aligned}
&\gamma_{ii}=\Gamma_{ii}(\alpha_1,\alpha_2)+\alpha_3 K_{ii}(\alpha_1,\alpha_2),\\
&\gamma_{ij}=\Gamma_{ij}(\alpha_1,\alpha_2)+\alpha_3 K_{ij}(\alpha_1,\alpha_2),\\
&\gamma_{i3}=\Gamma_{i3}(\alpha_1,\alpha_2),\quad \gamma_{3i}=\Gamma_{3i}(\alpha_1,\alpha_2),\quad \gamma_{33}=0.
\end{aligned} \tag{14}$$

$$\begin{aligned}
&\chi_{ii}=\kappa_{ii}(\alpha_1,\alpha_2),\quad \chi_{ij}=\kappa_{ij}(\alpha_1,\alpha_2),\\
&\chi_{i3}=\kappa_{i3}(\alpha_1,\alpha_2)+\alpha_3 l_{i3}(\alpha_1,\alpha_2),\\
&\chi_{33}=\kappa_{33}(\alpha_1,\alpha_2),\qquad \chi_{3i}=0,
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
&\Gamma_{ii}=\frac{1}{A_i}\frac{\partial u_i}{\partial\alpha_i}+\frac{1}{A_iA_j}\frac{\partial A_i}{\partial\alpha_j}u_j+\frac{w}{R_i},\\
&\Gamma_{ij}=\frac{1}{A_i}\frac{\partial u_j}{\partial\alpha_i}-\frac{1}{A_iA_j}\frac{\partial A_i}{\partial\alpha_j}u_i-(-1)^j\Omega_3,\\
&K_{ii}=\frac{1}{A_i}\frac{\partial \psi_i}{\partial\alpha_i}+\frac{1}{A_iA_j}\frac{\partial A_i}{\partial\alpha_j}\psi_j,\\
&K_{ij}=\frac{1}{A_i}\frac{\partial \psi_j}{\partial\alpha_i}-\frac{1}{A_iA_j}\frac{\partial A_i}{\partial\alpha_j}\psi_i-(-1)^j\iota,\\
&\Gamma_{i3}=-\vartheta_i+(-1)^j\Omega_j,\qquad \Gamma_{3i}=\psi_i-(-1)^j\Omega_j,\\
&\vartheta_i=-\frac{1}{A_i}\frac{\partial w}{\partial\alpha_i}+\frac{u_i}{R_i},
\end{aligned} \tag{16}$$

$$\begin{aligned}
&\kappa_{ii}=\frac{1}{A_i}\frac{\partial \Omega_i}{\partial\alpha_i}+\frac{1}{A_iA_j}\frac{\partial A_i}{\partial\alpha_j}\Omega_j+\frac{\Omega_3}{R_i},\\
&\kappa_{ij}=\frac{1}{A_i}\frac{\partial \Omega_j}{\partial\alpha_i}-\frac{1}{A_iA_j}\frac{\partial A_i}{\partial\alpha_j}\Omega_i,\\
&\kappa_{i3}=\frac{1}{A_i}\frac{\partial \Omega_3}{\partial\alpha_i}-\frac{\Omega_i}{R_i},\quad l_{i3}=\frac{1}{A_i}\frac{\partial \iota}{\partial\alpha_i}.
\end{aligned} \tag{17}$$

Further, on the basis of generalized Hook's law (2), equilibrium equations (1) and accepted hypotheses, following formulas for components of force and moment stresses will be obtained:

$$\begin{aligned}
&\sigma_{ii}=\frac{E}{1-\nu^2}\left(\Gamma_{ii}+\nu\Gamma_{jj}\right)+\alpha_3\frac{E}{1-\nu^2}\left(K_{ii}+\nu K_{jj}\right),\\
&\sigma_{ij}=\left[(\mu+\alpha)\Gamma_{ij}+(\mu-\alpha)\Gamma_{ji}\right]+\alpha_3\left[(\mu+\alpha)K_{ij}+(\mu-\alpha)K_{ji}\right],\\
&\sigma_{i3}=(\mu+\alpha)\Gamma_{i3}+(\mu-\alpha)\Gamma_{3i},\\
&\sigma_{33}=\frac{q_3^+-q_3^-}{2}+\frac{\alpha_3}{2h}\left(q_3^++q_3^-\right),\\
&\sigma_{3i}=\overset{0}{\sigma}_{3i}(\alpha_1,\alpha_2)+\alpha_3\left\{-\frac{1}{A_iA_j}\left[\frac{\partial\left(A_j\overset{0}{\sigma}_{ii}\right)}{\partial\alpha_i}+\frac{\partial\left(A_i\overset{0}{\sigma}_{ji}\right)}{\partial\alpha_j}\right]+\right.\\
&\qquad\left.+\frac{1}{A_iA_j}\frac{\partial A_j}{\partial\alpha_i}\overset{0}{\sigma}_{jj}-\frac{1}{A_iA_j}\frac{\partial A_i}{\partial\alpha_j}\overset{0}{\sigma}_{ij}-\frac{\sigma_{i3}}{R_i}\right\}+\\
&\qquad+\left(\frac{\alpha_3^2}{2}-\frac{h^2}{6}\right)\left\{-\frac{1}{A_iA_j}\left[\frac{\partial\left(A_j\overset{1}{\sigma}_{ii}\right)}{\partial\alpha_i}+\frac{\partial\left(A_i\overset{1}{\sigma}_{ji}\right)}{\partial\alpha_j}\right]+\right.\\
&\qquad\left.+\frac{1}{A_iA_j}\frac{\partial A_j}{\partial\alpha_i}\overset{1}{\sigma}_{jj}-\frac{1}{A_iA_j}\frac{\partial A_i}{\partial\alpha_j}\overset{1}{\sigma}_{ij}\right\},\\
&\mu_{ii}=\frac{4\gamma(\beta+\gamma)}{\beta+2\gamma}\kappa_{ii}+\frac{2\gamma\beta}{\beta+2\gamma}\kappa_{jj}+\frac{\beta}{\beta+2\gamma}\overset{0}{\mu}_{33},\\
&\mu_{ij}=(\gamma+\varepsilon)\kappa_{ij}+(\gamma-\varepsilon)\kappa_{ji},\\
&\iota=\frac{\beta+\gamma}{\gamma(3\beta+2\gamma)}\overset{0}{\mu}_{33}-\frac{\beta}{2\gamma(3\beta+2\gamma)}\left(\mu_{11}+\mu_{22}\right),\\
&\mu_{3i}=\frac{m_i^+-m_i^-}{2}+\frac{\alpha_3}{2h}\left(m_i^++m_i^-\right),\\
&\mu_{33}=\overset{0}{\mu}_{33}(\alpha_1,\alpha_2)+\alpha_3\left\{-\frac{1}{A_1A_2}\left(\frac{\partial\left(A_2\overset{0}{\mu}_{13}\right)}{\partial\alpha_1}+\frac{\partial\left(A_1\overset{0}{\mu}_{23}\right)}{\partial\alpha_2}\right)+\right.\\
&\qquad\left.+\left(\frac{\mu_{11}}{R_1}+\frac{\mu_{22}}{R_2}\right)-\left(\overset{0}{\sigma}_{12}-\overset{0}{\sigma}_{21}\right)\right\}+\left(\frac{\alpha_3^2}{2}-\frac{h^2}{6}\right)\cdot\\
&\qquad\cdot\left\{-\frac{1}{A_1A_2}\left(\frac{\partial\left(A_2\overset{1}{\mu}_{13}\right)}{\partial\alpha_1}+\frac{\partial\left(A_1\overset{1}{\mu}_{23}\right)}{\partial\alpha_2}\right)-\left(\overset{1}{\sigma}_{12}-\overset{1}{\sigma}_{21}\right)\right\},\\
&\mu_{i3}=\left[\frac{4\gamma\varepsilon}{\gamma+\varepsilon}\kappa_{i3}\right]+\alpha_3\left[\frac{4\gamma\varepsilon}{\gamma+\varepsilon}l_{i3}\right].
\end{aligned} \tag{18}$$

Here $\overset{0}{\sigma}_{ii}, \overset{0}{\sigma}_{ij}, \overset{0}{\mu}_{i3}, \overset{1}{\sigma}_{ii}, \overset{1}{\sigma}_{ij}, \overset{1}{\mu}_{i3}$ are constant and linear by α_3 parts of force stresses σ_{ii}, σ_{ij} and moment stresses μ_{i3}.

In order to bring three-dimensional problem of the micropolar theory of elasticity to two-dimensional one, which has been already done for displacements, deformations, bending-torsions, force and moment stresses, instead of the components of the tensors of force and moment stresses, statically equivalent to them integral characteristics-forces $T_{ii}, S_{ij}, N_{i3}, N_{3i}$, moments $M_{ii}, H_{ij}, L_{ii}, L_{ij}, L_{i3}, L_{33}$ and hypermoments Λ_{i3} are introduced, which are expressed as follows with the help of assumption 4:

$$T_{ii} = \int_{-h}^{h} \sigma_{ii} d\alpha_3, \quad S_{ij} = \int_{-h}^{h} \sigma_{ij} d\alpha_3, \; N_{i3} = \int_{-h}^{h} \sigma_{i3} d\alpha_3,$$
$$N_{3i} = \int_{-h}^{h} \sigma_{3i} d\alpha_3, \quad M_{ii} = \int_{-h}^{h} \alpha_3 \sigma_{ii} d\alpha_3, \quad H_{ij} = \int_{-h}^{h} \alpha_3 \sigma_{ij} d\alpha_3,$$
$$L_{ii} = \int_{-h}^{h} \mu_{ii} d\alpha_3, \quad L_{ij} = \int_{-h}^{h} \mu_{ij} d\alpha_3, \tag{19}$$
$$L_{33} = \int_{-h}^{h} \mu_{33} d\alpha_3, \quad L_{i3} = \int_{-h}^{h} \mu_{i3} d\alpha_3, \quad \Lambda_{i3} = \int_{-h}^{h} \alpha_3 \mu_{i3} d\alpha_3.$$

The basic system of equations of micropolar elastic thin shells with free fields of displacements and rotations will be expressed as follows:

Equilibrium equations:

$$\frac{1}{A_i}\frac{\partial T_{ii}}{\partial \alpha_i} + \frac{1}{A_i A_j}\frac{\partial A_j}{\partial \alpha_i}\left(T_{ii} - T_{jj}\right) + \frac{1}{A_j}\frac{\partial S_{ji}}{\partial \alpha_j} +$$
$$+ \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}\left(S_{ji} + S_{ij}\right) + \frac{N_{i3}}{R_i} = -\left(q_i^+ + q_i^-\right),$$
$$\frac{1}{A_i}\frac{\partial M_{ii}}{\partial \alpha_i} + \frac{1}{A_i A_j}\frac{\partial A_j}{\partial \alpha_i}\left(M_{ii} - M_{jj}\right) + \frac{1}{A_j}\frac{\partial H_{ji}}{\partial \alpha_j} +$$
$$+ \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}\left(H_{ji} + H_{ij}\right) - N_{3i} = -h\left(q_i^+ - q_i^-\right),$$
$$\frac{T_{11}}{R_1} + \frac{T_{22}}{R_2} - \frac{1}{A_1 A_2}\left[\frac{\partial\left(A_2 N_{13}\right)}{\partial \alpha_1} + \frac{\partial\left(A_1 N_{23}\right)}{\partial \alpha_2}\right] = q_3^+ + q_3^-, \tag{20}$$
$$\frac{1}{A_i}\frac{\partial L_{ii}}{\partial \alpha_i} + \frac{1}{A_i A_j}\frac{\partial A_j}{\partial \alpha_i}\left(L_{ii} - L_{jj}\right) + \frac{1}{A_j}\frac{\partial L_{ji}}{\partial \alpha_j} +$$
$$+ \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}\left(L_{ji} + L_{ij}\right) + \frac{L_{i3}}{R_i} +$$
$$+ (-1)^j\left(N_{j3} - N_{3j}\right) = -\left(m_i^+ + m_i^-\right),$$
$$\frac{L_{11}}{R_1} + \frac{L_{22}}{R_2} - \frac{1}{A_1 A_2}\left[\frac{\partial\left(A_2 L_{13}\right)}{\partial \alpha_1} + \frac{\partial\left(A_1 L_{23}\right)}{\partial \alpha_2}\right] - \tag{21}$$
$$-\left(S_{12} - S_{21}\right) = \left(m_3^+ + m_3^-\right),$$
$$L_{33} - \frac{1}{A_1 A_2}\left[\frac{\partial\left(A_2 \Lambda_{13}\right)}{\partial \alpha_1} + \frac{\partial\left(A_1 \Lambda_{23}\right)}{\partial \alpha_2}\right] - \left(H_{12} - H_{21}\right) = h\left(m_3^+ - m_3^-\right).$$

Elasticity relations:

$$T_{ii} = \frac{2Eh}{1-\nu^2}\left[\Gamma_{ii} + \nu\Gamma_{jj}\right], \quad S_{ij} = 2h\left[(\mu+\alpha)\Gamma_{ij} + (\mu-\alpha)\Gamma_{ji}\right],$$
$$M_{ii} = \frac{2Eh^3}{3\left(1-\nu^2\right)}\left[K_{ii} + \nu K_{jj}\right],$$
$$H_{ij} = \frac{2h^3}{3}\left[(\mu+\alpha)K_{ij} + (\mu-\alpha)K_{ji}\right], \tag{22}$$
$$N_{i3} = 2h(\mu+\alpha)\Gamma_{i3} + 2h(\mu-\alpha)\Gamma_{3i},$$
$$N_{3i} = 2h(\mu+\alpha)\Gamma_{3i} + 2h(\mu-\alpha)\Gamma_{i3},$$
$$L_{ii} = 2h\left[\frac{4\gamma(\beta+\gamma)}{\beta+2\gamma}\kappa_{ii} + \frac{2\gamma\beta}{\beta+2\gamma}\kappa_{jj}\right] + \frac{\beta}{\beta+2\gamma}L_{33},$$
$$L_{ij} = 2h\left[(\gamma+\varepsilon)\kappa_{ij} + (\gamma-\varepsilon)\kappa_{ji}\right],$$
$$L_{33} = 2h\left[(\beta+2\gamma)\iota + \beta\left(\kappa_{11} + \kappa_{22}\right)\right], \tag{23}$$
$$L_{i3} = 2h\left[\frac{4\gamma\varepsilon}{\gamma+\varepsilon}\kappa_{i3}\right], \quad \Lambda_{i3} = \frac{2h^3}{3}\left[\frac{4\gamma\varepsilon}{\gamma+\varepsilon}l_{i3}\right].$$

Geometric relations (16), (17) should be added to the equilibrium equations (20), (21) and elastiicity relations (22), (23).

"Softened" boundary conditions are introduced on the boundary contour Γ of the middle surface of the shell, considering that this contour matches with the coordinate line $\alpha_1 = const$.

$$T_{11} = T_{11}^* \text{ or } u_1 = u_1^*, \quad S_{12} = S_{12}^* \text{ or } u_2 = u_2^*,$$
$$N_{13} = N_{13}^* \text{ or } w = w^*, \tag{24}$$
$$M_{11} = M_{11}^* \text{ or } K_{11} = K_{11}^*, \quad H_{12} = H_{12}^* \text{ or } K_{12} = K_{12}^*.$$
$$L_{11} = L_{11}^* \text{ or } \kappa_{11} = \kappa_{11}^*, \quad L_{12} = L_{12}^* \text{ or } \kappa_{12} = \kappa_{12}^*,$$
$$L_{13} = L_{13}^* \text{ or } \kappa_{13} = \kappa_{13}^*, \; \Lambda_{13} = \Lambda_{13}^* \text{ or } l_{13} = l_{13}^*. \tag{25}$$

The system of equations (20)-(23), (16), (17) of micropolar elastic thin shells with free fields of displacements and rotations is a system of differential equations of 18th order with 9 boundary conditions (24), (25) on each contour of the middle surface of the shell. It is a system of 52 equations for 52 unknown functions: $u_i, w, \psi_i, \Omega_i, \Omega_3, \iota, \vartheta_i, T_{ii}, S_{ij}, N_{i3}, N_{3i}$, $M_{ii}, H_{ij}, L_{ii}, L_{ij}, L_{33}, L_{i3}, \Lambda_{i3}, \Gamma_{ii}, \Gamma_{ij}, \Gamma_{i3}, \Gamma_{3i}, K_{ii}, K_{ij}, \kappa_{ii}$, $\kappa_{ij}, \kappa_{i3}, l_{i3}$.

Transverse shears and related deformations are completely taken into account in the model (20)-(23), (16), (17), (24), (25) of micropolar elastic thin shells with free fields of displacements and rotations.

If we put $\alpha = 0$ in the model (20)-(23), (16), (17), (24), (25), then system of equations and boundary conditions of Timoshenko's classical theory of elastic shells will be separated [4] (with some difference related to the static hypothesis 3).

IV. Energy balance equation, theorem of Betty's type, functional and variation equation of microplar elastic thin shells with free fields of displacements and rotations.

On the basis of the above accepted hypotheses we'll pass to two-dimensional continuum in the energetic relation (6) and in the main functional (10), making integration by α_3 from $-h$ to $+h$. As a result we obtain:

1. Energy balance equation in the averaged form:

$$\iint\limits_S W_0 A_1 A_2 d\alpha_1 d\alpha_2 = A_0, \tag{26}$$

where

$$W_0 = \frac{1}{2}\big(T_{11}\Gamma_{11} + T_{22}\Gamma_{22} + S_{12}\Gamma_{12} + S_{21}\Gamma_{21} + M_{11}K_{11} + M_{22}K_{22} + H_{12}K_{12} + H_{21}K_{21} + N_{13}\Gamma_{13} + N_{31}\Gamma_{31} + N_{23}\Gamma_{23} + N_{32}\Gamma_{32} + L_{11}k_{11} + L_{22}k_{22} + L_{33}k_{33} + L_{12}k_{12} + L_{21}k_{21} + L_{13}k_{13} + L_{23}k_{23} + \Lambda_{13}l_{13} + \Lambda_{23}l_{23}\big),$$

or on the basis of (22), (23)

$$\begin{aligned} W_0 = \frac{1}{2}\Big\langle & \frac{2Eh}{1-\nu^2}\left(\Gamma_{11}^2 + \Gamma_{22}^2 + 2\nu\Gamma_{11}\Gamma_{22}\right) + \\ & + \frac{2Eh^3}{3(1-\nu^2)}\left(K_{11}^2 + K_{22}^2 + 2\nu K_{11}K_{22}\right) + \left[2h(\mu+\alpha)\left(\Gamma_{12}^2 + \Gamma_{21}^2\right) + \right. \\ & \left. + 2(\mu-\alpha)\Gamma_{12}\Gamma_{21}\right] + \\ & + \frac{2h^3}{3}\left[(\mu+\alpha)\left(K_{12}^2 + K_{21}^2\right) + 2(\mu-\alpha)K_{12}K_{31}\right] + \\ & + 2h\left[(\mu+\alpha)\left(\Gamma_{13}^2 + \Gamma_{31}^2 + \Gamma_{23}^2 + \Gamma_{32}^2\right) + 2(\mu-\alpha)\left(\Gamma_{13}\Gamma_{31} + \Gamma_{23}\Gamma_{32}\right)\right] + \\ & + 2h\left[(\beta+2\gamma)\left(k_{11}^2 + k_{22}^2 + k_{33}^2\right)^2 + 2\beta\left(k_{11}k_{22} + k_{11}k_{33} + k_{22}k_{33}\right)\right] + \\ & + 2h\left[(\gamma+\varepsilon)\left(k_{12}^2 + k_{21}^2\right) + \right. \\ & \left. + 2(\gamma-\varepsilon)k_{12}k_{21}\right] + 2h\frac{4\gamma\varepsilon}{\gamma+\varepsilon}\left(k_{13}^2 + k_{23}^2\right) + \frac{2h^3}{3}\frac{4\gamma\varepsilon}{\gamma+\varepsilon}\left(l_{13}^2 + l_{23}^2\right)\Big\rangle. \end{aligned} \tag{27}$$

A_0 is the work of external forces and moments:

$$\begin{aligned} A_0 = \frac{1}{2}\Bigg\{ & \int\limits_{l_1}\left(S_{21}^0 u_1 + H_{21}^0\psi_1 + T_{22}^0 u_2 + M_{22}^0\psi_2 + N_{23}^0 w + L_{21}^0\Omega_1 + \right. \\ & \left. + L_{22}^0\Omega_2 + L_{33}^0\Omega_3 + \Lambda_{23}^0\iota\right)A_1 d\alpha_1 + \\ & + \int\limits_{l_2}\left(T_{11}^0 u_1 + M_{11}^0\psi_1 + S_{12}^0 u_1 + H_{12}^0\psi_2 + N_{13}^0 w + L_{11}^0\Omega_1 + \right. \\ & \left. + L_{12}^0\Omega_2 + L_{23}^0\Omega_3 + \Lambda_{23}^0\iota\right)A_2 d\alpha_2 + \\ & + \iint\limits_S\Big[\left(q_1^+ + q_1^-\right)u_1 + \left(q_1^+ - q_1^-\right)h\psi_1 + \\ & + \left(q_2^+ + q_2^-\right)u_2 + \left(q_2^+ - q_2^-\right)h\psi_2 + \\ & + \left(q_3^+ + q_3^-\right)w + \left(m_1^+ + m_1^-\right)\Omega_1 + \left(m_2^+ + m_2^-\right)\Omega_2 + \left(m_3^+ + m_3^-\right)\Omega_3 + \\ & + \left(m_3^+ + m_3^-\right)h\iota\Big]A_1 A_2 d\alpha_1 d\alpha_2\Bigg\}. \end{aligned} \tag{28}$$

On the basis of energetic balance equation [8], uniqueness theorem, existence theorem and other energetic theorems can be proved in the theory of micropolar elastic thin shells, also Ritz and Bubnov-Galyorkins' variation methods can be justified for solution of the boundary-value problem (20)-(23), (16), (17), (24), (25).

2. Formulas of Green type:

$$\begin{aligned} & T_{11} = \frac{\partial W_0}{\partial \Gamma_{11}},\ S_{12} = \frac{\partial W_0}{\partial \Gamma_{12}},\ \ldots,\ M_{11} = \frac{\partial W_0}{\partial K_{11}}, \\ & H_{12} = \frac{\partial W_0}{\partial K_{12}},\ \ldots,\ L_{11} = \frac{\partial W_0}{\partial k_{11}},\ldots \end{aligned} \tag{29}$$

3. Betty's theorem is true in the theory of micropolar elastic thin shells:

In order to prove this theorem we'll multiply equation $(20)_1$ on u_1', $(20)_2$ – on u_2', $(20)_3$ – on w', $(20)_4$ – on ψ_1', $(20)_5$ – on ψ_2', $(21)_1$ – on Ω_1, $(21)_2$ – on Ω_2, $(21)_3$ – on Ω_3, $(21)_4$ – on ι. Further we'll summarize these equations and we'll integrate them in region (S) of the shell middle surface. After some transformations, we'll obtain:

$$\begin{aligned} & -\int\limits_{l_1}\left(S_{21}u_1' + T_{22}u_2' + N_{23}w' + H_{21}\psi_1' + M_{22}\psi_2' + L_{21}\Omega_1' + L_{22}\Omega_2' + \right. \\ & \left. + L_{23}\Omega_3' + \Lambda_{23}\iota'\right)A_1 d\alpha_1 + \int\limits_{l_2}\left(T_{11}u_1' + S_{12}u_2' + N_{13}w' + M_{11}\psi_1' + \right. \\ & \left. + H_{12}\psi_2' + L_{11}\Omega_1' + L_{12}\Omega_2' + L_{13}\Omega_3' + \Lambda_{13}\iota'\right)A_2 d\alpha_2 - \\ & -\iint\limits_{(S)}\left(T_{11}\Gamma_{11}' + T_{22}\Gamma_{22}' + S_{12}\Gamma_{12}' + S_{21}\Gamma_{21}' + N_{13}\Gamma_{13}' + \right. \\ & + N_{31}\Gamma_{31}' + N_{23}\Gamma_{23}' + N_{32}\Gamma_{32}' + \\ & + M_{11}K_{11}' + M_{22}K_{22}' + H_{12}K_{12}' + H_{21}K_{21}' + L_{11}k_{11}' + L_{22}k_{22}' + \\ & + L_{12}k_{12}' + L_{21}k_{21}' + L_{13}k_{13}' + L_{23}k_{23}' + \\ & \left. + \Lambda_{13}l_{13}' + \Lambda_{23}l_{23}' + L_{33}k_{33}'\right)A_1 A_2 d\alpha_1 d\alpha_2 = \\ & = -\iint\limits_{(S)}\Big[\left(q_1^+ + q_1^-\right)u_1' + \left(q_2^+ + q_2^-\right)u_2' + \left(q_3^+ + q_3^-\right)w' + \\ & + h\left(q_1^+ - q_1^-\right)\psi_1' + h\left(q_2^+ - q_2^-\right)\psi_2' + \\ & + \left(m_1^+ + m_1^-\right)\Omega_1' + \left(m_2^+ + m_2^-\right)\Omega_2' + \left(m_3^+ + m_3^-\right)\Omega_3' + h\left(m_3^+ - m_3^-\right)\iota'\Big]\cdot \\ & \cdot A_1 A_2 d\alpha_1 d\alpha_2. \end{aligned} \tag{30}$$

On the basis of elasticity relations (22), (23) we'll obtain following relation:

$$\begin{aligned} & T_{11}\Gamma_{11}' + T_{22}\Gamma_{22}' + S_{12}\Gamma_{12}' + S_{21}\Gamma_{21}' + N_{13}\Gamma_{13}' + N_{31}\Gamma_{31}' + \\ & + N_{23}\Gamma_{23}' + N_{32}\Gamma_{32}' + M_{11}K_{11}' + M_{22}K_{22}' + H_{12}K_{12}' + \\ & + H_{21}K_{21}' + L_{11}k_{11}' + L_{22}k_{22}' + L_{12}k_{12}' + L_{21}k_{21}' + \\ & + L_{13}k_{13}' + L_{23}k_{23}' + \Lambda_{13}l_{13}' + \Lambda_{23}l_{23}' + L_{33}k_{33}' = \\ & = \frac{2Eh}{1-\nu^2}\left(\Gamma_{11}\Gamma_{11}' + \Gamma_{22}\Gamma_{22}'\right) + \\ & + \frac{2Eh\nu}{1-\nu^2}\left(\Gamma_{22}\Gamma_{11}' + \Gamma_{11}\Gamma_{22}'\right) + 2h(\mu+\alpha)\left(\Gamma_{12}\Gamma_{12}' + \Gamma_{21}\Gamma_{21}' + \right. \\ & \left. + \Gamma_{13}\Gamma_{13}' + \Gamma_{23}\Gamma_{23}' + \Gamma_{31}\Gamma_{31}' + \Gamma_{32}\Gamma_{32}'\right) + \\ & + 2h(\mu-\alpha)\left(\Gamma_{21}\Gamma_{12}' + \Gamma_{12}\Gamma_{21}' + \Gamma_{31}\Gamma_{13}' + \Gamma_{13}\Gamma_{31}' + \Gamma_{32}\Gamma_{23}' + \Gamma_{23}\Gamma_{32}'\right) + \\ & + \frac{2Eh^3}{3(1-\nu^2)}\left(K_{11}K_{11}' + K_{22}K_{22}'\right) + \frac{2Eh^3\nu}{3(1-\nu^2)}\left(K_{22}K_{11}' + K_{11}K_{22}'\right) + \end{aligned}$$

$$
\begin{aligned}
&+\frac{2h^3}{3}(\mu+\alpha)(K_{12}K'_{12}+K_{21}K'_{21})+\\
&+\frac{2h^3}{3}(\mu-\alpha)(K_{21}K'_{12}+K_{12}K'_{21})+\\
&+2h(\beta+2\gamma)(k_{11}k'_{11}+k_{22}k'_{22}+k_{33}k'_{33})+\\
&+2h\beta(k_{11}k'_{22}+k_{33}k'_{22}+k_{11}k'_{33}+k_{22}k'_{33}+\\
&+k_{22}k'_{11}+k_{33}k'_{11})+2h(\gamma+\varepsilon)(k_{12}k'_{12}+k_{21}k'_{21})+\\
&+2h(\gamma-\varepsilon)(k_{21}k'_{12}+k_{12}k'_{21})+\\
&+2h\frac{4\gamma\varepsilon}{\gamma+\varepsilon}k_{13}k'_{13}+2h\frac{4\gamma\varepsilon}{\gamma+\varepsilon}k_{23}k'_{23}+\\
&+\frac{2h^3}{3}\frac{4\gamma\varepsilon}{\gamma+\varepsilon}l_{13}l'_{13}+\frac{2h^3}{3}\frac{4\gamma\varepsilon}{\gamma+\varepsilon}l_{23}l'_{23}.
\end{aligned}
\qquad (31)
$$

Taking into consideration the symmetry of relation (31), Betty's theorem will be obtained from (30) for the general theory of micropolar elastic thin shells:

$$
\begin{aligned}
&\int_{l_1}\Bigg[\left(\int_{-h}^{h}\overset{0}{\sigma}_{22}\,d\alpha_3\right)u'_2+\left(\int_{-h}^{h}\overset{0}{\sigma}_{21}\,d\alpha_3\right)u'_1+\left(\int_{-h}^{h}\overset{0}{\sigma}_{23}\,d\alpha_3\right)w'+\\
&+\left(\int_{-h}^{h}\overset{0}{\sigma}_{22}\,\alpha_3 d\alpha_3\right)\psi'_2+\left(\int_{-h}^{h}\overset{0}{\sigma}_{21}\,\alpha_3 d\alpha_3\right)\psi'_1+\left(\int_{-h}^{h}\overset{0}{\mu}_{21}\,d\alpha_3\right)\Omega'_1+\\
&+\left(\int_{-h}^{h}\overset{0}{\mu}_{22}\,d\alpha_3\right)\Omega'_2+\left(\int_{-h}^{h}\overset{0}{\mu}_{23}\,d\alpha_3\right)\Omega'_3+\\
&+\left(\int_{-h}^{h}\overset{0}{\mu}_{23}\,d\alpha_3\right)\iota'\Bigg]A_1 d\alpha_1+\int_{l_2}\Bigg[\left(\int_{-h}^{h}\overset{0}{\sigma}_{11}\,d\alpha_3\right)u'_1+\left(\int_{-h}^{h}\overset{0}{\sigma}_{12}\,d\alpha_3\right)u'_2+\\
&+\left(\int_{-h}^{h}\overset{0}{\sigma}_{13}\,d\alpha_3\right)w'++\left(\int_{-h}^{h}\overset{0}{\sigma}_{11}\,\alpha_3 d\alpha_3\right)\psi'_1+\left(\int_{-h}^{h}\overset{0}{\sigma}_{12}\,\alpha_3 d\alpha_3\right)\psi'_2+\\
&+\left(\int_{-h}^{h}\overset{0}{\mu}_{11}\,d\alpha_3\right)\Omega'_1+\left(\int_{-h}^{h}\overset{0}{\mu}_{12}\,d\alpha_3\right)\Omega'_2+\\
&+\left(\int_{-h}^{h}\overset{0}{\mu}_{13}\,d\alpha_3\right)\Omega'_3+\left(\int_{-h}^{h}\overset{0}{\mu}_{13}\,\alpha_3 d\alpha_3\right)\iota'\Bigg]A_2 d\alpha_2+\\
&\iint_{(S)}\Big[(q_1^{+}+q_1^{-})u'_1+(q_2^{+}+q_2^{-})u'_2+(q_3^{+}+q_3^{-})w'+h(q_1^{+}-q_1^{-})\psi'_1+\\
&+h(q_2^{+}-q_2^{-})\psi'_2+(m_1^{+}+m_1^{-})\Omega'_1+(m_2^{+}+m_2^{-})\Omega'_2+\\
&+(m_3^{+}+m_3^{-})\Omega'_3+h(m_3^{+}-m_3^{-})\iota'\Big]A_1A_2\,d\alpha_1 d\alpha_2=\\
&=\int_{l_1}\Bigg[\left(\int_{-h}^{h}\overset{0}{\sigma}{}'_{22}\,d\alpha_3\right)u_2+\left(\int_{-h}^{h}\overset{0}{\sigma}{}'_{21}\,d\alpha_3\right)u_1+\left(\int_{-h}^{h}\overset{0}{\sigma}{}'_{23}\,d\alpha_3\right)w'+\\
&+\left(\int_{-h}^{h}\overset{0}{\sigma}{}'_{22}\,\alpha_3 d\alpha_3\right)\psi_2+\left(\int_{-h}^{h}\overset{0}{\sigma}{}'_{21}\,\alpha_3 d\alpha_3\right)\psi_1+\left(\int_{-h}^{h}\overset{0}{\mu}{}'_{21}\,d\alpha_3\right)\Omega_1+\\
&+\left(\int_{-h}^{h}\overset{0}{\mu}{}'_{22}\,d\alpha_3\right)\Omega_2+\left(\int_{-h}^{h}\overset{0}{\mu}{}'_{23}\,d\alpha_3\right)\Omega_3+\\
&+\left(\int_{-h}^{h}\overset{0}{\mu}{}'_{23}\,\alpha_3 d\alpha_3\right)\iota\Bigg]A_1 d\alpha_1+\\
&+\int_{l_2}\Bigg[\left(\int_{-h}^{h}\overset{0}{\sigma}{}'_{11}\,d\alpha_3\right)u_1+\left(\int_{-h}^{h}\overset{0}{\sigma}{}'_{12}\,d\alpha_3\right)u_2+\left(\int_{-h}^{h}\overset{0}{\sigma}{}'_{13}\,d\alpha_3\right)w+\\
&+\left(\int_{-h}^{h}\overset{0}{\sigma}{}'_{11}\,\alpha_3 d\alpha_3\right)\psi_1+\left(\int_{-h}^{h}\overset{0}{\sigma}{}'_{12}\,\alpha_3 d\alpha_3\right)\psi_2+\\
&+\left(\int_{-h}^{h}\overset{0}{\mu}{}'_{11}\,d\alpha_3\right)\Omega_1+\left(\int_{-h}^{h}\overset{0}{\mu}{}'_{12}\,d\alpha_3\right)\Omega_2+\left(\int_{-h}^{h}\overset{0}{\mu}{}'_{13}\,d\alpha_3\right)\Omega_3+\\
&+\left(\int_{-h}^{h}\overset{0}{\mu}{}'_{13}\,\alpha_3 d\alpha_3\right)\iota\Bigg]A_2 d\alpha_2+\\
&\iint_{(S)}\Big[(q_1'^{+}+q_1'^{-})u_1+(q_2'^{+}+q_2'^{-})u_2+(q_3'^{+}+q_3'^{-})w+\\
&+h(q_1'^{+}-q_1'^{-})\psi_1+h(q_2'^{+}-q_2'^{-})\psi_2+(m_1'^{+}+m_1'^{-})\Omega_1+\\
&+(m_2'^{+}+m_2'^{-})\Omega_2+(m_3'^{+}+m_3'^{-})\Omega_3+h(m_3'^{+}-m_3'^{-})\iota'\Big]A_1A_2\,d\alpha_1 d\alpha_2.
\end{aligned}
\qquad (32)
$$

4. Expression for the averaged functional:

$$
\begin{aligned}
I_0=\iint_S\Bigg\langle W_0-\Bigg\{&T_{11}\left[\Gamma_{11}-\left(\frac{1}{A_1}\frac{\partial u_1}{\partial\alpha_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}u_2+\frac{w}{R_1}\right)\right]+\\
&+M_{11}\left[K_{11}-\left(\frac{1}{A_1}\frac{\partial\psi_1}{\partial\alpha_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\psi_2\right)\right]+\\
&+T_{22}\left[\Gamma_{22}-\left(\frac{1}{A_2}\frac{\partial\psi_2}{\partial\alpha_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\psi_1\right)\right]+\\
&+S_{12}\left[\Gamma_{12}-\left(\frac{1}{A_1}\frac{\partial u_2}{\partial\alpha_1}-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}u_1-\Omega_3\right)\right]+\\
&+M_{12}\left[K_{12}-\left(\frac{1}{A_1}\frac{\partial\psi_2}{\partial\alpha_1}-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\psi_1-\iota\right)\right]+\\
&+S_{21}\left[\Gamma_{21}-\left(\frac{1}{A_1}\frac{\partial u_1}{\partial\alpha_1}-\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}u_2+\Omega_3\right)\right]+\\
&+M_{21}\left[K_{21}-\left(\frac{1}{A_2}\frac{\partial\psi_1}{\partial\alpha_2}-\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\psi_2+\iota\right)\right]+\\
&+N_{31}[\Gamma_{31}-(\psi_1-\Omega_2)]+N_{13}\left[\Gamma_{13}-\left(\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1}-\frac{u_1}{R_1}+\Omega_2\right)\right]+\\
&+N_{32}[\Gamma_{32}-(\psi_2+\Omega_1)]+N_{23}\left[\Gamma_{23}-\left(\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2}-\frac{u_2}{R_2}-\Omega_1\right)\right]+\\
&+L_{11}\left[k_{11}-\left(\frac{1}{A_1}\frac{\partial\Omega_1}{\partial\alpha_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\Omega_2+\frac{\Omega_3}{R_1}\right)\right]+
\end{aligned}
$$

$$+L_{22}\left[k_{22}-\left(\frac{1}{A_2}\frac{\partial\Omega_2}{\partial\alpha_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\Omega_1+\frac{\Omega_3}{R_1}\right)\right]+L_{33}(k_{33}-\iota)+$$
$$+L_{32}\left[k_{12}-\left(\frac{1}{A_1}\frac{\partial\Omega_2}{\partial\alpha_1}-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\Omega_1\right)\right]+ \quad (33)$$
$$+L_{21}\left[k_{21}-\left(\frac{1}{A_2}\frac{\partial\Omega_1}{\partial\alpha_2}-\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\Omega_2\right)\right]+$$
$$+L_{13}\left[k_{13}-\left(\frac{1}{A_2}\frac{\partial\Omega_3}{\partial\alpha_1}-\frac{\Omega_1}{R_1}\right)\right]+\Lambda_{13}\left[l_{13}-\frac{1}{A_1}\frac{\partial\iota}{\partial\alpha_1}\right]+$$
$$+L_{23}\left[k_{23}-\left(\frac{1}{A_2}\frac{\partial\Omega_3}{\partial\alpha_2}-\frac{\Omega_3}{R_2}\right)\right]+\Lambda_{23}\left[l_{23}-\frac{1}{A_2}\frac{\partial\iota}{\partial\alpha_2}\right]\Bigg\}\Bigg\rangle\cdot$$
$$\cdot A_1A_2d\alpha_1d\alpha_2-$$
$$-\iint_S\left(q_1^+u_1+q_1^+h\psi_1+q_2^+u_2+q_2^+h\psi_2+q_3^+w+m_1^+\Omega_1+\right.$$
$$\left.+m_2^+\Omega_2+m_3^+\Omega_3+m_3^+h\iota\right)A_1A_2d\alpha_1d\alpha_2+$$
$$+\iint_S\left(q_1^-u_1-q_1^-h\psi_1+q_2^-u_2-q_2^-h\psi_2+q_3^-w+m_1^-\Omega_1+\right.$$
$$\left.+m_2^-\Omega_2+m_3^-\Omega_3-m_3^-h\iota\right)A_1A_2d\alpha_1d\alpha_2+$$
$$+\int_{l_1'}\left(S_{21}^0u_1+T_{22}^0u_2+H_{21}^0\psi_1+M_{22}^0\psi_2+N_{23}^0w+L_{21}^0\Omega_1+\right.$$
$$\left.+L_{22}^0\Omega_2+L_{23}^0\Omega_3+\Lambda_{23}^0\iota\right)A_1d\alpha_1+$$
$$+\int_{l_1''}\left[S_{21}\left(u_1-u_1^0\right)+H_{21}\left(\psi_1-\psi_1^0\right)+T_{22}\left(u_2-u_2^0\right)+\right.$$
$$+M_{22}\left(\psi_2-\psi_2^0\right)+N_{23}\left(w-w^0\right)+L_{21}\left(\Omega_1-\Omega_1^0\right)+$$
$$\left.+L_{22}\left(\Omega_2-\Omega_2^0\right)+L_{23}\left(\Omega_3-\Omega_3^0\right)+\Lambda_{23}\left(\iota-\iota^0\right)\right]A_1d\alpha_1+$$
$$+\int_{l_2'}\left(T_{11}^0u_1+M_{11}^0\psi_1+S_{12}^0u_2+H_{12}^0\psi_2+N_{13}^0w+L_{11}^0\Omega_1+\right.$$
$$\left.+L_{12}^0\Omega_2+L_{13}^0\Omega_3+\Lambda_{13}^0\iota\right)A_2d\alpha_2+$$
$$+\int_{l_2''}\left[T_{11}\left(u_1-u_1^0\right)+M_{11}\left(\psi_1-\psi_1^0\right)+S_{12}\left(u_2-u_2^0\right)+H_{12}\left(\psi_2-\psi_2^0\right)+\right.$$
$$+N_{13}\left(w-w_0\right)+L_{11}\left(\Omega_1-\Omega_1^0\right)+L_{12}\left(\Omega_2-\Omega_2^0\right)+$$
$$\left.+L_{13}\left(\Omega_3-\Omega_3^0\right)+\Lambda_{13}\left(\iota-\iota^0\right)\right]A_2d\alpha_2.$$

If we make the variation of I_0 by the all unknown functional arguments, we'll obtain main equations and boundary conditions ((20)-(23), (16), (17), (24), (25)) of micropolar elastic thin shells with free fields of displacements and rotations from the equation $\delta I_0 = 0$.

It should be noted that the above formulated variation problem corresponds the general variation principle of micropolar elastic thin shells. Hence, extreme principles of Lagrange and Kastilianos' type of micropolar elastic thin shells will be also obtained as private cases. Methods of their approximate solution can be used for each variation equation (particularly, Ritz and Galyorkins' methods), which will reduce the boundary-value problem of the theory of micropolar elastic thin shells to the solution of system of algebraic equations.

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2. S.H. Sargsyan, "General theory of micropolar elastic thin shells", *Physical mesomechanics,* vol. 15, N1-2, 2012, pp. 69-79.

3. S.H. Sargsyan, "Theory of micropolar elastic thin shells", *Applied mathematics and mechanics,* vol.76, 2012, pp. 235-249.

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