Elements of mathematical phenomenology in dynamics of multi-body systems with fractionally damped discrete continuous layers

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***Abstract*—**Dynamics of deformable linear viscoelastic multi-bodies (beams. plates, membranes and belts with the same boundary conditions) coupled by standard light fractional order discrete continuous layers is considered using Petrović's theory of elements of mathematical phenomenology. Starting with coupled fractional order differential equations in terms of transverse displacements of linear elastic beams which are coupled by fractional order discrete continuous layers with the corresponding boundary conditions, a system of coupled ordinary fractional order differential equations is derived in terms of eigen amplitude functions. Independent main eigen modes and a set of characteristic numbers of the eigen time functions corresponding to eigen amplitude functions are obtained. Using Petrović's theory of mathematical analogy and qualitative analogy, properties of the main eigen modes and characteristic numbers of the time functions of vibrations of multi-plates as well as multi-membranes coupled by fractional order discrete continuous layers are studied. Energy analysis in a fractionally damped discrete continuous layer is carried out, and a generalized function its energy dissipation is defined.

***Keywords*—**Deformable multi-body system, fractionally damped discrete continuous layer, partial fractional order differential equations, phenomenological mapping, qualitative analogy, mathematical analogy.

I. Introduction

The fundamental ideas of Petrović proposed in his two monographs, *Elements of Mathematical Phenomenology* [1] and *Phenomenological Mappings* [2] published in Serbian in 1911 and 1933, respectively, from time to time appear in current scientific publications. The sixth, last chapter of [1], is entitled *Phenomenological analogies*, and it describes the fundamentals of mathematical analogies and qualitative analogy, which will be utilized in the present research for studying dynamics of such different hybrid multi-body deformable systems as beams, plates, membranes, and belts coupled by fractionally damped discrete continuous layers. Each of the considered hybrid systems contains a series of the same type deformable bodies (beams, plates or membranes) with the same boundary conditions and coupled by discrete continuous layers with linear viscoelastic properties governed by the generalized fractional order like Kelvin-Voigt model, resulting in the fact that the dynamic behaviour of such a hybrid system is described by the corresponding set of coupled partial fractional order differential equations.

In numerous papers, simple and complex rheological elements with elastic, viscoelastic, and plastic properties are utilized. All these elements could be arranged in parallel or in series, and/or in different hybrid combinations of these simple mass neglected elements. The behaviour of each such model could be described by its specific constitutive stress-strain or force-displacement relation involving the following constants of materials [3-6]: coefficients of linear elasticity or coefficients of linear rigidity, coefficients of non-linear cubic nonlinearity, relaxation times, retardation times, as well as fractional parameters characteristic for fractional order viscoelastic models. Values of all these coefficients are obtained experimentally.

II. Discrete continuous fractional order layer

Let us consider a hybrid system of deformable multi-bodies coupled by discrete continuous fractionally damped layers. The discrete continuous layer represents a set of standard light fractional order spring-dashpot elements homogeneously distributed between each of two adjacent deformable bodies. Each of the standard light fractional order elements within the discrete continuous layer is oriented axially in a common transversal direction of the deformable body displacements. During the motion of deformable bodies in the transversal direction each of the standard light fractional order elements experiences extension or compression equal to the difference between two displacements of the corresponding body points:

for plates and membranes

$$\Delta w_{k+1,k}(x,y,t)=w_{k+1}(x,y,t)-w_k(x,y,t)$$

and for beams and belts

$$\Delta w_{k+1,k}(x,t)=w_{k+1}(x,t)-w_k(x,t),$$

where $w_k(x,y,t)$ and $w_{k+1}(x,y,t)$ are the transversal displacements of middle surface points of thin ideal elastic

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plates (or membranes), x and y are the coordinates of the plate (or membrane) middle surface points, k and $k+1$ are the orders of the plates (or membranes) in a hybrid system, and $w_k(x,t)$ and $w_{k+1}(x,t)$ are the transversal displacements of neutral line points of an ideal elastic prismatic beam (or belt), x is the coordinate along the neutral line of beams (or belts), and k and $k+1$ are the orders of the beams (or belts) in a hybrid system.

Constitutive generalized force – extension relation of a fractional order spring-dashpot element according to the generalized fractional order like Kelvin-Voigt model could be written in the following form:

$$Q_{\alpha,k}(t) = -\left\{c_{0(k,k+1)}\left[w_{k+1}(x,y,t)-w_k(x,y,t)\right]\right\} \\ -\left\{c_{\alpha(k,k+1)}D_t^\alpha\left[w_{k+1}(x,y,t)-w_k(x,y,t)\right]\right\} = -Q_{\alpha,k+1}(t), \tag{1}$$

where $D_t^\alpha[\bullet]$ is the fractional order differential operator of the α^{th} derivative with respect to time t [3-10]

$$D_t^\alpha[\bullet] = \frac{d^\alpha[\bullet]}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{[\bullet]}{(t-\tau)^\alpha}d\tau \ , \tag{2}$$

$\Gamma(1-\alpha)$ is the Euler Gamma function, $0<\alpha\le 1$, $c_{0(k,k+1)}$ and $c_{\alpha(k,k+1)}$ are rigidity coefficients, in so doing $c_{0(k,k+1)}$ are prolonged moduli of elasticity, $c_{\alpha(k,k+1)} = \tau^\alpha c_{0(k,k+1)}$, and τ is the retardation time.

At $\alpha = 1$, the Voigt model (1) with a fractional time-derivative (2), i.e. the generalized Voigt model, goes over into the Voigt model with a conventional time-derivative, i.e. the classical Voigt model, since the Riemann-Liouville fractional derivative (2) goes over into the conventional first-order derivative with respect to time t. At $\alpha = 0$, the generalized Voigt model (1) loses the physical sense (see the paper of the Guest Editors of this Special Issue [11], as well as their state-of-the-art article [4] for details).

Each fractional order element possesses the potential energy

$$\tilde{\mathbf{E}}_\mathrm{p} = \frac{1}{2}c_{0(k,k+1)}\left[w_{k+1}(x,y,t)-w_k(x,y,t)\right]^2. \tag{3}$$

Full potential energy $\mathbf{E}_\mathrm{p}$ of a discrete continuous fractional order spring-dashpot layer for any α within the segment $0<\alpha\le 1$ for plate or membrane takes the form

$$\mathbf{E}_\mathrm{p} = \iint_S \tilde{\mathbf{E}}_\mathrm{p}dS = \frac{1}{2}c_{0(k,k+1)}\iint_S\left[w_{k+1}(x,y,t)-w_k(x,y,t)\right]^2 dxdy \tag{4}$$

where S is the plate (or membrane) middle surface area of the discrete continuous layer distribution.

This fractional order element in the dynamic state is no conservative element, and thus the dissipation of mechanical energy could be expressed in the form [10]

$$\tilde{\Phi}_{0<\alpha\le 1} = \frac{1}{2}c_{0<\alpha\le 1(k,k+1)}\left\{D_t^\alpha\left[w_{k+1}(x,y,t)-w_k(x,y,t)\right]\right\}^2 \ . \tag{5}$$

III. Models and Partial Fractional Order Differential Equations of Multi-Body System Dynamics

Three membranes and three beams coupled by discrete continuous fractionally damped layers into hybrid deformable multi-body systems are presented, respectively, in Figure 1 (a) and (b).

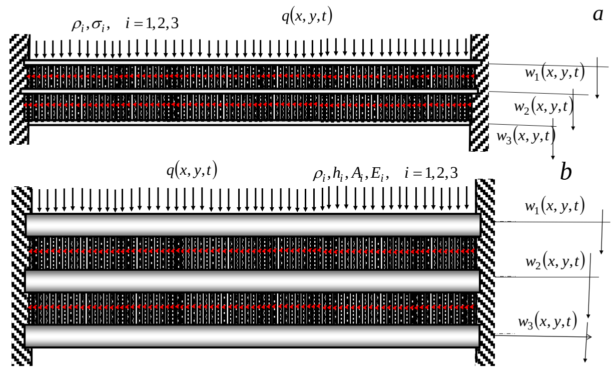

Figure 1 (a) Three membranes and (b) three beams coupled by discrete continuous fractionally damped layers into hybrid deformable multi-body systems

A. Governing partial differential equations of a hybrid deformable multi-beam system

Let us consider transverse motions of a hybrid deformable multi-beam system presented in Figure 1 (b). For this purpose, we will study a deformable three-beam system in which beams are homogeneous prismatic and pure elastic and are coupled by fractional order spring-dashpot discrete continuous layers described in Sect. II. For all three beams boundary conditions are the same.

Transverse vibrations of this three-beam system are decribed by the following three coupled partial fractional order differential equations [12]-[15]:

$$\begin{aligned} \rho_1 A_1\frac{\partial^2 w_1(x,t)}{\partial t^2} &= -B_1\frac{\partial^4 w_1(x,t)}{\partial x^4} + c_{0(1.2)}\left[w_2(x,t)-w_1(x,t)\right] \\ &\quad + c_{\alpha(1,2)}D_t^\alpha\left[w_2(x,t)-w_1(x,t)\right] + q_1(x,t), \\ \rho_2 A_2\frac{\partial^2 w_2(x,t)}{\partial t^2} &= -B_2\frac{\partial^4 w_2(x,t)}{\partial x^4} - c_{0(1.2)}\left[w_2(x,t)-w_1(x,t)\right] - \\ &\quad - c_{\alpha(1,2)}D_t^\alpha\left[w_2(x,t)-w_1(x,t)\right] + c_{0(2.3)}\left[w_3(x,t)-w_2(x,t)\right] + \\ &\quad + c_{\alpha(2,3)}D_t^\alpha\left[w_3(x,t)-w_2(x,t)\right] - q_2(x,t), \\ \rho_3 A_3\frac{\partial^2 w_3((x,t)}{\partial t^2} &= -B_3\frac{\partial^4 w_3(x,t)}{\partial x^4} - c_{0(2.3)}\left[w_3(x,t)-w_2(x,t)\right] - \\ &\quad - c_{\alpha(2,3)}D_t^\alpha\left[w_3(x,t)-w_2(x,t)\right] - q_3(x,t), \end{aligned} \tag{6}$$

where $B_k = E_k I_{zk}$ ($k = 1,2,3$) are the coefficients of flexural rigidity of the beams.

For complete description of this hybrid system, it is necessary to define and add the corresponding boundary and initial conditions. All three beams are of the same length and are subjected to the same boundary conditions. This fact permits us to suppose that eigen amplitude functions for all three beams are described via the one and same form $\mathrm{W}_m(x)$ ($m = 1,2,3,4...\infty$), and the solution is expected in the following form:

$$w_k\left(x,t\right)=\sum_{m=1}^{\infty}\mathrm{W}_m(x)T_{k(m)}(t)\,,\ (k=1,2,3)\,, \tag{7}$$

and that distributed external excitations along the beam lengths are

$$\frac{q_k(x,t)}{\rho_k}=\sum_{m=1}^{M}h_{0k,m}\,\mathrm{W}_m(x)\sin(\Omega_{km}t+\vartheta_{k,m})\ \ (k=1,2,3)\,. \tag{8}$$

Substituting the proposed solution (7) into the system of partial fractional order differential equations (6), multiplying each equation by $\mathrm{W}_r\left(x\right)$ ($r=1,2,3,4...,\infty$), and integrating all terms over the length of the beams with due account for the orthogonality of the amplitude eigen functions, as a result we obtain the set of the ordinary fractional order differential equations in terms of eigen time functions $T_{k(m)}(t)$ ($k=1,2,3$; $m=1,2,3,4,....,\infty$) in the following form:

$$\begin{aligned}
&\ddot{T}_{1(m)}(t)+c_1^2k_m^2T_{1(m)}(t)+a_{0(1,2)}^2T_{1(m)}\left(t\right)\\
&\quad+a_{\alpha(1,2)}^2D_t^{\alpha}\left[T_{1(m)}\left(t\right)\right]-a_{0(1,2)}^2T_{2(m)}(t)\\
&\quad-a_{\alpha(1,2)}^2D_t^{\alpha}\left[T_{2(m)}(t)\right]=h_{01,m}\sin\left(\Omega_{1,m}t+\vartheta_{1,m}\right),\\
&\ddot{T}_{2(m)}(t)+c_2^2k_m^2T_{2(m)}(t+\left[\tilde{a}_{0(1,2)}^2+a_{0(2,3)}^2\right]T_{2(m)}(t)+\\
&\quad+\left[\tilde{a}_{\alpha(1,2)}^2+a_{\alpha(2,3)}^2\right]D_t^{\alpha}\left[T_{2(m)}(t)\right]-\tilde{a}_{0(1,2)}^2T_{1(m)}(t)-\\
&\quad-\tilde{a}_{\alpha(1,2)}^2D_t^{\alpha}\left[T_{1(m)}(t)\right]-a_{0(2,3)}^2T_{3(m)}(t)-\\
&\quad-a_{\alpha(2,3)}^2D_t^{\alpha}\left[T_{3(m)}(t)\right]=h_{02,m}\sin\left(\Omega_{2m}t+\vartheta_{2,m}\right),\\
&\ddot{T}_{3(m)}(t)+c_3^2k_m^2T_{3(m)}(t)+\left[\tilde{a}_{0(2,3)}^2+a_{0(3,4)}^2\right]T_{3(m)}(t)\\
&\quad-\tilde{a}_{0(2,3)}^2T_{2(m)}(t)+\left[\tilde{a}_{\alpha(2,3)}^2+a_{\alpha(3,4)}^2\right]D_t^{\alpha}\left[T_{3(m)}(t)\right]\\
&\quad-\tilde{a}_{\alpha(2,3)}^2D_t^{\alpha}\left[T_{2(m)}(t)\right]\approx h_{03,m}\sin\left(\Omega_{3m}t+\vartheta_{3,m}\right).
\end{aligned} \tag{9}$$

For solving the infinite number of sets of equations (9) in terms of $T_{k(m)}(t)$ ($k=1,2,3$; $m=1,2,3,4,....,\infty$), it is necessary to find the modal matrix of the system of linear differential equations reduced from (9) and the corresponding eigen normal coordinates of this system for the case when $h_{0k,m}=0$ ($k=1,2,3$; $m=1,2,3,4,....,\infty$). Introducing the modal matrix $\mathbf{R}_{(m)}$ ($m=1,2,3,4,....,\infty$) of the linear system of eigen time functions of free linear vibrations with the m-th eigen amplitude shape in the following form:

$$\mathbf{R}_{(m)}=\left(\left\{K_{(m)3k}^{(s)}\right\}\right)=\left(K_{(m)3k}^{(s)}\right)_{\rightarrow\ s=1,2,3}^{\downarrow\ k=1,2,3}\ \ (m=1,2,3,4,....,\infty) \tag{10}$$

and the corresponding eigen main time coordinates ξ_s

$$\left\{\xi_{(m)s}\right\}=\left\{C_{(m)s}\cos\left(\tilde{\omega}_{(m)s}t+\beta_{(m)s}\right)\right\},\ \ (s=1,2,3), \tag{11}$$

where $\tilde{\omega}_{(m)s}$ ($s=1,2,3$; $m=1,2,3,4,....,\infty$) are the eigen circular frequencies of the linear part of equations (9), $C_{(m)s}$ and $\beta_{(m)s}$ ($s=1,2,3$; $m=1,2,3,4,....,\infty$) are integral constants defined by the initial conditions, initial coordinate and velocities values, the formula for transformation of the generalized eigen time function coordinates of the linear system takes the following form:

$$\left\{T_{k(m)}\right\}=\sum_{s=1}^{s=3}\left\{K_{(m)3k}^{s}\right\}C_{(m)s}\cos\left(\tilde{\omega}_{(m)s}t+\beta_{(m)s}\right)=\mathbf{R}_{(m)}\left\{\xi_{(m)s}\right\}. \tag{12}$$

Considering (12), equations (9) are reduced to the form:

$$\ddot{\xi}_{(m)s}\left(t\right)+\tilde{\omega}_{m(s)}^2\xi_{(m)s}\left(t\right)+\tilde{\omega}_{\alpha(m)(s)}^2D_t^{\alpha}\left[\xi_{(m)s}\left(t\right)\right]=0\,, \tag{13}$$

where $\tilde{\omega}_{\alpha(m)(s)}^2=\tau^{\alpha}\tilde{\omega}_{m(s)}^2$.

Generally speaking, equation (13) describes the dynamics of a fractional order like Kelvil-Voigt oscillator which have been studied by many researchers [3-5,16,17].

From equations (13) it follows that this system contains infinite numbers of subsets, each of which involves three independent ordinary fractional order differential equations in terms of the coordinates $\xi_{(m)s}(t)$ ($s=1,2,3$, $m=1,2,3,4,....,\infty$), which are the eigen main time coordinates of the eigen time functions $T_{k(m)}$ ($k=1,2,3$, $m=1,2,3,4,....,\infty$) of one eigen amplitude shape function $\mathbf{W}_m\left(x\right)$ ($m=1,2,3,4,....,\infty$). These coordinates as solutions of the corresponding subset of fractional order differential equations (13) present eigen modes of free vibrations of the beams. Then, two sets of the characteristic numbers, $\tilde{\omega}_{m(s)}^2$ and $\tilde{\omega}_{\alpha(nm)(s)}^2=\tau^{\alpha}\tilde{\omega}_{m(s)}^2$, govern two particular solutions of the ordinary fractional order differential equations (13) as two complement modes, the cosine-like mode $\left[\xi_{(m)s}\left(t\right)\right]_{\cos\left(\tilde{\omega}_{(m)(s)}t+\alpha_{(m)(s)}\right)}$ and the sine-like mode $\left[\xi_{(m)s}\left(t\right)\right]_{\sin\left(\tilde{\omega}_{(m)(s)}t+\alpha_{(m)(s)}\right)}$ expressed in terms of time series [12, 17, 19]

$$\begin{aligned}
&\left[\xi_{(m)s}\left(t\right)\right]_{\cos\left(\tilde{\omega}_{m(s)}t+\alpha_{(m)(s)}\right)}\\
&=\sum_{k=0}^{\infty}\left(-1\right)^k\tilde{\omega}_{\alpha(m)(s)}^{2k}t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{\left(\mp1\right)^j\tilde{\omega}_{\alpha(m)(s)}^{2j}t^{-\alpha j}}{\tilde{\omega}_{m(s)}^{2j}\Gamma\left(2k+1-\alpha j\right)}\,,
\end{aligned} \tag{14}$$

$$\begin{aligned}
&\left[\xi_{(m)s}\left(t\right)\right]_{\sin\left(\tilde{\omega}_{m(s)}t+\alpha_{(m)(s)}\right)}\\
&=\sum_{k=0}^{\infty}\left(-1\right)^k\tilde{\omega}_{\alpha(m)(s)}^{2k}t^{2k+1}\sum_{j=0}^{k}\binom{k}{j}\frac{\left(\mp1\right)^j\tilde{\omega}_{\alpha(m)(s)}^{-2j}t^{-\alpha j}}{\tilde{\omega}_{m(s)}^{2j}\Gamma\left(2k+2-\alpha j\right)}\,.
\end{aligned} \tag{15}$$

Then the solution of the ordinary fractional order differential equations (13) takes in following form:

$$\begin{aligned}
\xi_{(m)s}\left(t\right)&=\xi_{(m)s}\left(0\right)\left[\xi_{(m)s}\left(t\right)\right]_{\cos\left(\tilde{\omega}_{m(s)}t+\alpha_{(m)(s)}\right)}\\
&+\dot{\xi}_{(nm)s}\left(0\right)\left[\xi_{(m)s}\left(t\right)\right]_{\sin\left(\tilde{\omega}_{m(s)}t+\alpha_{(m)(s)}\right)}\,,
\end{aligned} \tag{16}$$

where $\xi_{(m)s}\left(0\right)$ and $\dot{\xi}_{(m)s}\left(0\right)$ are the integral constants determined by the initial conditions. Considering the proposed solution (7) and obtained eigen fractional modes of time functions $T_{k(m)}(t)$ ($k=1,2,3$; $m=1,2,3,4,....,\infty$) of the fractional order system (9) for $h_{0k,m}=0$ ($k=1,2,3$; $m=1,2,3,4,....,\infty$), is possible to writte the following matrix column expression for free transverse displacements of three beam neutral axis points:

$$\left\{w_k\left(x,t\right)\right\}=\sum_{m=1}^{\infty}W_m(x)\mathbf{R}_{(m)}\left\{\xi_{(m)s}\left(t\right)\right\},$$

or

$$\left\{w_k\left(x,t\right)\right\}=\sum_{m=1}^{\infty}W_m(x)\mathbf{R}_{(m)}\left\{\xi_{(m)s}\left(0\right)\right\}\left[\xi_{(m)s}\left(t\right)\right]_{\cos\left(\tilde{\omega}_{m(s)}t+\alpha_{(m)(s)}\right)}$$
$$+\sum_{m=1}^{\infty}W_m(x)\mathbf{R}_{(m)}\left\{\dot{\xi}_{(nm)s}\left(0\right)\right\}\left[\xi_{(m)s}\left(t\right)\right]_{\sin\left(\tilde{\omega}_{m(s)}t+\alpha_{(m)(s)}\right)}\quad . \quad (17)$$

From obtained results for the independent fractional order differential equations (13) in terms of the independent eigen main coordinates $\xi_{(m)s}(t)$, ($s=1,2,3$; $m=1,2,3,4,....,\infty$), we can identify the corresponding quasi-like analogy between free vibrations of harmonic oscillators connected in a conservative chain mechanical system and free vibrations of fractional order like Kelvin-Voigt oscillators connected in a non-conservative chain system with finite number of degrees-of-freedom. Similarly is it possible to identify the analogy between the time functions in a multi-beam system with beams coupled by ideal elastic discrete continuous layers and by fractionally damped discrete continuous layers. It could be concluded that there exists structural analogy between multi-beam systems and the corresponding component vibrations modes. The same conclusions are valid for forced vibrations under harmonic external excitation distributed along the length of beams.

B. Governing partial differential equations of a hybrid deformable multi-plate system

Let us consider transverse oscillations of a hybrid deformable three-plate system presented in Figure 1 (b), where in thin homogeneous prismatic and pure elastic plates are coupled by fractionally damped discrete continuous layers described in Sect. II. For all three plates boundary conditions are the same.

The transverse vibrations of this three-plate system are decribed by the following coupled partial fractional order differential equations [13]-[15]:

$$\begin{aligned}
&\rho_1 h_1\frac{\partial^2 w_1(x,y,t)}{\partial t^2}=-D_1\Delta\Delta w_1(x,y,t)+c_{0(1,2)}\left[w_2\left(x,y,t\right)-w_1\left(x,y,t\right)\right]\\
&\quad+c_{\alpha(1,2)}D_t^{\alpha}\left[w_2\left(x,y,t\right)-w_1\left(x,y,t\right)\right]+q_1\left(x,y,t\right),\\
&\rho_2 A_2\frac{\partial^2 w_2(x,y,t)}{\partial t^2}=-D_2\Delta\Delta w_2(x,y,t)-c_{0(1,2)}\left[w_2\left(x,y,t\right)-w_1\left(x,y,t\right)\right]\\
&\quad-c_{\alpha(1,2)}D_t^{\alpha}\left[w_2\left(x,y,t\right)-w_1\left(x,y,t\right)\right]+c_{0(2,3)}\left[w_3\left(x,t\right)-w_2\left(x,t\right)\right]\\
&\quad+c_{\alpha(2,3)}D_t^{\alpha}\left[w_3\left(x,t\right)-w_2\left(x,t\right)\right]-q_2\left(x,y,t\right),\\
&\rho_3 A_3\frac{\partial^2 w_3(x,y,t)}{\partial t^2}=-D_3\Delta\Delta w_3(x,y,t)-c_{0(2,3)}\left[w_3\left(x,y,t\right)-w_2\left(x,y,t\right)\right]\\
&\quad-c_{\alpha(2,3)}D_t^{\alpha}\left[w_3\left(x,y,t\right)-w_2\left(x,y,t\right)\right]-q_3\left(x,y,t\right),
\end{aligned}\qquad (18)$$

where D_k $(k=1,2,3)$ are the flexural rigidities of the plates.

For complete description of this hybrid system, it is necessary to define and add the corresponding boundary and initial conditions. All three plates are of the same contours and are subjected to the same boundary conditions. This fact permits us to suppose that eigen amplitude functions for all three plates could be taken in one and the same form $\mathbf{W}_{nm}(x,y)$ ($n,m=1,2,3,4...\infty$), and the solution is supposed in the following form [15,18]:

$$w_k\left(x,y,t\right)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathbf{W}_{nm}(x,y)T_{k(nm)}(t)\ \ (k=1,2,3), \qquad (19)$$

with the distributed external excitation along beam lengths

$$\frac{q_k(x,y,t)}{\rho}=\sum_{m=1}^{M}\sum_{n=1}^{N}h_{0k,nm}\mathbf{W}_{nm}(x,y)\sin(\Omega_{knm}t+\vartheta_{k,nm})\,. \qquad (20)$$

Substituting the proposed solution (19) into the system of partial fractional order differential equations (18), multiplying each equations by $\mathbf{W}_{rs}(x,y)$ ($s,r=1,2,3,4...,\infty$), and integrating all terms along the middle surface of the plates with due account for the orthogonality conditions of the amplitude eigen functions $\mathbf{W}_{nm}(x,y)$ and $\mathbf{W}_{rs}(x,y)$, as a result we obtain a system of the ordinary fractional order differential equations in terms of the eigen time functions $T_{k(nm)}(t)$ ($k=1,2,3$; $n,m=1,2,3,4,....,\infty$) in the following form:

$$\begin{aligned}
&\ddot{T}_{1(nm)}(t)+c_1^2k_{nm}^2T_{1(nm)}(t)+a_{0(1,2)}^2T_{1(nm)}\left(t\right)\\
&\quad+a_{\alpha(1,2)}^2D_t^{\alpha}\left[T_{1(nm)}\left(t\right)\right]-a_{0(1,2)}^2T_{2(nm)}(t)\\
&\quad-a_{\alpha(1,2)}^2D_t^{\alpha}\left[T_{2(nm)}(t)\right]=h_{01,nm}\sin\left(\Omega_{1,nm}t+\vartheta_{1,nm}\right),\\
&\ddot{T}_{2(nm)}(t)+c_2^2k_{nm}^2T_{2(nm)}(t+\left[\tilde{a}_{0(1,2)}^2+a_{0(2,3)}^2\right]T_{2(nm)}(t)+\\
&\quad+\left[\tilde{a}_{\alpha(1,2)}^2+a_{\alpha(2,3)}^2\right]D_t^{\alpha}\left[T_{2(nm)}(t)\right]-\tilde{a}_{0(1,2)}^2T_{1(nm)}(t)-\\
&\quad-\tilde{a}_{\alpha(1,2)}^2D_t^{\alpha}\left[T_{1(nm)}(t)\right]-a_{0(2,3)}^2T_{3(nm)}(t)-\\
&\quad-a_{\alpha(2,3)}^2D_t^{\alpha}\left[T_{3(nm)}(t)\right]=h_{02,nm}\sin\left(\Omega_{2nm}t+\vartheta_{2,nm}\right),\\
&\ddot{T}_{3(nm)}(t)+c_3^2k_{nm}^2T_{3(nm)}(t)+\left[\tilde{a}_{0(2,3)}^2+a_{0(3,4)}^2\right]T_{3(nm)}(t)\\
&\quad-\tilde{a}_{0(2,3)}^2T_{2(nm)}(t)+\left[\tilde{a}_{\alpha(2,3)}^2+a_{\alpha(3,4)}^2\right]D_t^{\alpha}\left[T_{3(nm)}(t)\right]\\
&\quad-\tilde{a}_{\alpha(2,3)}^2D_t^{\alpha}\left[T_{2(nm)}(t)\right]\approx h_{03,nm}\sin\left(\Omega_{3nm}t+\vartheta_{3,nm}\right).
\end{aligned}\qquad (21)$$

The comparison of the set of ordinary fractional order differential equations (21) in eigen time functions $T_{k(nm)}(t)$ ($k=1,2,3$; $n,m=1,2,3,4,....,\infty$) with one eigen amplitude shape function $\mathbf{W}_{nm}(x,y)$ written for the three-plate system and the set of ordinary fractional order differential equations (9) in eigen time functions $T_{k(m)}(t)$ ($k=1,2,3$; $m=1,2,3,4,....,\infty$) with one eigen amplitude shape function $\mathbf{W}_m(x)$ written for the three-beam system reveals the mathematical analogy between the eigen time functions for three-plate system $T_{k(nm)}(t)$ and three-beam system $T_{k(m)}(t)$. There are also another types of analogies such as structural analogy, mathematical analogy, and phenomenological analogy, resulting in phenomenological mappings between the main eigen time coordinates $\xi_{(nm)s}(t)$ ($s=1,2,3$; $n,m=1,2,3,4,....,\infty$) of there-plate system and the main eigen time coordinates $\xi_{(m)s}(t)$ ($s=1,2,3$; $m=1,2,3,4,....,\infty$) of the three-beam system, according to which we could obtain for the three-plate system the following:

$$\ddot{\xi}_{(nm)s}(t)+\tilde{\omega}^2_{nm(s)}\xi_{(nm)s}(t)+\tilde{\omega}^2_{\alpha(nm)(s)}D_t^\alpha\left[\xi_{(nm)s}(t)\right]=0\,, \quad (22)$$

where $\tilde{\omega}^2_{nm(s)}$ are the squares of the eigen circular frequencies of the corresponding linear part of equations (18) and $\tilde{\omega}^2_{\alpha(nm)(s)}=\tau^\alpha\tilde{\omega}^2_{nm(s)}$.

Thus, once again the problem is reduced to equation (22) describing the dynamics of the fractional order like Kelvin-Voigt ocsillators.

From the obtained system of ordinary fractional order differential equations (22) at $h_{0k,nm}=0$ ($k=1,2,3$; $n,m=1,2,3,4,....,\infty$), we can conclude that this system contains infinite numbers of subsets, each involving three independent ordinary fractional order differential equations in terms of the coordinates $\xi_{(nm)s}(t)$ ($s=1,2,3$; $n,m=1,2,3,4,....,\infty$) which are the main eigen time coordinates of the eigen time functions $T_{k(nm)}(t)$ ($k=1,2,3$; $n,m=1,2,3,4,....,\infty$) with one eigen amplitude shape function $\mathbf{W}_{nm}(x,y)$ ($n,m=1,2,3,4,....,\infty$) describing free transverse vibrations of the three-plate system. For obtaining necessary solutions for free vibrations of the three-plate system, it is possible to use the solutions the for three-beam system (14)-(17) substituting there the characteristic numbers $\tilde{\omega}^2_{nm(s)}$ and $\tilde{\omega}^2_{\alpha(nm)(s)}=\tau^\alpha\tilde{\omega}^2_{nm(s)}$ ($s=1,2,3$; $n,m=1,2,3,....,\infty$) and the corresponding eigen amplitude function $\mathbf{W}_{nm}(x,y)$ of the three-plate system.

Finally let us enumerate the analogous eigen functions and characteristic numbers of three-body systems compiled of plates and beams connected with each other by viscoelastic layers, the damping features of which are described by the fractional order like Kelvin-Voigt model (1):

(1) $\mathbf{W}_{nm}(x,y)$ ($n,m=1,2,3,4,....,\infty$) are analogous with $\mathbf{W}_{m}(x,y)$, $m=1,2,3,4,....,\infty$;

(2) $T_{k(nm)}(t)$ ($k=1,2,3$; $n,m=1,2,3,4,....,\infty$) are analogous with $T_{k(m)}(t)$ ($k=1,2,3$; $m=1,2,3,4,....,\infty$);

(3) $\xi_{(nm)s}(t)$ ($s=1,2,3$; $n,m=1,2,3,4,....,\infty$) are analogous with $\xi_{(m)s}(t)$ ($s=1,2,3$; $m=1,2,3,4,....,\infty$);

(4) $\tilde{\omega}^2_{nm(s)}$ and $\tilde{\omega}^2_{\alpha(nm)(s)}=\tau^\alpha\tilde{\omega}^2_{nm(s)}$ are analogous with $\tilde{\omega}^2_{m(s)}$ and $\tilde{\omega}^2_{\alpha(m)(s)}=\tau^\alpha\tilde{\omega}^2_{m(s)}$;

(5) $\left[\xi_{(nm)s}(t)\right]_{\cos\left(\tilde{\omega}_{(nm)(s)}t+\alpha_{(nm)(s)}\right)}$ are analogous with $\left[\xi_{(m)s}(t)\right]_{\cos\left(\tilde{\omega}_{(m)(s)}t+\alpha_{(m)(s)}\right)}$;

(6) $\left[\xi_{(nm)s}(t)\right]_{\sin\left(\tilde{\omega}_{(nm)(s)}t+\alpha_{(nm)(s)}\right)}$ are analogous with $\left[\xi_{(m)s}(t)\right]_{\sin\left(\tilde{\omega}_{(m)(s)}t+\alpha_{(m)(s)}\right)}$;

(7) $\ddot{\xi}_{(nm)s}(t)+\tilde{\omega}^2_{nm(s)}\xi_{(nm)s}(t)+\tilde{\omega}^2_{\alpha(nm)(s)}D_t^\alpha\left[\xi_{(nm)s}(t)\right]=0$ ($s=1,2,3,....,M$; $n,m=1,2,3,....,\infty$) are analogous with

$$\ddot{\xi}_{(m)s}(t)+\tilde{\omega}^2_{m(s)}\xi_{(m)s}(t)+\tilde{\omega}^2_{\alpha(m)(s)}D_t^\alpha\left[\xi_{(m)s}(t)\right]=0$$

($s=1,2,3,....,M$, $m=1,2,3,4,....,\infty$).

It should be noted that similar results and discussion could be found in the state-of-the-art article by Rossikhin and Shitikova [3], wherein in Sects 4.2.6 and 4.3.2 multi-beam and multi-plate systems have been investigated, respectively, considering the viscoelastic features of fractionally damped layers. See also the papers by Hedrih [13,14] dealing with this matter.

All above listed analogies valid for free vibratory regimes could be extended over forced oscillatory regimes of transverse motions of both three-beam and three-plate hybrid systems. Thus, characteristic equations for the case of forced vibrations could be written in the following form:

$$\left\langle\ddot{\xi}_{(nm)1}(t)+\tilde{\omega}^2_{nm(1)}\xi_{(nm)1}(t)+\tilde{\omega}^2_{\alpha(nm)(1)}D_t^\alpha\left[\xi_{(nm)1}(t)\right]\right\rangle=$$

$$=\frac{\begin{vmatrix} h_{01,nm}\sin\left(\Omega_{1,nm}t+\vartheta_{1,nm}\right) & \mathbf{K}^{(2)}_{(nm)31} & \mathbf{K}^{(3)}_{(nm)31}\\ h_{02,nm}\sin\left(\Omega_{2,nm}t+\vartheta_{2,nm}\right) & \mathbf{K}^{(2)}_{(nm)32} & \mathbf{K}^{(3)}_{(nm)32}\\ h_{03,nm}\sin\left(\Omega_{3,nm}t+\vartheta_{3,nm}\right) & \mathbf{K}^{(2)}_{(nm)33} & \mathbf{K}^{(3)}_{(nm)33}\end{vmatrix}}{\begin{vmatrix}\mathbf{K}^{(1)}_{(nm)31} & \mathbf{K}^{(2)}_{(nm)31} & \mathbf{K}^{(3)}_{(nm)31}\\ \mathbf{K}^{(1)}_{(nm)32} & \mathbf{K}^{(2)}_{(nm)32} & \mathbf{K}^{(3)}_{(nm)32}\\ \mathbf{K}^{(1)}_{(nm)33} & \mathbf{K}^{(2)}_{(nm)33} & \mathbf{K}^{(3)}_{(nm)33}\end{vmatrix}}$$

$$\left\langle\ddot{\xi}_{(nm)2}(t)+\tilde{\omega}^2_{nm(2)}\xi_{(nm)s}2(t)+\tilde{\omega}^2_{\alpha(nm)(2)}D_t^\alpha\left[\xi_{(nm)2}(t)\right]\right\rangle=$$

$$=\frac{\begin{vmatrix}\mathbf{K}^{(1)}_{(nm)31} & h_{01,nm}\sin\left(\Omega_{1,nm}t+\vartheta_{1,nm}\right) & \mathbf{K}^{(3)}_{(nm)31}\\ \mathbf{K}^{(1)}_{(nm)32} & h_{02,nm}\sin\left(\Omega_{3,nm}t+\vartheta_{2,nm}\right) & \mathbf{K}^{(3)}_{(nm)32}\\ \mathbf{K}^{(1)}_{(nm)33} & h_{03,nm}\sin\left(\Omega_{3,nm}t+\vartheta_{3,nm}\right) & \mathbf{K}^{(3)}_{(nm)33}\end{vmatrix}}{\begin{vmatrix}\mathbf{K}^{(1)}_{(nm)31} & \mathbf{K}^{(2)}_{(nm)31} & \mathbf{K}^{(3)}_{(nm)31}\\ \mathbf{K}^{(1)}_{(nm)32} & \mathbf{K}^{(2)}_{(nm)32} & \mathbf{K}^{(3)}_{(nm)32}\\ \mathbf{K}^{(1)}_{(nm)33} & \mathbf{K}^{(2)}_{(nm)33} & \mathbf{K}^{(3)}_{(nm)33}\end{vmatrix}} \quad (23)$$

$$\left\langle \ddot{\xi}_{(nm)3}\left(t\right)+\tilde{\omega}^2_{nm(3)}\xi_{(nm)3}\left(t\right)+\tilde{\omega}^2_{\alpha(nm)(3)}D_t^{\alpha}\left[\xi_{(nm)3}\left(t\right)\right]\right\rangle =$$

$$= \frac{\begin{vmatrix} \mathbf{K}^{(1)}_{(nm)31} & \mathbf{K}^{(2)}_{(nm)31} & h_{01,nm}\sin\left(\Omega_{1,nm}t+\vartheta_{1,nm}\right) \\ \mathbf{K}^{(1)}_{(nm)32} & \mathbf{K}^{(2)}_{(nm)32} & h_{02,nm}\sin\left(\Omega_{2,nm}t+\vartheta_{2,nm}\right) \\ \mathbf{K}^{(1)}_{(nm)33} & \mathbf{K}^{(2)}_{(nm)33} & h_{03,nm}\sin\left(\Omega_{3,nm}t+\vartheta_{3,nm}\right) \end{vmatrix}}{\begin{vmatrix} \mathbf{K}^{(1)}_{(nm)31} & \mathbf{K}^{(2)}_{(nm)31} & \mathbf{K}^{(3)}_{(nm)31} \\ \mathbf{K}^{(1)}_{(nm)32} & \mathbf{K}^{(2)}_{(nm)32} & \mathbf{K}^{(3)}_{(nm)32} \\ \mathbf{K}^{(1)}_{(nm)33} & \mathbf{K}^{(2)}_{(nm)33} & \mathbf{K}^{(3)}_{(nm)33} \end{vmatrix}}$$

Note that equations (23) could be solved using the Laplace transform method or by the generalized Lagrange method of variation of constants [19].

C. *Governing partial differential equations of a hybrid deformable multi-membrane system*

Let us now consider transversal oscillations of a hybrid deformable multi-membrane system presented in Figure 1 (a). For this purpose, we will study a hybrid deformable three-membrane system, wherein membranes are thin homogeneous and pure elastic [15] and coupled by fractional order discrete continuum layers described in Sect. II.

Transverse vibrations of such a three-membrane system are decribed by the following coupled partial fractional order differential equations:

$$\rho_1\frac{\partial^2 w_1(x,y,t)}{\partial t^2}=\rho_1 c_1^2\Delta w_1(x,y,t)+c_{0(1.2)}\left[w_2\left(x,y,t\right)-w_1\left(x,y,t\right)\right] + c_{\alpha(1,2)}D_t^{\alpha}\left[w_2\left(x,y,t\right)-w_1\left(x,y,t\right)\right]+q_1\left(x,y,t\right),$$

$$\rho_2 A_2\frac{\partial^2 w_2(x,y,t)}{\partial t^2}=\rho_2 c_2^2\Delta w_2(x,y,t)-c_{0(1.2)}\left[w_2\left(x,y,t\right)-w_1\left(x,y,t\right)\right] - c_{\alpha(1,2)}D_t^{\alpha}\left[w_2\left(x,y,t\right)-w_1\left(x,y,t\right)\right]+c_{0(2.3)}\left[w_3\left(x,t\right)-w_2\left(x,t\right)\right] + c_{\alpha(2,3)}D_t^{\alpha}\left[w_3\left(x,t\right)-w_2\left(x,t\right)\right]-q_2\left(x,y,t\right), \quad (24)$$

$$\rho_3\frac{\partial^2 w_3(x,y,t)}{\partial t^2}=\rho_3 c_3^2\Delta w_3(x,y,t)-c_{0(2.3)}\left[w_3\left(x,y,t\right)-w_2\left(x,y,t\right)\right] - c_{\alpha(2,3)}D_t^{\alpha}\left[w_3\left(x,y,t\right)-w_2\left(x,y,t\right)\right]-q_3\left(x,y,t\right).$$

For complete description of this hybrid system, it is necessary to define and add the corresponding boundary and initial conditions. All three membranes are of the same dimensions and are subjected to the same boundary conditions. This fact permits us, as it has been done for the previously considered systems, to suppose that eigen amplitude functions for all three membranes are of one and the same form $\mathbf{W}_{nm}(x,y)$ ($n,m=1,2,3,4...\infty$), and the solution is supposed in the following form [15,18]:

$$w_k\left(x,y,t\right)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathbf{W}_{nm}(x,y)T_{k(nm)}(t) \quad (k=1,2,3) \quad (25)$$

with the distributed external excitation along the membrane surfaces

$$\frac{q_k(x,y,t)}{\rho}=\sum_{m=1}^{M}\sum_{n=1}^{N}h_{0k,nm}W_{nm}(x,y)\sin(\Omega_{knm}t+\vartheta_{k,nm}). \quad (26)$$

Substituting the proposed solution (25) into the system of partial fractional order differential equations (24), multiplying each equation by $\mathbf{W}_{sr}(x,y)$ ($s,r=1,2,3,4...,\infty$) with due account for the conditions of orthogonality of the eigen amplitude functions, $\mathbf{W}_{nm}(x,y)$ and $\mathbf{W}_{sr}(x,y)$, as a result we obtain a system of the ordinary fractional order differential equations in terms of the eigen time functions $T_{k(nm)}(t)$ ($k=1,2,3$; $n,m=1,2,3,4,....,\infty$) in the forms of (21) as for the case of the three-plate system. Then, all discussion presented above concerning the analogies (qualitative, structural and mathematical) between the kinetic parameters, characteristic numbers, eigen time functions with main eigen time coordinates and main eigen modes of free vibrations of the three-beam and three-plate systems is valid for the three-membrane system as well. In doing so it is necessary only to pay attention for a proper choice of characteristic values defining the geometric and materials parameters of the system under consideration.

D. *Discussion*

On the basis of the above presented results, it is possible to formulate some theorems. Two of them have been already published in [20], and now we could formulate three more theorems.

Theorem 1: *Considered system of the coupled fractional order partial differential equations describing transverse vibrations of a deformable multi-body system (involving beams, or plates, or membranes) with fractionally damped continuous layers is quasi-linear, and the main eigen time coordinates of the corresponding system of linear differential equations in terms of eigen time functions with one eigen amplitude shape are analogous to the main eigen coordinates of the ftactional order differential equations describing vibrations of fractional order likeKelvin-Voigt oscillators.*

Theorem 2: *Dynamics of a hybrid system, which contains* N *deformable bodies (beams, plates or membranes) coupled by discrete continuous fractionally damped layers with the same boundary conditions and with the displacements* $w_k(x,y,t)$ *and* $w_{k+1}(x,y,t)$ *is described by the corresponding system of coupled partial fractional order differential equations in each eigen amplitude mode* $\mathbf{W}_{nm}(x,y)$ *from the infinite number set of the corresponding* N*-frequency eigen time functions* $T_{k(nm)}\left(t\right)=\sum_{s=1}^{s=N}\mathbf{K}^{(s)}_{(nm)Nk}\xi_{(nm)s}\left(t\right)$, *where* $\xi_{(nm)s}\left(t\right)$ ($s=1,2,3...,N$) *are independent main eigen time modes of the corresponding subsystem in eigen amplitude mode*

$\mathbf{W}_{nm}(x,y)$. *These main eigen time modes* $\xi_{(nm)s}(t)$ ($s=1,2,3...,N$) *are described by the system of the independent ordinary fractional order differential equations of the form:*

$$\ddot{\xi}_{(nm)s}(t)+\tilde{\omega}^2_{nm(s)}\xi_{(nm)s}(t)+\tilde{\omega}^2_{\alpha(nm)(s)}D^{\alpha}_{t}\left[\xi_{(nm)s}(t)\right]=0 \qquad (27)$$

with two sets of characteristic values, $\tilde{\omega}^2_{nm(s)}$ *and* $\tilde{\omega}^2_{\alpha(nm)(s)}=\tau^{\alpha}\tilde{\omega}^2_{nm(s)}$, *where* $\tilde{\omega}^2_{nm(s)}$ *are squared eigen circular frequencies of free vibrations and* τ *is system's retardation time.*

Proof of this theorem 2 is evident from Sect. III.

Theorem 3: *Generalized forces* Q_{w_k} *and* $Q_{w_{k+1}}$ *of interaction of two deformable bodies coupled by a standard discrete continuous layer with known kinetic* $\mathbf{E}_{\mathrm{k}}$ *and potential* $\mathbf{E}_{\mathrm{p}}$ *energies and known generalized function* $\Phi_{0<\alpha\le 1}$ *of the fractional order dissipation of the system energy for the displacements* $w_k(x,y,t)$ *and* $w_{k+1}(x,y,t)$ *at the points of contacts of deformable bodies with the discrete continuous layer could be expressed in terms of energy and its fractional order dissipation within the discrete continuous fractionally damped layer and written in the following forms:*

$$Q_{w_k}=-\left\langle \frac{d}{dt}\frac{\partial \mathbf{E}_{\mathrm{k}}}{\partial\left(\frac{\partial w_k(x,y,t)}{\partial t}\right)}-\frac{\partial \mathbf{E}_{\mathrm{k}}}{\partial w_k(x,y,t)}\right\rangle \qquad (28)$$

$$-\frac{\partial \mathbf{E}_{\mathrm{p}}}{\partial w_k(x,y,t)}-\frac{\partial \Phi_{0<\alpha\le 1}}{\partial\left(D^{\alpha}_{t}\left[w_k(x,y,t)\right]\right)}=Q_{w_k(x,y,t)},$$

$$Q_{w_{k+1}}=-\left\langle \frac{d}{dt}\frac{\partial \mathbf{E}_{\mathrm{k}}}{\partial\left(\frac{\partial w_{k+1}(x,y,t)}{\partial t}\right)}-\frac{\partial \mathbf{E}_{\mathrm{k}}}{\partial w_{k+1}(x,y,t)}\right\rangle \qquad (29)$$

$$-\frac{\partial \mathbf{E}_{\mathrm{p}}}{\partial w_{k+1}(x,y,t)}-\frac{\partial \Phi_{0<\alpha\le 1}}{\partial\left(D^{\alpha}_{t}\left[w_{k+1}(x,y,t)\right]\right)}=Q_{w_{k+1}(x,y,t)}.$$

Proof of this theorem 3 is evident from Sect. II.

IV. Conclusion

Transverse free vibrations of a hybrid multi-body system involving plates or membranes are investigated on the basic of the results obtained for free transverse vibrations of a multi-beam system.

The results obtained in the given paper for the case of free transverse vibrations of the systems combined from three deformable bodies (beams, plates or membranes) with the same contour and boundary conditions, which are coupled via viscoelastic layers, the properties of which are described by the fractional order derivative and constitutive relation like Kelvin-Voigt model, could be easily generalized for the case of hybrid systems involving any finite number of elastic bodies.

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