Axisymmetric solution to time-fractional heat conduction equation in an infinite cylinder under local heating and associated thermal stresses

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Dedicated to the 100 anniversary of Academician Yury N. Rabotnov

***Abstract*—**The theory of thermal stresses based on the time-fractional heat conduction equation with the Caputo derivative is used to investigate axisymmetric thermal stresses in an infinite cylinder under local heating of its surface. The representation of stresses in terms of the displacement potential and the biharmonic Love function is employed. The solution is obtained using the integral transform technique (the Laplace transform with respect to time, the exponential Fourier transform with respect to the axial coordinate and the finite Hankel transform with respect to the radial coordinate).

***Keywords*—**Fractional calculus, Mittag–Leffler function, non-Fourier heat conduction, thermal stresses.

I. INTRODUCTION

Investigation of different physical phenomena in media with complex internal structure has led to considering differential equations with derivatives of fractional order. Numerical applications of the fractional calculus to problems of mechanics can be found in the literature: fractional relaxation-oscillation [1], rheology [2]–[4], creep [5], hereditary mechanics of solids [6], [7], the Brownian motion [8], stress and strain localization in solids [9], the fractional dynamical systems [10] (see also [11]–[16] and references therein). The time-fractional derivatives describe time nonlocality (the memory effects), the space-fractional derivatives represent space nonlocality (the long-range interaction).

The first theory of thermal stresses based on a fractional heat conduction equation with the time-fractional derivative of the order $0 < \alpha \le 2$ was proposed in [17]. Later on fractional thermoelasticity was generalized to take into account also space-fractional derivatives in the heat conduction equation [18]–[20]. Theories of thermal stresses based on the time-fractional [21]–[23] and space-time fractional telegraph equation [24]–[26] were also proposed.

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In this paper, we consider the axisymmetric thermoelasticity problem for a long cylinder subjected to local heating at the boundary surface. Heat conduction is described by the time-fractional equation with the Caputo derivative of the order $0 < \alpha \le 2$. The solution is obtained using the Laplace transform with respect to time t, the exponential Fourier transform with respect to the axial coordinate z, and the finite Hankel transform with respect to the radial coordinate r. The representation of the stress tensor components in terms of the displacement potential Φ and the biharmonic Love function Ψ is employed. Earlier the solutions to the time-fractional heat conduction equation in a cylinder were analyzed in [27] and [28]. Radial heat conduction in a cylinder and associated thermal stresses were studied in [29].

II. STATEMENT OF THE PROBLEM

A quasi-static uncoupled theory of thermal stresses based on the time-fractional heat conduction equation is governed by the following system of equations:
the equilibrium equation in terms of displacements

$$\mu\Delta\mathbf{u} + (\lambda + \mu)\nabla(\nabla\cdot\mathbf{u}) = \beta K\nabla T, \tag{1}$$

the stress-strain-temperature relation

$$\boldsymbol{\sigma} = 2\mu\,\mathbf{e} + (\lambda\,\mathrm{tr}\,\mathbf{e} - \beta KT)\mathbf{I}, \tag{2}$$

and the heat conduction equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\Delta T, \quad 0 < \alpha \le 2, \tag{3}$$

where $\mathbf{u}$ is the displacement vector, $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{e}$ denotes the linear strain tensor

$$\mathbf{e} = \frac{1}{2}\left(\nabla\mathbf{u} + \mathbf{u}\nabla\right), \tag{4}$$

λ and μ are Lamé constants, $K = \lambda + 2\mu/3$ is the modulus of dilatation, β is the thermal coefficient of volumetric expansion, a is the thermal diffusivity coefficient, $\mathbf{I}$ stands for the unit tensor, ∇ is the gradient operator, and Δ denotes the Laplace operator.

In the heat conduction equation (3), we use the Caputo fractional derivative [12], [30], [31]

$$\frac{d^{\alpha} f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n f(\tau)}{d\tau^n} d\tau, \quad n-1 < \alpha < n, \tag{5}$$

where $\Gamma(\alpha)$ is the gamma function.

Just as in the classical theory of thermal stresses [32] and [33], we can use the representation of the stress tensor $\boldsymbol{\sigma}$ in terms of the displacement potential Φ and the Galerkin vector $\mathbf{G}$:

$$\boldsymbol{\sigma}^{(1)} = 2\mu\left(\nabla\nabla\Phi - \mathbf{I}\Delta\Phi\right), \tag{6}$$

$$\boldsymbol{\sigma}^{(2)} = 2\mu\left[\left(\nu\,\mathbf{I}\,\Delta - \nabla\nabla\right)\nabla\cdot\mathbf{G} + (1+\nu)\Delta\left(\nabla\mathbf{G} + \mathbf{G}\nabla\right)\right]. \tag{7}$$

Here ν is the Poisson ratio.

The part of stresses due to the displacement potential describes the influence of the temperature field

$$\Delta\Phi = mT, \qquad m = \frac{1+\nu}{1-\nu}\,\frac{\beta}{3}; \tag{8}$$

the stresses expressed in terms of the biharmonic Galerkin vector

$$\Delta\Delta\mathbf{G} = 0 \tag{9}$$

allow us to satisfy the prescribed boundary conditions for the components of the total stress tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}$.

III. Temperature Field in a Long Cylinder

Consider the axisymmetric time-fractional heat conduction equation in a long cylinder

$$\frac{\partial^{\alpha} T}{\partial t^{\alpha}} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right), \tag{10}$$

$$0 \le r < R, \quad -\infty < z < \infty, \quad 0 < t < \infty, \quad 0 < \alpha \le 2,$$

under zero initial conditions

$$t = 0: \quad T = 0, \qquad 0 < \alpha \le 2, \tag{11}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = 0, \quad 1 < \alpha \le 2, \tag{12}$$

and the Dirichlet boundary condition corresponding to the local heating of a surface $r = R$

$$r = R: \quad T = \begin{cases} T_0, & |z| < l, \\ 0, & |z| > l. \end{cases} \tag{13}$$

The zero condition at infinity

$$\lim_{z\to\pm\infty} T(r,z,t) = 0 \tag{14}$$

is also assumed.

The integral transform technique will be used to solve (10)–(14). Recall that the Caputo fractional derivative for its Laplace transform rule requires the knowledge of the initial value of a function and its integer derivatives of order $k = 1, 2, \ldots, n-1$:

$$L\left\{\frac{d^{\alpha} f(t)}{d\,t^{\alpha}}\right\} = s^{\alpha} L\{f(t)\} - \sum_{k=0}^{n-1} s^{\alpha-1-k} f^{(k)}(0^+), \quad n-1 < \alpha < n, \tag{15}$$

where s is the transform variable.

Application of the Laplace transform with respect to time to (10)–(13) gives

$$s^{\alpha} T^* = a\left(\frac{\partial^2 T^*}{\partial r^2} + \frac{1}{r}\frac{\partial T^*}{\partial r} + \frac{\partial^2 T^*}{\partial z^2}\right), \tag{16}$$

$$r = R: \quad T^* = \begin{cases} T_0/s, & |z| < l, \\ 0, & |z| > l, \end{cases} \tag{17}$$

where the asterisk denotes the Laplace transform.

The exponential Fourier transform with respect to the axial coordinate z leads to the following equation:

$$s^{\alpha} \tilde{T}^* = a\left(\frac{\partial^2 \tilde{T}^*}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{T}^*}{\partial r} - \eta^2 \tilde{T}\right), \tag{18}$$

and the boundary condition

$$r = R: \quad \tilde{T}* = \frac{\sqrt{2} T_0 \sin(l\eta)}{\sqrt{\pi}\ s\eta}, \tag{19}$$

where the tilde denotes the Fourier transform, and η is the transform variable.

Next, we use the finite Hankel transform [34]

$$H\{f(r)\} = \hat{f}(\xi_k) = \int_0^R f(r) J_0(r\xi_k) r\, dr, \tag{20}$$

$$H^{-1}\{\hat{f}(\xi_k)\} = f(r) = \frac{2}{R^2} \sum_{k=1}^{\infty} \hat{f}(\xi_k) \frac{J_0(r\xi_k)}{[J_1(R\xi_k)]^2}, \tag{21}$$

where the hat marks the Hankel transform, $J_n(r)$ is the Bessel function of the order n, $R\xi_k$ are the positive roots of the zeroth order Bessel function

$$J_0(R\xi_k) = 0, \tag{22}$$

and

$$H\left\{\frac{d^2 f(r)}{dr^2} + \frac{1}{r}\frac{df(r)}{dr}\right\} = -\xi_k^2\ \hat{f}(\xi_k) + R\xi_k J_1(R\xi) f(R). \tag{23}$$

From (18), (19) and (23) we get

$$\hat{\tilde{T}}*(\xi_k, \eta, s) = \frac{\sqrt{2} aRT_0}{\sqrt{\pi}} \frac{\sin(l\eta)\ \xi_k J_1(R\xi_k)}{\eta} \times \frac{1}{s[s^\alpha + a(\xi_k^2 + \eta^2)]} \tag{24}$$

or

$$\hat{\tilde{T}}*(\xi_k, \eta, s) = \frac{\sqrt{2} RT_0}{\sqrt{\pi}} \frac{\sin(l\eta)\ \xi_k J_1(R\xi_k)}{\eta\left(\xi_k^2 + \eta^2\right)} \times \left[\frac{1}{s} - \frac{s^{\alpha-1}}{s^\alpha + a\left(\xi_k^2 + \eta^2\right)}\right]. \tag{25}$$

The inverse Laplace transform results in

$$\hat{\tilde{T}}(\xi_k, \eta, t) = \frac{\sqrt{2} RT_0}{\sqrt{\pi}} \frac{\sin(l\eta)\ \xi_k J_1(R\xi_k)}{\eta\left(\xi_k^2 + \eta^2\right)} \times \left\{1 - E_\alpha\left[-a\left(\xi_k^2 + \eta^2\right)t^\alpha\right]\right\}, \tag{26}$$

where $E_\alpha(z)$ is the Mittag-Leffler function in one parameter α having the series representation

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0, \quad z \in C, \tag{27}$$

and the following formula [12], [30], [31]

$$L^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha + b}\right\} = E_\alpha\left(-bt^\alpha\right) \tag{28}$$

has been used.

Inversion of the finite Hankel transform gives

$$\tilde{T}(r, \eta, t) = \frac{2\sqrt{2} T_0}{\sqrt{\pi} R} \sum_{k=1}^{\infty} \frac{\xi_k J_0(r\xi_k)}{\left(\xi_k^2 + \eta^2\right) J_1(R\xi_k)} \times \left\{1 - E_\alpha\left[-a\left(\xi_k^2 + \eta^2\right)t^\alpha\right]\right\} \frac{\sin(l\eta)}{\eta} \tag{29}$$

and finally

$$T(r, z, t) = \frac{2T_0}{\pi R} \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} \frac{\xi_k J_0(r\xi_k)}{\left(\xi_k^2 + \eta^2\right) \eta\, J_1(R\xi_k)} \times \left\{1 - E_\alpha\left[-a\left(\xi_k^2 + \eta^2\right)t^\alpha\right]\right\} \sin(l\eta)\cos(z\eta)\, d\eta. \tag{30}$$

It should be emphasized that the relation [34]

$$\frac{2}{R} \sum_{k=1}^{\infty} \frac{\xi_k J_0(r\xi_k)}{\left(\xi_k^2 - \beta^2\right) J_1(R\xi_k)} = \frac{J_0(r\beta)}{J_0(R\beta)} \tag{31}$$

for $\beta = i\eta$ can be rewritten as

$$\frac{2}{R} \sum_{k=1}^{\infty} \frac{\xi_k J_0(r\xi_k)}{\left(\xi_k^2 + \eta^2\right) J_1(R\xi_k)} = \frac{I_0(r\eta)}{I_0(R\eta)}, \tag{32}$$

where $I_n(r)$ is the modified Bessel function of order n. Hence (30) takes the form

$$T(r, z, t) = \frac{T_0}{\pi} \int_{-\infty}^{\infty} \frac{I_0(r\eta)}{I_0(R\eta)} \frac{\sin(l\eta)\cos(z\eta)}{\eta} d\eta - \frac{2T_0}{\pi R} \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} \frac{\xi_k J_0(r\xi_k)}{\left(\xi_k^2 + \eta^2\right) J_1(R\xi_k)} \times E_\alpha\left[-a\left(\xi_k^2 + \eta^2\right)t^\alpha\right] \frac{\sin(l\eta)\cos(z\eta)}{\eta} d\eta. \tag{33}$$

At the boundary surface $r = R$, the first integral in (33) satisfies the boundary condition (13), whereas the second one equals zero.

IV. Thermal Stresses in a Long Cylinder

Equation (8) in the cylindrical coordinates takes the form

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial z^2} = mT, \tag{34}$$

or, after applying the Fourier and finite Hankel transforms,

$$\hat{\tilde{\Phi}} = -\frac{m}{\xi_k^2 + \eta^2}\hat{\tilde{T}}, \tag{35}$$

which gives

$$\tilde{\Phi} = -C\frac{\sin(l\eta)}{\eta}\sum_{k=1}^{\infty} P_k \frac{J_0(r\xi_k)}{J_1(R\xi_k)}, \tag{36}$$

where

$$C = \frac{2\sqrt{2}mT_0}{R\sqrt{\pi}}, \tag{37}$$

$$P_k = \frac{\xi_k}{\left(\xi_k^2 + \eta^2\right)^2}\left\{1 - E_\alpha\left[-a\left(\xi_k^2 + \eta^2\right)t^\alpha\right]\right\}. \tag{38}$$

The components of the tensor $\boldsymbol{\sigma}^{(1)}$ are expressed as [32]

$$\sigma_{rr}^{(1)} = 2\mu\left[\frac{\partial^2 \Phi}{\partial r^2} - \Delta\Phi\right], \tag{39}$$

$$\sigma_{\theta\theta}^{(1)} = 2\mu\left[\frac{1}{r}\frac{\partial \Phi}{\partial r} - \Delta\Phi\right], \tag{40}$$

$$\sigma_{zz}^{(1)} = 2\mu\left[\frac{\partial^2 \Phi}{\partial z^2} - \Delta\Phi\right], \tag{41}$$

$$\sigma_{rz}^{(1)} = 2\mu\frac{\partial^2 \Phi}{\partial r \partial z} \tag{42}$$

or in the Fourier transform domain

$$\tilde{\sigma}_{rr}^{(1)} = -2\mu C\frac{\sin(l\eta)}{\eta} \times \sum_{k=1}^{\infty} P_k \frac{\xi_k J_1(r\xi_k) + r\eta^2 J_0(r\xi_k)}{rJ_1(R\xi_k)}, \tag{43}$$

$$\tilde{\sigma}_{\theta\theta}^{(1)} = 2\mu C\frac{\sin(l\eta)}{\eta} \times \sum_{k=1}^{\infty} P_k \frac{\xi_k J_1(r\xi_k) - r\left(\xi_k^2 + \eta^2\right)J_0(r\xi_k)}{rJ_1(R\xi_k)}, \tag{44}$$

$$\tilde{\sigma}_{zz}^{(1)} = -2\mu C\frac{\sin(l\eta)}{\eta}\sum_{k=1}^{\infty} P_k \frac{\xi_k^2 J_0(r\xi_k)}{J_1(R\xi_k)}, \tag{45}$$

$$\tilde{\sigma}_{rz}^{(1)} = -2\mu\, iC\sin(l\eta)\sum_{k=1}^{\infty} P_k \frac{\xi_k J_1(r\xi_k)}{J_1(R\xi_k)}. \tag{46}$$

To satisfy the boundary conditions for the total stress tensor, we should consider the stress tensor $\boldsymbol{\sigma}^{(2)}$ expressed in terms of the Love function [35]:

$$\sigma_{rr}^{(2)} = 2\mu\frac{\partial}{\partial z}\left(\nu\,\Delta\Psi - \frac{\partial^2 \Psi}{\partial r^2}\right), \tag{47}$$

$$\sigma_{\theta\theta}^{(2)} = 2\mu\frac{\partial}{\partial z}\left(\nu\,\Delta\Psi - \frac{1}{r}\frac{\partial \Psi}{\partial r}\right), \tag{48}$$

$$\sigma_{zz}^{(2)} = 2\mu\frac{\partial}{\partial z}\left[(2-\nu)\Delta\Psi - \frac{\partial^2 \Psi}{\partial z^2}\right], \tag{49}$$

$$\sigma_{rz}^{(2)} = 2\mu\frac{\partial}{\partial r}\left[(1-\nu)\Delta\Psi - \frac{\partial^2 \Psi}{\partial z^2}\right]. \tag{50}$$

The biharmonic Love function

$$\left(\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{\partial^2 \Psi}{\partial z^2}\right)^2 = 0 \tag{51}$$

can be obtained as a particular case of the Galerkin vector having only the z-component $\mathbf{G} = (0, 0, \Psi)$.

The solution of (51) in the Fourier transform domain

$$\left(\frac{\partial^2 \tilde{\Psi}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{\Psi}}{\partial r} - \eta^2\tilde{\Psi}\right)^2 = 0 \tag{52}$$

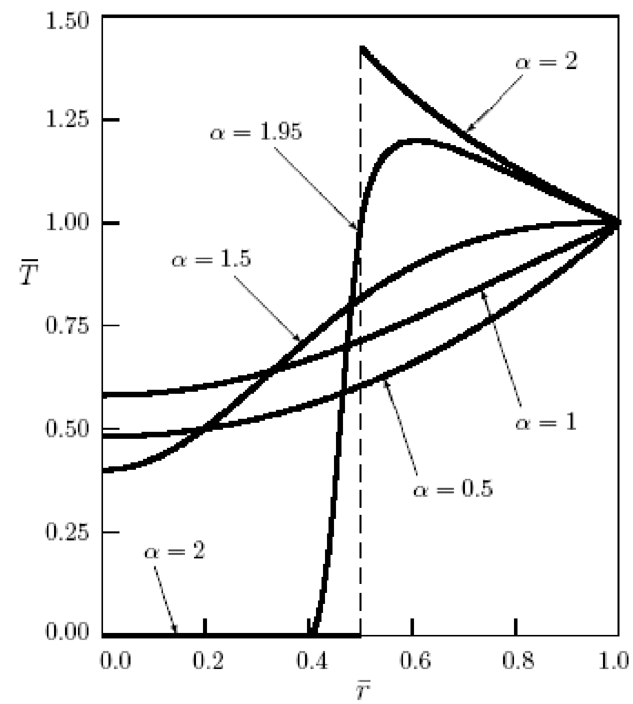

Fig. 1 the distance-dependence of a nondimensional temperature ($z=0$, $\kappa=0.5$, $\bar{l}=1$)

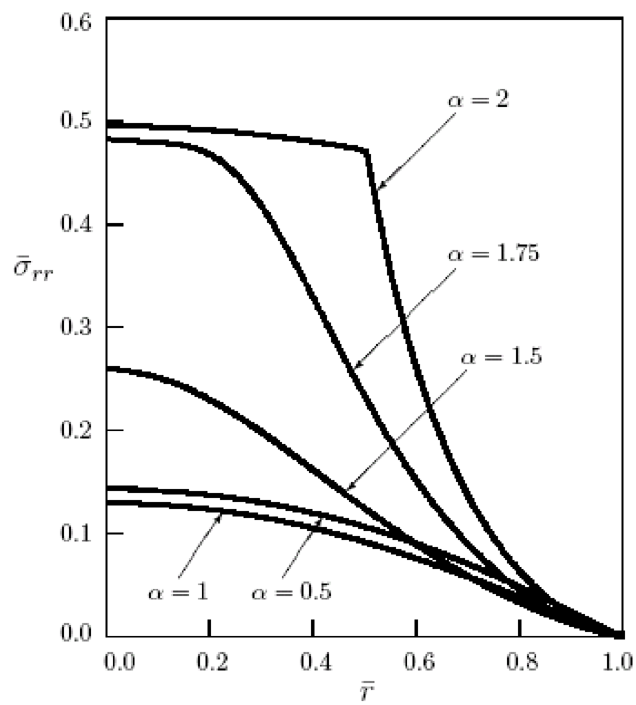

Fig. 2 the distance-dependence of the nondimensional stress component $\overline{\sigma}_{rr}$ ($z=0$, $\kappa=0.5$, $\bar{l}=1$)

finite at $r=0$ has the form [34]

$$\tilde{\Psi} = A(\eta) I_0(r\eta) + B(\eta) r\eta\, I_1(r\eta) \tag{53}$$

with $A(\eta)$ and $B(\eta)$ being the integration coefficients.

From (47) –(50) and (53) we get

$$\begin{aligned}\tilde{\sigma}_{rr}^{(2)} &= 2\mu\, i\{\eta^3 A(\eta)[I_0(r\eta) - (r\eta)^{-1} I_1(r\eta)] \\ &\quad + \eta^3 B(\eta)[(1-2\nu) I_0(r\eta) + r\eta I_1(r\eta)]\},\end{aligned} \tag{54}$$

$$\begin{aligned}\tilde{\sigma}_{\theta\theta}^{(2)} &= 2\mu\, i[(1-2\nu)\eta^3 B(\eta) I_0(r\eta) \\ &\quad + \eta^2 A(\eta) r^{-1} I_1(r\eta)],\end{aligned} \tag{55}$$

$$\begin{aligned}\tilde{\sigma}_{zz}^{(2)} &= -2\mu\, i\{\eta^3 A(\eta) I_0(r\eta) \\ &\quad + \eta^3 B(\eta)[2(2-\nu) I_0(r\eta) + r\eta I_1(r\eta)]\},\end{aligned} \tag{56}$$

$$\begin{aligned}\tilde{\sigma}_{rz}^{(2)} &= 2\mu\{\eta^3 A(\eta) I_1(r\eta) \\ &\quad + \eta^3 B(\eta)[2(1-\nu) I_1(r\eta) + r\eta I_0(r\eta)]\}.\end{aligned} \tag{57}$$

The surface of a cylinder is traction free. Hence

$$r = R: \quad \sigma_{rr}^{(1)} + \sigma_{rr}^{(2)} = 0, \tag{58}$$

$$r = R: \quad \sigma_{rz}^{(1)} + \sigma_{rz}^{(2)} = 0, \tag{59}$$

what allows us to determine the integration constants

$$\begin{aligned}A(\eta) &= \frac{iC\sin(l\eta)}{\eta^3 D(\eta)} \sum_{k=1}^{\infty} \xi_k P_k \\ &\quad \times \{[R^2\eta^2 + 2(1-\nu)] I_1(R\eta) + 2(1-\nu) R\eta I_0(R\eta)\},\end{aligned} \tag{60}$$

$$\mathrm{B}(\eta) = -\frac{iC\sin(l\eta)}{\eta^3 D(\eta)} R\eta I_0(R\eta) \sum_{k=1}^{\infty} \xi_k P_k, \tag{61}$$

where

$$D(\eta) = [R^2\eta^2 + 2(1-\nu)] I_1^2(R\eta) - R^2\eta^2 I_0^2(R\eta). \tag{62}$$

The final expressions for the stress tensor components have the following form:

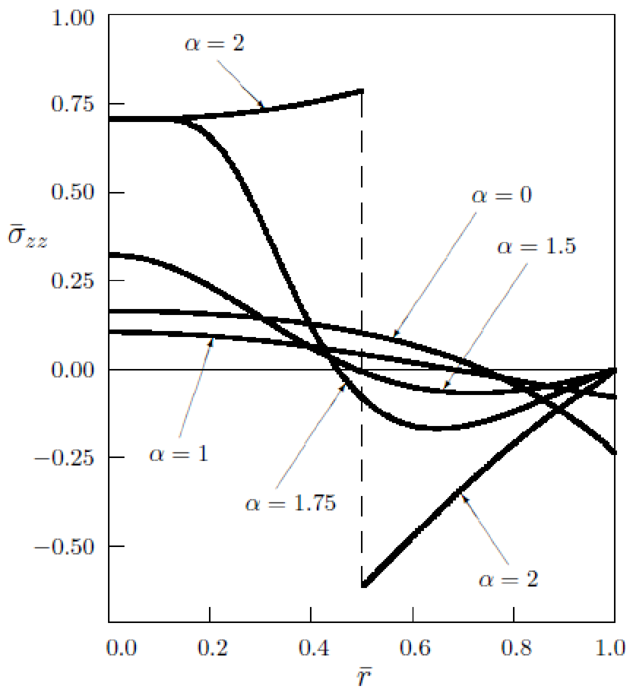

Fig. 3 the distance-dependence of the nondimensional stress component $\bar{\sigma}_{zz}$ ($z = 0$, $\kappa = 0.5$, $\bar{l} = 1$)

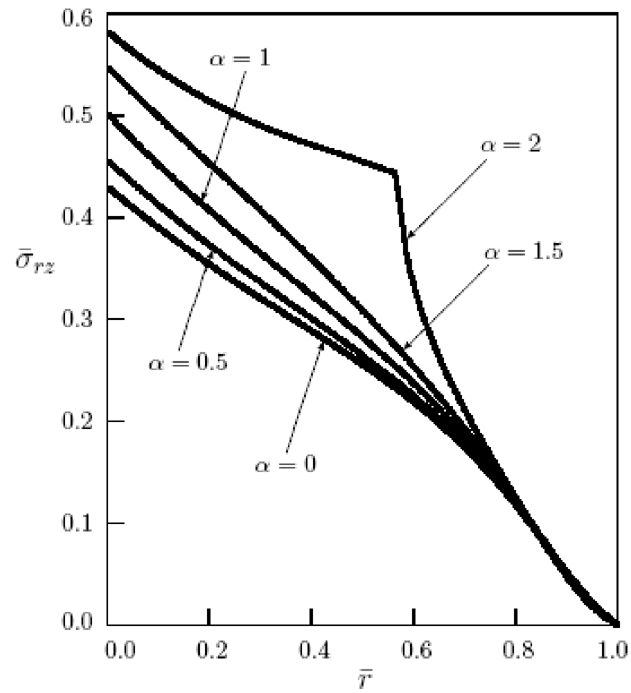

Fig. 4 the distance-dependence of the nondimensional stress component $\bar{\sigma}_{rz}$ ($z = 0.75$, $\kappa = 0.5$, $\bar{l} = 1$)

$$\sigma_{rr} = C_1 \int_{-\infty}^{\infty} \frac{\sin(l\eta)\cos(z\eta)}{r\eta} \times \sum_{k=1}^{\infty} P_k \left[\frac{\xi_k S(\eta)}{D(\eta)} - \frac{\xi_k J_1(r\xi_k) + r\eta^2 J_0(r\xi_k)}{J_1(R\xi_k)} \right] d\eta, \tag{63}$$

$$\sigma_{\theta\theta} = C_1 \int_{-\infty}^{\infty} \frac{\sin(l\eta)\cos(z\eta)}{r\eta} \sum_{k=1}^{\infty} P_k \times \left[\frac{\xi_k U(\eta)}{D(\eta)} + \frac{\xi_k J_1(r\xi_k) - r(\xi_k^2 + \eta^2) J_0(r\xi_k)}{J_1(R\xi_k)} \right] d\eta, \tag{64}$$

$$\sigma_{zz} = C_1 \int_{-\infty}^{\infty} \sin(l\eta)\cos(z\eta) \times \sum_{k=1}^{\infty} \xi_k P_k \left[\frac{V(\eta)}{D(\eta)} - \frac{\xi_k J_0(r\xi_k)}{\eta J_1(R\xi_k)} \right] d\eta, \tag{65}$$

$$\sigma_{rz} = C_1 \int_{-\infty}^{\infty} \sin(l\eta)\sin(z\eta) \times \sum_{k=1}^{\infty} \xi_k P_k \left[\frac{W(\eta)}{D(\eta)} - \frac{J_1(r\xi_k)}{J_1(R\xi_k)} \right] d\eta, \tag{66}$$

where

$$C_1 = \frac{4\mu m T_0}{\pi R}, \tag{67}$$

$$\begin{aligned} S(\eta) = {} & \left[R^2\eta^2 + 2(1-\nu)\right] I_1(R\eta) I_1(r\eta) \\ & - Rr\eta^2 I_0(R\eta) I_0(r\eta) \\ & + \left[Rr^2\eta^3 + 2(1-\nu)R\eta\right] I_0(R\eta) I_1(r\eta) \\ & - \left[R^2 r\eta^3 + 2(1-\nu) r\eta\right] I_0(r\eta) I_1(R\eta), \end{aligned} \tag{68}$$

$$\begin{aligned} U(\eta) = {} & (1-2\nu) Rr\eta^2 I_0(R\eta) I_0(r\eta) \\ & - \left[R^2\eta^2 + 2(1-\nu)\right] I_1(R\eta) I_1(r\eta) \\ & - 2(1-\nu) R\eta I_0(R\eta) I_1(r\eta), \end{aligned} \tag{69}$$

$$\begin{aligned} V(\eta) = {} & \left[R^2\eta^2 + 2(1-\nu)\right] I_1(R\eta) I_0(r\eta) \\ & - Rr\eta^2 I_0(R\eta) I_1(r\eta) - 2R\eta\, I_0(R\eta) I_0(r\eta), \end{aligned} \tag{70}$$

$$\mathrm{W}(\eta)=\left[R^2\eta^2+2(1-\nu)\right]I_1(R\eta)I_1(r\eta) \\ -Rr\eta^2 I_0(R\eta)I_0(r\eta). \tag{71}$$

In the particular case of classical thermoelasticity ($\alpha=1$), the Mittag-Leffler function

$$E_1\left(-a\xi_k^2 t\right)=\exp\left(-a\xi_k^2 t\right), \tag{72}$$

and the obtained solution coincides with that presented in [32]. Two other particular cases, with $\alpha=2$ (the ballistic heat conduction described by the wave equation for temperature) and with $\alpha=0$ (the localized heat conduction described by the Helmholtz equation for temperature), are obtained when

$$E_2\left(-a\xi_k^2 t^2\right)=\cos\left(\sqrt{a}\xi_k t\right) \tag{73}$$

and

$$E_0\left(-a\xi_k^2\right)=\frac{1}{1+a\xi_k^2}, \tag{74}$$

respectively.

V. Numerical Calculations

The results of numerical calculations are shown in Figs. 1-4. To evaluate the Mittag-Leffler function, we have applied the algorithm suggested in [36]. The following nondimensional quantities

$$\bar{T}=\frac{T}{T_0}, \qquad \bar{\sigma}_{ij}=\frac{1}{2\mu m T_0}\sigma_{ij}, \qquad \bar{r}=\frac{r}{R}, \\ \bar{z}=\frac{z}{R}, \qquad \bar{l}=\frac{l}{R}, \qquad \kappa=\frac{\sqrt{a}t^{\alpha/2}}{R} \tag{75}$$

have been introduced. In computations we have assumed $\nu=0.25$. Temperature T and the stress components σ_{rr}, $\sigma_{\theta\theta}$, and σ_{zz} are even functions in z; in calculations we have taken $z=0$. The stress component σ_{rz} is an odd function in z; calculations were carries out for $\bar{z}=0.75$.

VI. Conclusion

In the case $0<\alpha<1$, the time-fractional heat conduction equation interpolates the elliptic Helmholtz equation and the classical heat conduction equation. Because the fractional heat conduction equation in the case $1\le\alpha\le2$ interpolates the standard heat conduction equation ($\alpha=1$) and the wave equation ($\alpha=2$), the proposed theory interpolates the classical theory of thermal stresses and thermoelasticity without energy dissipation introduced by Green and Naghdi [37]. In the case of the wave equation ($\alpha=2$), temperature T (Fig. 1) and the stress component σ_{zz} (Fig. 3) have jumps at the wave front corresponding to $\bar{r}=1-\kappa$ with $0<\kappa<1$. The curves for α approaching 2 approximate this jump. It should be emphasized that the response of the time-fractional heat conduction equation with $1\le\alpha\le2$ to a localized disturbance spreads infinitely fast [38] and [39]. On the other hand, the fundamental solution to the time-fractional heat conduction equation possesses a maximum that disperses with a finite speed similar to the behavior of the fundamental solution to the wave equation [38] and [39]. Fractional thermoelasticity offers considerable possibilities for better describing thermal stresses in porous materials, fractals, random and disordered media, and other solids with complicated internal structure. In the framework of fractional thermoelasticity, several problems for various geometries were solved in [40]–[46].

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