The initial period of mixed-mode crack growth in viscoelastic composite with Rabotnov's relaxation law

Mikhailo F. Selivanov, Yuri O. Chornoivan

Dedicated to the 100 anniversary of Academician Yury N. Rabotnov

Abstract—A numerical algorithm is presented to study the initial period of crack growth in a viscoelastic composite under the mixed-mode loading. Viscoelastic properties of the composite are described using linear viscoelasticity operators with Yu. N. Rabotnov's kernel. An example of calculation is given for a composite with viscoelastic components.

Keywords—Crack, delayed fracture, fractional derivatives and integrals, mixed-mode loading, Rabotnov's operators, viscoelasticity.

I. Introduction

REINFORCED composite materials and multiphase polymer systems have been a subject of growing interest as they become widely used in constructions. These materials can modernize the present-day civil infrastructure because of their strength, resistance to the influence of the environment, and efficiency from the viewpoint of cost; they are used for increasing the strength as compared with homogeneous polymer materials. It is important for structural design to develop efficient methods for the prediction and modeling of such material behavior using principles of the mechanics of composite materials. Many properties and characteristics of a material can be changed by appropriately choosing the substances of its components and the values of their volume fractions. In numerous structural materials, the viscoelastic nature of their mechanical behavior is very important, and, therefore, this requires knowing their long-term properties. These properties are crucial to investigate delayed fracture of structural elements and determine their durability.

The parameters of long-term deformation of viscoelastic composites can be effectively determined using the methods of the mechanics of composite materials and the linear theory of viscoelasticity, proceeding from experimental data on the long-term deformation of the materials of components of a composite. This approach for investigating the long-term deformation of composites makes it possible to model their mechanical properties quite exactly on the basis of known properties of the materials of components, the volume content of these components, and the type of reinforcement, without manufacturing the composite itself. Micromechanical methods for predicting the effective characteristics of an elastic composite based on the properties of the materials of components and their volume fraction have been developed fairly successfully up to now. It was found that the method of empirical describing of these characteristics using fractional calculus very effective from the theoretical and engineering points of view [1].

The wide application of this method in the modern theory of viscoelasticity cannot be even imagined without Yu. N. Rabotnov's theory [2, 3] of Volterra integral operators with weakly singular kernels that could be interpreted in terms of fractional integrals and derivatives.

A combined use of the integral operators of viscoelasticity with Rabotnov's kernel to describe viscoelastic properties of composite and the concepts of nonlinear fracture mechanics allows us to obtain an effective solution of several problems in the fracture mechanics of composites [4]–[9].

Modern fracture mechanics uses energy, stress and deformation criteria to describe the process of fracture for materials of different types. The deformation criteria can be effective for the elasto-plastic materials with considerable plastic zones near the crack front. The stress criterion based on SIF can give an inappropriate precision when used for such materials. Furthermore, deformation criterion, namely the COD-criterion, is widely used to study the subcritical growth of the cracks in viscoelastic materials. This criterion allows one to obtain kinetic equations of slow crack growth in viscoelastic media (see survey [6]).

In the present work, we investigate an initial period of a crack in linearly viscoelastic reinforced composite under mixed-mode loading. The deformable properties of the composite as a whole are described as a sum of Rabotnov's operators.

M. F. Selivanov, S.P. Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, Nesterov st. 3, Kyiv, 03057 Ukraine (corresponding author, phone: 380-44-454-7769; e-mail: fract@inmech.kiev.ua).

Y. O. Chornoivan, S.P. Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, Nesterov st. 3, Kyiv, 03057 Ukraine (e-mail: yurchor@ukr.net).

II. On a Representation of Operator Integral Function

A. An Operator Function Representation for the Same Values of Fractional Order Exponent

As it can be seen from the literature on the linear viscoelasticity, there are two major approaches to solve the problems of stress-strain determination in bodies made of viscoelastic composites [10]. These are the method based on finding the corresponding stresses or strains as original functions corresponding to their transforms obtained by a Laplace–Carson-type transformation [11], and the method based on the Volterra principle [3]. Investigations of the rheological properties of viscoelastic composites by the first of the methods are performed mainly in the domain of transforms, and the inverse transform to the domain of original functions is seldom carried out and is done predominantly with numerous restrictions and simplifications and preassigned accuracy. There are two ways of transforming from the domain of transforms to the time domain. The first does not impose restrictions on the influence function, and the transformation to the time domain is performed by approximate methods, which depend substantially on the behavior of the influence function for the materials of components. Another way is to find the solution and its parameters in the transformed domain in a preassigned form, which enables a relatively simple inverse transform to the solution in the time domain. Construction of the solution in such a way is possible when the viscoelastic properties are described within the framework of the standard mechanical model, where the influence function is a linear combination of decreasing exponents (Prony–Dirichlet series), or according to the generalized fractional derivative model, where the influence function is a linear combination of the Mittag-Leffler functions [10]. Here, the deformable properties of components are described within the framework of the standard mechanical model.

Both physical and mathematical requirements to modeling the material rheological properties within the framework of the linear theory of viscoelasticity make the correct description of these properties to be an uneasy task. However, it is practically impossible to satisfy all these requirements, and, hence, each of the models presented in the literature has both advantages and shortcomings and is applied in accordance with the problem to be solved. An important advantage of the description of viscoelastic properties of the components of a composite within the framework of the standard mechanical model is its universality: this model describes fairly exactly the time variation of the properties of numerous linearly viscoelastic materials. From the mathematical viewpoint, this accuracy is reached by increasing the number of terms of the Prony series, i.e., the number of rheological parameters. However, as a result, the influence functions lose their smoothness, which is not natural. To remove this shortcoming, one can seek the influence function not as a linear combination of exponents but as a linear combination of the Mittag-Leffler functions, which describes the process of deformation with the use of a smaller number of terms. Naturally, such a description complicates the engineering application of the results obtained, but the number of rheological parameters of the material under investigation becomes less. This is advantageous also in view of the considerations given in what follows.

Based on the characteristics of deformation of a composite, one can solve the problems of determining the stress-strain state near stress concentrators (cracks, holes) located in the body of a composite material. Stresses or displacements in the neighborhood of cracks can be determined from the elastic solution, where elastic constants are replaced with the operators of linear viscoelasticity.

All investigations in the field of the linear theory of viscoelasticity are based on the constitutive relation in the form of a heredity integral or a convolution integral, first obtained by Boltzmann

$$\sigma\left(t\right) = \int_{-\infty}^{t} C\left(t-\tau\right) d\varepsilon(\tau)\,, \tag{1}$$

where the convolution kernel $C(t)$ is a function of relaxation. The functions of stresses $\sigma(t)$ and strains $\varepsilon(t)$ are time functions of the class of Heaviside functions.

As it was proposed by Yu. N. Rabotnov [2], the relaxation function can be found as

$$C(t) = E_0 - \sum\nolimits_{k=1}^{n} E_k \left[1 - E_\alpha\left(-\left(t / \rho_k\right)^\alpha\right)\right]$$

or

$$C(t) = E_\infty + \sum\nolimits_{k=1}^{n} E_k E_\alpha\left(-\left(t / \rho_k\right)^\alpha\right),$$

where

$$E_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma\left[1+\alpha n\right]} \tag{2}$$

is the Mittag-Leffler function of order α $(0 < \alpha \le 1)$ and Γ is the Euler gamma function

The quantity $E_0 = C(0)$ in (1) is the instantaneous modulus of elasticity, and the functional characteristics of the relaxation rate $T(t) = -C'(t) / E_0$ is a relaxation kernel or influence function (the function of relaxation rate) [3]. In the case when the creep function is represented in the form (1) as

$$\frac{d}{dt}\left[E_\alpha\left(-\left(t / \rho\right)^\alpha\right)\right] = -\rho^{-\alpha} \sum_{m=0}^{\infty} \frac{\left(-\rho^{-\alpha}\right)^m t^{m\alpha-(1-\alpha)}}{\Gamma\left[(m+1)\alpha\right]}$$

the influence function is $T(t) = \sum_{k=1}^{n} \lambda_k R_{\alpha'}\left(\beta_k, t\right)$, where

$\beta_k = \rho_k^{-\alpha}$, $\lambda_k = \dfrac{E_k}{E_0}\rho_k^{-\alpha}$, $\alpha' = 1-\alpha$ and

$$R_\alpha(\beta,t)=\sum_{m=0}^{\infty}\frac{\left(-\beta\right)^m t^{m(1-\alpha)-\alpha}}{\Gamma\left[(m+1)(1-\alpha)\right]}, \quad \beta>0$$

is the Rabotnov fractional exponential function, and α is its fractional order exponent. For $0<\alpha\le 1$, the Mittag-Leffler function is often erroneously called as the exponent of fractional order [1]. However, (1) unlike to the Rabotnov fractional exponential function, which is a weakly singular function at $t=0$, the Mittag-Leffler function is regular everywhere, and (2) if the Rabotnov function is used as relaxation kernels, then resolvent kernels coincide. Only one more function possesses this unique feature, namely: conventional exponent.

Then the relaxation function is

$$C(t)=E_0\left(1-\int_{-\infty}^{t}\sum_{k=1}^{n}\lambda_k R_\alpha(\beta_k,t-\tau)d\tau\right), \tag{3}$$

or written in operator form

$$C^*=E_0\left[1-\sum_{k=1}^{n}\lambda_k R_\alpha^*(\beta_k)\right]. \tag{4}$$

The result of the operator R^* application to a function of the strain can be written as

$$R^*\cdot\varepsilon(t)\equiv R(t)*\varepsilon(t)=\int_{-\infty}^{t}R(t-\tau)\varepsilon(\tau)d\tau,$$

where $R(t)$ is the kernel of the operator R^*.

Then it is possible to use the resolvent operator algebra and operator continued fractions to obtain any viscoelastic solution via the correspondence principle [11], or Volterra's principle (in the case of operator solution). One is referred to [12] for the details.

The only obstacle in application of this method to study deformation of composite materials is that all operators should have the same value of the fractional order exponent. However, this could be passed using an approximation described below.

B. Adjustment of Rabotnov's Operators to a Common Fractional Order Exponents

Using the operator form of the relaxation function (4), it is possible to write the relaxation functions for the composite components as

$$C_{\alpha_i}^*\cdot 1=E_0^{(i)}\left[1-\sum_{k=1}^{n}\lambda_k^{(i)}R_{\alpha_i}^*(\beta_k^{(i)})\right]. \tag{5}$$

Further, it is possible to use the Rabotnov operator representation [3]

$$R_\alpha^*(\beta)=\frac{I_\alpha^*}{1+\beta I_\alpha^*},$$

where I_α^* is the Abel operator with the kernel

$$I_\alpha(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}$$

and $\Gamma(\alpha)$ is the Euler gamma function.

We represent the solution of the problem of linear viscoelasticity in the operator domain as a function of the Abel operator

$$F(I_\alpha^*)=F\left[(I_\alpha^*)^{\frac{\alpha_1}{\alpha}},(I_\alpha^*)^{\frac{\alpha_2}{\alpha}},\ldots,(I_\alpha^*)^{\frac{\alpha_N}{\alpha}}\right]\equiv F_\alpha^*. \tag{6}$$

Let a continued fraction $c_0+\operatorname*{D}_{k=1}^{\infty}\frac{c_k x}{1}$ converges uniformly to the function $F(x)$. Then it is possible to interpret the function F_α^* from (6) as an operator continued fraction (OCF)

$$c_0+\operatorname*{D}_{k=1}^{\infty}\frac{c_k I_\alpha^*}{1}. \tag{7}$$

The fraction

$$\psi_1^{n\,*}=c_0+\operatorname*{D}_{k=1}^{n}\frac{c_k I_\alpha^*}{1}$$

is the *n*-th approximant of (6). This approximant can be obtained for $x_i\to x_0=0$ by equivalent transformations of the fraction [13]

$$\psi_1^{n\,*}=V_0+\operatorname*{D}_{k=1}^{n}\frac{I_\alpha^*-x_k}{V_k} \tag{8}$$

via an approximation of the corresponding function of a real variable

$$\psi_1^n(x)=V_0+\operatorname*{D}_{k=1}^{n}\frac{x-x_k}{V_k}$$

Thus, the operators $\psi_1^{n\,*}$ approximate F_α^*.

Further we use the following notation for a fragment of OCF (8):

$$\psi_m^{n\,*}=V_{m-1}+\operatorname*{D}_{k=m}^{n}\frac{I_\alpha^*-x_k}{V_k}. \tag{9}$$

Using the algebra of resolvent operators, a fragment of OCF $\psi_m^{n\,*}$ defined by (9) could be represented in the form [14]

$$\psi_m^{n\,*}=a_m^n+\sum_{i=1}^{L_{n-m}}a_{m,i}^n R^*(\lambda_{m,\,i}^n), \quad L_s=\left[\frac{s}{2}\right]+1,$$

where

$$a_m^n=V_{m-1}-\frac{x_m}{a_{m+1}^n}, \quad a_n^n=V_{n-1}-\frac{x_n}{V_n},$$

$a_{m,\,i}^n$ and $\lambda_{m,\,i}^n$ are determined from recurrent relations as follows:

– for a fragment with an odd number of terms ($m=n-2l$)

(i) with one term ($l=0$)

$$a_{n,\,1}^n=\frac{1}{V_{n-1}V_n-x_n}, \quad \lambda_{n,\,1}^n=\lambda;$$

(ii) with three or more terms ($l>0$)

$$a_{m,\,1}^n=\frac{1}{V_{m-1}\,a_{m+1}^n-x_m}\left(1-\sum_{i=1}^{L_{n-m-1}}\frac{b_{m+1,\,i}^n}{\lambda-\mu_{m+1,\,i}^n}\right), \quad \lambda_{m,\,1}^n=\lambda;$$

$$a^n_{m,\,i} = \frac{b^n_{m+1,\,i-1}}{V_{m-1}a^n_{m+1} - x_m}\left(x_m + \frac{h}{\lambda - \mu^n_{m+1,\,i-1}}\right),$$

$$\lambda^n_{m+1,\,i-1} = \mu^n_{m+1,\,i-1}, \quad i = 2,3,\ldots,L_{n-m}.$$

– for a fragment with an even number of terms ($m = n - 2l - 1$, $l \geq 0$):

$$a^n_{m,\,i} = \frac{b^n_{m+1,\,i}}{V_{m-1}\,a^n_{m+1} - x_m}\left(x_m + \frac{h}{\lambda - \mu^n_{m+1,\,i}}\right),$$

$$\lambda^n_{m+1,\,i} = \mu^n_{m+1,\,i}, \quad i = 1,2,\ldots,L_{n-m}.$$

In these relations, $\mu^n_{m+1,\,i}$ ($i = 1,2,\ldots,L_{n-m}$) are the roots of the equation

$$1 - \sum_{i=1}^{L_{n-m}} \frac{a^n_{m+1,\,i}}{\lambda^n_{m+1,\,i} - \mu} = 0\,,$$

and $b^n_{m+1,\,i}$ ($i = 1,2,\ldots,L_{n-m}$) is the solution of the system of equations

$$1 - \sum_{i=1}^{L_{n-m-1}} \frac{b^n_{m+1,\,i}}{\lambda^n_{m+1,\,j} - \mu^n_{m+1,\,i}} = 0\,, \quad j = 1,2,\ldots,L_{n-m}.$$

Thus, it is possible to represent the approximation of the operator function with the help of the *n*-th convergent as

$$F^*_\alpha \approx \lambda_0 + \sum_{i=1}^{L_{n-1}} \lambda_i R^*_\alpha(\beta_i)\,, \quad L_n = \left[\frac{n}{2}\right] + 1\,.$$

The order of the convergent necessary for the approximation of the operator function can be chosen depending on the accuracy of determining the function $F^* \cdot 1$.

An important point in determining the parameters of a fractional exponential function is the optimal choice of its parameter α. This parameter characterizes the creep or relaxation rate on a time interval where it differs substantially from zero. From the geometrical viewpoint, it is the slope of the inclined part of the creep or relaxation curve in logarithmic coordinates.

Even in a problem for the composite consisting of several components having viscoelastic properties we could construct a series of fractional exponential functions that describes the deformation of other components [15]. Thus, the solutions of the boundary problems of linear viscoelasticity and fracture mechanics can be constructed with the use of only one base operator, responsible for the time variation of the stress-strain state.

III. Calculation of the Initial Period Duration for a Mixed-Mode Crack in a Viscoelastic Composite

A. Viscoelastic properties of the composite under consideration

Consider a viscoelastic laminate which can be represented as an orthotropic material after homogenization. Then the generalized Hooke's law for the principal directions can be written as

$$\begin{aligned} \varepsilon_{11} &= a_{11}\sigma_{11} + a_{12}\sigma_{22}, \\ \varepsilon_{22} &= a_{12}\sigma_{11} + a_{22}\sigma_{22}, \\ \gamma_{12} &= a_{66}\tau_{12}, \end{aligned} \tag{10}$$

where a_{ij} take the form of $a_{11} = 1/E_{11}$, $a_{12} = -\nu_{21}/E_{11}$, $a_{22} = 1/E_{22}$, and $a_{66} = 1/G_{12}$.

To determine the effective viscoelastic moduli of the composite (E_{11}, E_{22}, G_{12}, and ν_{21}), the experimental data for the material relaxation can be used, as well as the characteristics obtained from the results using homogenization theory for the composite materials of a known structure.

Technical constants are found using [16] as functions of shear moduli G_i and Poisson's ratios ν_i of isotropic components (a subscript index $i = 1$ corresponds to the characteristics of reinforcement and $i = 2$ corresponds to the characteristics of matrix):

$$E_{11} = c_1E_1 + c_2E_2 + \frac{8G_1c_1c_2(\nu_1 - \nu_2)}{d_1},$$

$$E_{22} = \left[\frac{\nu_{21}^2}{E_1} + \frac{\varkappa_2 + 1}{8G_2}\left(\frac{2 + (\varkappa_1 - 1)\tilde{G}}{d_1} - \frac{2c_1(1 - \tilde{G})}{d_2}\right)\right]^{-1},$$

$$G_{12} = G^0_{12}\left\{1 + n^2(n-1)\frac{G^0_{12}}{G_2}\left(\frac{\tilde{G} - 1}{c_2\tilde{G} + 1 + c_1}\right)^2 \right.$$
$$\left. \times \frac{\sin^n\alpha_n}{\pi^n}\left[c_1^2 - c_1^{2n}\left(\frac{\tilde{G} - 1}{\tilde{G} + 1}\right)^2\right]\right\},$$

$$G^0_{12} = G_2\frac{1 + c_1 + c_2\tilde{G}}{c_2 + (1 + c_1)\tilde{G}}, \qquad G_{23} = G_2\frac{d_2}{c_2\varkappa_2 + (1 + \varkappa_2c_1)\tilde{G}},$$

$$\nu_{21} = \nu_2 - (\varkappa_2 + 1)(\nu_2 - \nu_1)c_1 / d_1, \tag{11}$$

where

$$d_1 = 2 + c_1(\varkappa_2 - 1) + c_2(\varkappa_1 - 1)\tilde{G},$$
$$d_2 = \varkappa_2 + c_1 + c_2\tilde{G}, \varkappa_i = 3 - 4\nu_i \tilde{G} = G_2 / G_1,$$

$_n = 2\pi / n$, and n is a number that determines the type of fibers packing (e.g., $n = 4$ corresponds to tetragonal packing).

Experimental data for components were taken from [17] and adopted to the proposed approach. For the sake of simplicity, it was assumed that is is enough to keep one term in (3) ($n = 1$). Reinforcement has the following characteristics:

$$K_0 = 85.8\ \text{GPa}, \quad K_\infty = 11.7\ \text{GPa}, \quad \alpha_K = 0.8,$$
$$\lambda_K = 9 \cdot 10^{-4}\sec^{-\alpha_K}, \quad G_0 = 29.3 \cdot 10^4\,\text{GPa},$$
$$G_\infty = 0.117\ \text{GPa}, \quad \alpha_G = 0.9, \quad \lambda_G = 1.4 \cdot 10^{-3}\sec^{-\alpha_G},$$

where K is the bulk modulus, and G is the shear modulus. Subscripts "0" and "∞" denote the instantaneous and long-term values, correspondently.

Matrix characteristics are as follows:

$$K_0 = 5.41\ \text{GPa}, \quad K_\infty = 0.251\ \text{GPa}, \quad \alpha_K = 0.86,$$
$$\lambda_K = 3\cdot 10^{-2}\sec^{-\alpha_K}, \quad G_0 = 2.51\ \text{GPa}, \quad \alpha_G = 0.69,$$
$$G_\infty = 2.93\ \text{MPa}, \quad \lambda_G = 9\cdot 10^{-1}\sec^{-\alpha_G}.$$

B. *Crack modelling and the results*

Consider a crack in a viscoelastic orthotropic composite. To take non-linear deformation at the crack tips into account, we use the Dugdate model [12, 18] with the following cohesive law at the process zones (Fig. 1):

$$\sigma_{22} = \sigma_1\left(1-\frac{\delta_{\mathrm{I}}}{\delta_{\mathrm{I}}^*}\right), \quad \tau_{12} = \tau_1\left(1-\frac{\delta_{\mathrm{II}}}{\delta_{\mathrm{II}}^*}\right).$$

where σ_1 and τ_1 are some limiting values of stresses, δ_I and δ_{II} are crack opening in the direction of the corresponding axis, and δ_I^* and δ_{II}^* are the critical values of the opening.

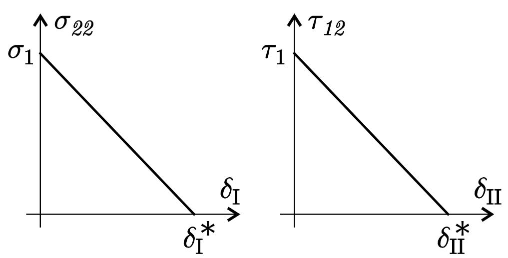

Fig.1 Cohesive law at the process zone

Stresses in the right process zone can be found as linear functions of the coordinate t as

$$\sigma_{22}(t) = \frac{(d-t)\sigma_0 + (t-b)\sigma_1}{d-b},$$
$$\tau_{12}(t) = \frac{(d-t)\tau_0 + (t-b)\tau_1}{d-b}$$
$$(b \le x \le d),$$

and symmetrically for the left zone.

Consider the case when the composite reinforcement direction coincides with the normal to the crack line direction. From the solution of an elastic problem, one can find that

$$\delta_{\mathrm{I}}(x) = \frac{L'}{\pi}\left[J_0(x)\sigma_0 + J_1(x)\sigma_1\right],$$
$$\delta_{\mathrm{II}}(x) = \frac{L''}{\pi}\left[J_0(x)\tau_0 + J_1(x)\tau_1\right],$$

where

$$L' = 2\sqrt{a_{11}\left[L_2 + a_{66}\right]}, \quad L'' = 2\sqrt{a_{22}\left[L_2 + a_{66}\right]},$$
$$L_2 = 2\left(\sqrt{a_{11}a_{22}} + a_{12}\right),$$

$$J_0(x) = -J_1(x) + (x-b)C_\Delta(b,x) - (x+b)C_\Delta(-b,x),$$
$$J_1(x) = T(x)\,/\,(d-b),$$
$$C_\Delta(\xi,x) = \ln\left|\frac{\sqrt{(d+\xi)(d-x)} - \sqrt{(d+x)(d-\xi)}}{\sqrt{(d+\xi)(d-x)} + \sqrt{(d+x)(d-\xi)}}\right|,$$
$$T(x) = \frac{1}{2}\left[(x+b)^2 C_\Delta(-b,x) + (x-b)^2 C_\Delta(b,x)\right] + \sqrt{(d^2-b^2)(d^2-x^2)}.$$

Until the opening displacement at the crack tip reaches its critical value, the problem parameters σ_0, τ_0, τ_1, and d can be determined from a system of four equations

$$\begin{cases} \sigma_1\left[1-\dfrac{L'}{\pi\delta_{\mathrm{I}}^*}(J_0(b)\sigma_0 + J_1(b)\sigma_1)\right] = \sigma_0 \\ \tau_1\left[1-\dfrac{L''}{\pi\delta_{II}^*}(J_0(b)\tau_0 + J_1(b)\tau_1)\right] = \tau_0, \\ N_0\sigma_0 + N_1\sigma_1 = \dfrac{\pi}{2}\sigma_y^\infty \\ N_0\tau_0 + N_1\tau_1 = \dfrac{\pi}{2}\tau_{xy}^\infty \end{cases} \qquad (12)$$

where the last two equations are the conditions of the stress finiteness at $x = \pm d$, and

$$N_0 = \frac{d\arccos\dfrac{b}{d} - \sqrt{d^2-b^2}}{d-b},$$
$$N_1 = -\frac{b\arccos\dfrac{b}{d} - \sqrt{d^2-b^2}}{d-b}.$$

As the opening at the crack tips reaches its critical value, σ_0 and τ_0 vanishes to zero, and (12) becomes

$$\begin{cases} L'J_1(b)\sigma_1 = \pi\delta_{\mathrm{I}}^* \\ L''J_1(b)\dfrac{\tau_{xy}^\infty}{\sigma_y^\infty}\sigma_1 = \pi\delta_{\mathrm{II}}^* \\ N_1\sigma_1 = \dfrac{\pi}{2}\sigma_y^\infty \end{cases} \qquad (13)$$

with $J_1(b) = \dfrac{2b^2}{d-b}\ln\dfrac{b}{d} + b + d.$

From the last equation of (12) one can determine d. Then the first and the second equations can be used in a combination with an experimental dependency $F(\delta_{\mathrm{I}}, \delta_{\mathrm{II}}) = 0$. For our example, this dependency can be taken as follows: $(\delta_I^*)^2 + (\delta_{II}^*)^2 = \delta_e^2$.

This allows us to determine the duration of initiation period of the crack growth. This duration t_0 should satisfy the following equation:

$$\left(\frac{\tau_{xy}^\infty}{\sigma_y^\infty}L'\right)^2 + (L'')^2 = \left(\frac{\pi\delta_e}{\sigma_1 J_1(b)}\right)^2.$$

For our example in Fig. 2, the following parameters are used: external loading $\sigma_y^\infty = 4$ MPa, $\tau_{xy}^\infty = 1, 1.5$, and 2 MPa (for curves 1, 2 and 3, respectively), $\sigma_1 = 35$ MPa, and $\delta_e = 3 \cdot 10^5$ m.

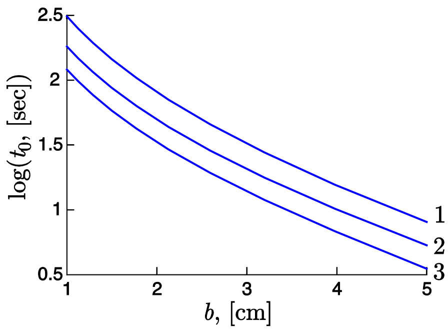

Fig. 2 Crack initiation period duration vs. crack half-length for the different values of τ_{xy}^∞

IV. Conclusion

The proposed algorithm can be used to study a wide range of the fracture mechanics problems for linear viscoelastic materials. Its major advantages are the use of well-defined algebra of resolvent operators and the possibility of application to various problems when the crack opening can be determined from an elastic solution by the correspondence principle.

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