Stabilization of Telescopic Inverse Pendulum Verification by Physical Models

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***Abstract*—**Article deals with physical models of mathematical and simple and double physical pendulum for its stabilization possibilities verification in instable inverse position. Whereas classical stabilization by proximal joint horizontal movement of inverse mathematical pendulum is well known and in article is just why and when is possible, the case of stabilization by proximal joint vertical oscillation comings-out from chaos theory and is referred in article by multiport physical models and their behavior.

***Keywords*–**Mathematical pendulum, double physical pendulum, inverse position stabilization, multiport physical models.

I. Introduction

It is generally known that nonlinear dynamic systems may have, in contrast to linear systems, several balance states (singular points) and behavior in their neighborhoods is able to distinguish. In this contribution we will show what the nonlinear system singular point typical characteristics are and how we can affect them. Second part is then oriented on chaos and bifurcations principles theory results utilization for such way stabilization of inverted pendulum, which although its application is known from fourteenth century, only chaos theory clarified it.

II. Mathematical Pendulum Stabilization by Proximal Coupling Horizontal Movement

Fig.1 shows mathematical pendulum basic type ordering connected by proximal kinematic rotating pair to the material cart with one freedom degree in global axis x direction.

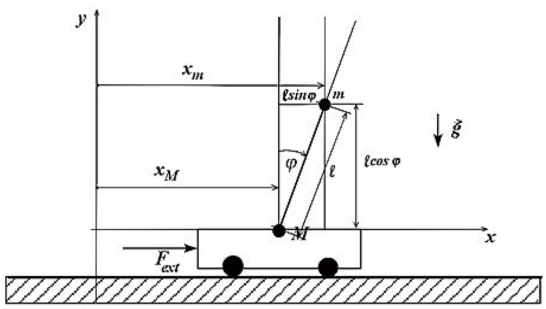

Fig.1 mathematic inverse pendulum ordering on material cart

Motional equations of this system are being:

$$\begin{aligned}&(M+m)\cdot\ddot{x}_M + m\cdot\ell\cdot\cos\varphi\cdot\ddot{\varphi} = F_{ext} + m\cdot\ell\cdot\sin\varphi\cdot\dot{\varphi}^2\\ &m\cdot\ell\cdot\cos\varphi\cdot\ddot{x}_M + m\cdot\ell^2\cdot\ddot{\varphi} = m\cdot\ell\cdot g\cdot\sin\varphi\end{aligned} \quad (1)$$

For state vector

$$\underline{x}^T = [\dot{x}_M \quad \dot{\varphi} \quad x_M \quad \varphi]$$

the linearized system in the surrounding point of $\underline{x} = \underline{0}$ and for outgoing quantity $\boldsymbol{\varphi}$ is

$$\underline{\dot{x}} = \overbrace{\begin{bmatrix} 0 & 0 & 0 & -\frac{mg}{M} \\ 0 & 0 & 0 & \frac{M+m}{M\cdot\ell}g \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}^{A} \cdot \begin{bmatrix} \dot{x}_M \\ \dot{\varphi} \\ x_M \\ \varphi \end{bmatrix} + \overbrace{\begin{bmatrix} \frac{1}{M} \\ -\frac{1}{M\cdot\ell} \\ 0 \\ 0 \end{bmatrix}}^{b} \cdot F_{ext}(t); \quad (2)$$

$$\underline{y} = \overbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}}^{c^T} \cdot \begin{bmatrix} \dot{x}_M \\ \dot{\varphi} \\ x_M \\ \varphi \end{bmatrix}$$

As for controllability matrix $\mathbf{Q}$ is

$$\det \mathbf{Q} = -\frac{\omega_0^2}{M^4\cdot\ell^2} \neq 0$$

the system is for finite **M** controllable.

But as for observability matrix $\mathbf{N}$ is $\mathbf{det}\,\mathrm{N} = \mathbf{0}$, the system isn't observable.

Matrix eigenvalues and eigenvectors of linearized system are

$$\lambda_1 = 0; \lambda_2 = 0; \lambda_3 = \sqrt{\frac{g\cdot(M+m)}{\ell\cdot M}}; \lambda_4 = -\sqrt{\frac{g\cdot(M+m)}{\ell\cdot M}}$$

$$\underline{s}_{1,2} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \quad \underline{s}_3 = \begin{bmatrix} -\frac{\frac{1}{M+m}}{\ell\cdot m} \\ -\frac{1}{\sqrt{\frac{g\cdot(M+m)}{\ell\cdot M}}} \\ \frac{\sqrt{M(M+m)}}{m\cdot\sqrt{g\cdot\ell}} \end{bmatrix}; \underline{s}_4 = \begin{bmatrix} -\frac{\frac{1}{M+m}}{\ell\cdot m} \\ \frac{1}{\sqrt{\frac{g\cdot(M+m)}{\ell\cdot M}}} \\ -\frac{\sqrt{M(M+m)}}{m\cdot\sqrt{g\cdot\ell}} \end{bmatrix} \quad (3)$$

Jordan's canonical form of system is

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$$\dot{x}_D(t)=\begin{bmatrix} -\sqrt{\frac{g(M+m)}{\ell\cdot M}} & 0 & 0 & 0\\ 0 & \sqrt{\frac{g(M+m)}{\ell\cdot M}} & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}\cdot x_D(t)+\begin{bmatrix}\frac{m}{2M(M+m)}\\ \frac{m}{2M(M+m)}\\ -\frac{1}{(M+m)}\\ \frac{1}{(M+m)}\end{bmatrix}\cdot F_{ext}(t) \qquad (4)$$

$$y=\begin{bmatrix}\frac{\sqrt{g\cdot M(M+m)}}{g\sqrt{\ell}\cdot m} & -\frac{\sqrt{g\cdot M(M+m)}}{g\sqrt{\ell}\cdot m} & 0 & 0\end{bmatrix}\cdot x_D(t)$$

Corresponding block diagram is

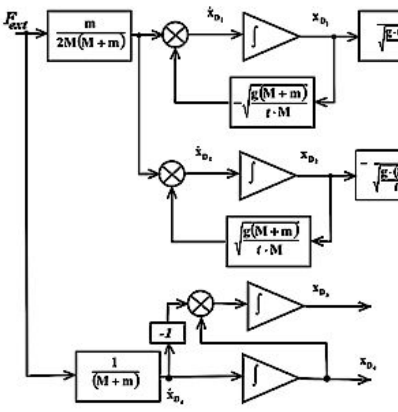


Fig.2 system diagram in Jordan's form

From diagram is see, why this system is controllable (all state quantities are influence able by input quantity $\mathbf{F_{ext}(t)}$).

And also why isn't observable. Output, ergo φ, depends only on first two state quantities, so other two state quantities aren't „ seen“ in output.

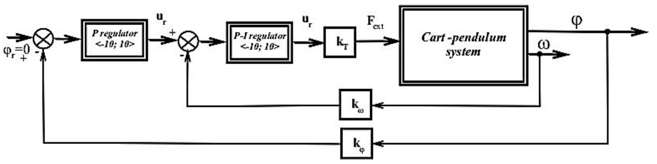


Fig.3 cascade controller diagram pendulum stabilization by force of cart movement

Fig. 3 shows diagram of hereof cart position control (mass M) and Fig. 4 shows its physical system multiport model including control law. In this model the compartment principle is used with released body in plane model with two rotational kinematic pairs.

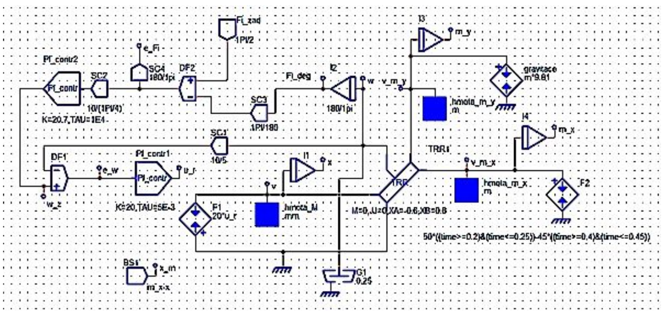


Fig.4 physical model of pendulum stabilization by means of cart movement set to the Dynast system [13].

III. Released Body Dynamics with Rotary Kinematic Pairs

In previous model setting, the model of released stiff body with two rotary kinematic pairs was used. Let present mentioned compartment description principle od Fig.5.

What is relation between point **A** absolute speed components and local homogenous coordinates $[\mathbf{x_A}, \mathbf{y_A}]$?

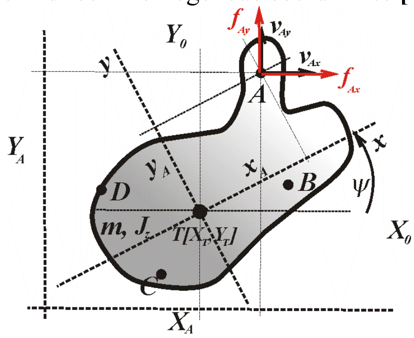


Fig.5 General released body in plane with four rotational kinematic pairs.

For its absolute velocity components (velocity in global reference coordinate system) pays:

$$\begin{bmatrix} v_{A_x}(t)\\ v_{A_y}(t)\\ 0\\ 1\end{bmatrix}=\begin{bmatrix}\frac{dX_T(t)}{dt}-\omega(t)\cdot[x_A\sin\psi(t)+y_A\cos\psi(t)]\\ \frac{dY_T(t)}{dt}+\omega(t)\cdot[x_A\cos\psi(t)-y_A\sin\psi(t)]\\ 0\\ 1\end{bmatrix} \qquad (5)$$

Point **A** absolute velocity is determinated by two components of across variables (velocity)

$\vec{v}_A=v_{A_x}\hat{\imath}+v_{A_y}\hat{\jmath}$ (where $\hat{\imath},\hat{\jmath}$ are global axes $\mathbf{X_0}$ and $\mathbf{Y_0}$ unitary vectors). Similarly for other points **B, C, D**.

For across variables the flow variables (generalized forces $\vec{F}_A=f_{A_x}\hat{\imath}+f_{A_y}\hat{\jmath}$) appertain (see Fig.5).

Fig.6 shows time dependencies of cart position, mathematical pendulum angle and its deviation from vertical position at response on outside perturbative force applied in axis **x** direction on mass **m**.

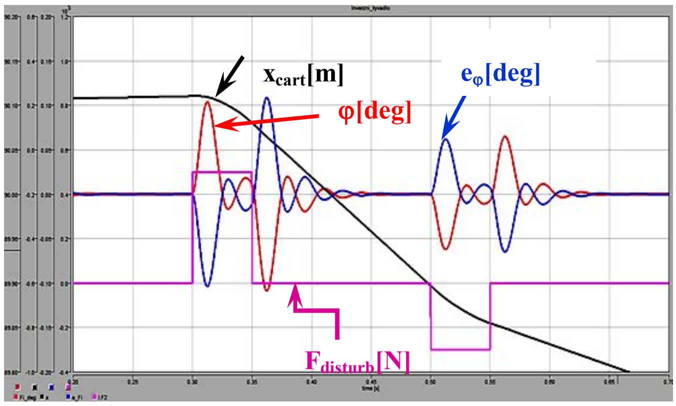


Fig.6 cart and pendulum behavior on outside perturbative force applied in direction **x** axis on mass **m**

IV. Mathematical Pendulum Stabilization by Proximal Coupling Vertical Movement

Interesting result will be further mentioned mathematical pendulum stabilization by proximal coupling vertical movement derived on chaos theory principle.

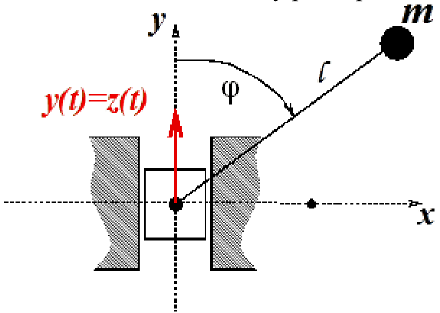


Fig.7 mathematical pendulum stabilization principle by proximal coupling vertical movement

System ordering is in Fig.7. Proximal joint of mathematical pendulum under consideration moves vertically harmoniously with certain amplitude and frequency.

Equation of motion describing existing motion is

$$\ddot{\varphi} = \frac{1}{\ell}\cdot\left[\ddot{z}(t)+g\right]\cdot\sin(\varphi) \tag{6}$$

For $z(t) = -A\cdot\cos(\omega\cdot t)$ we obtain the motion equation

$$\ddot{\varphi} - \frac{1}{\ell}\cdot\left[A\cdot\omega^2\cdot\cos(\omega\cdot t)+g\right]\cdot\sin(\varphi) = 0 \tag{7}$$

For $\varphi << 1$ and substitutions

$$\omega_0 = \sqrt{\frac{g}{\ell}}\;;\; T = \omega\cdot t\;;\; \alpha = \frac{\omega_0^2}{\omega^2};\; \beta = \frac{A}{\ell}$$

we obtain the equation

$$\frac{d^2\varphi}{dT^2} + \left[\alpha - \beta\cdot\cos(T)\right]\cdot\varphi = 0 \tag{8}$$

So, this is Mathieu type's second-order not-autonomous system with periodical matrix of system and by Floquet's theorem is possible to express the solution in the form

$$\varphi(T) = e^{i\cdot\nu\cdot T}\cdot P(T) \tag{9}$$

where the exponent $\boldsymbol{\nu}$ depends on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and **P(T)** is periodic function with period 2π. If $\boldsymbol{\nu}$ is real, but not a rational number, then (9) is not periodic. If $\boldsymbol{\nu}$ is a rational number $\nu = k/m$, then (9) is periodic with period at most 4π.m (but not 2π or 4π). If $\boldsymbol{\nu}$ is an integer number then (9) is periodic with period 2π or 4π).

From [3] and [4] is possible by $\boldsymbol{\nu}$ and **P(T)** evaluation to determinate that for

$$\frac{\sqrt{2}\cdot\omega_0}{\omega} < \beta < 0.45 \tag{10}$$

the system is stable.

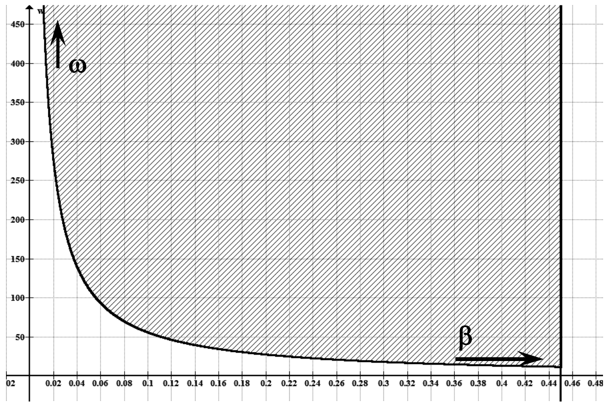


Fig.8 stability zone of inverted mathematical pendulum stabilized by proximal coupling vertical movement

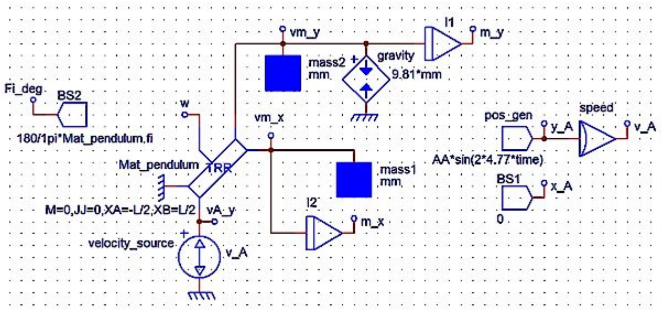


Fig.9 pendulum stabilization by means of proximal joint vertical movement physical model setting to the Dynast system

Fig.9 shows model setting to the physical simulation environment Dynast. Fig.10 shows mass at the end of immaterial arm behavior trajectory in 2D space at initial deviation about angle $\varphi(0) = -\pi/6;\; A = 0.42\text{m};$ Proximal joint oscillation frequency in y axis direction is two time pendulum oscillation frequency.

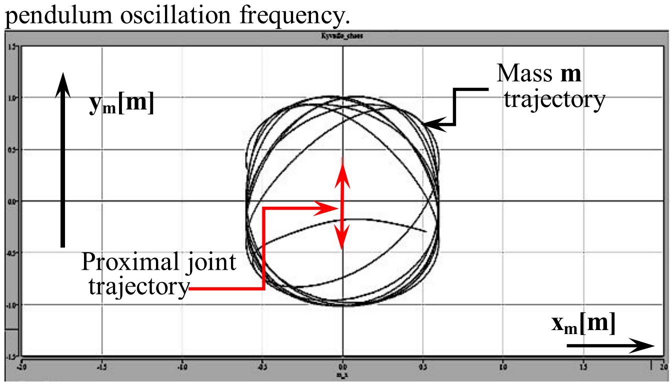


Fig.10 pendulum stabilization by means of proximal joint vertical movement physical model setting to the Dynast system

It's seen, that at sized oscillations amplitude A the pendulum begins be circulating around gripping axis.

At identical amplitude and specific frequency the arm oscillates around upper instable position according to Fig.11.

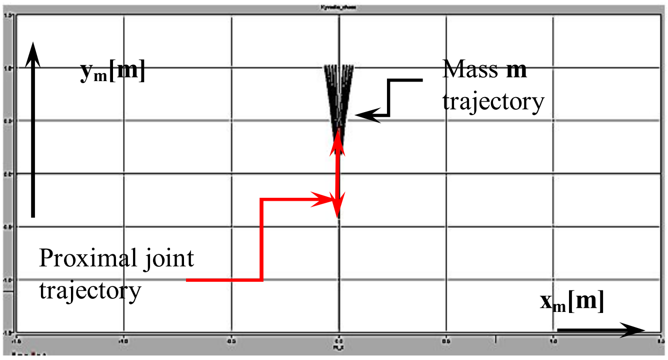


Fig.11 behavior trajectory of mass at the end of immaterial arm in 2D space at initial deviation about angle $\varphi(0) = \pi/2 - \pi/50$.

Detailed amplitude size and oscillation frequency reasoning and their computation already demonstrate Andrew Stephenson [1] in the year 1908.

V. Double Physical Pendulum Stabilization by Proximal Coupling Vertical Movement

We introduce next possibility of two physical pendulums stabilization bonded by rotary kinematic pair partly with each other, partly to vertically moving proximal joint of the first of them- Fig . 12.

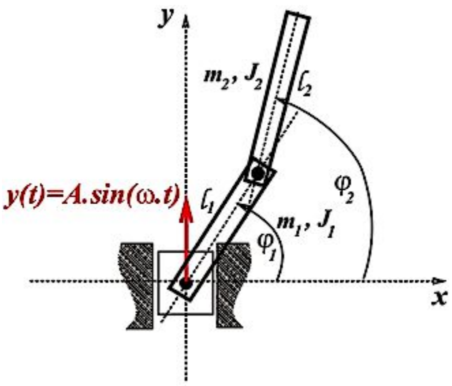


Fig.12 principle of double physical pendulums' stabilization by proximal coupling vertical movement

Already in the year 1738 Daniel Bernoulli shown, that telescopic pendulum (pendulum compound from **n** rigid articles) hung down is able to oscillate with any **n** natural frequencies, where with lowest frequency the pendulum articles swing more or less together, practically as if they form only one long pendulum and at highest frequency subsequently located pendulums oscillate in every instant in opposite directions.

Mathematical model of the system from Fig.12 is

$$\begin{gathered}\left(\frac{1}{3}\cdot m_1 + m_2\right)\cdot \ell_1^2 \cdot \ddot{\varphi}_1 + \frac{1}{2} m_2 \cdot \ell_1 \cdot \ell_2 \cdot \cos(\varphi_2 - \varphi_1)\cdot \ddot{\varphi}_2 - \frac{1}{2} m_2 \cdot \ell_1 \cdot \ell_2 \cdot \dot{\varphi}_2^2 \cdot \sin(\varphi_2 - \varphi_1) + \\ + \left(\frac{1}{2}\cdot m_1 + m_2\right)\cdot \ell_1 \cdot (\ddot{y} + g)\cdot \cos\varphi_1 = 0 \\ \frac{1}{2} m_2 \cdot \ell_1 \cdot \ell_2 \cdot \cos(\varphi_2 - \varphi_1)\cdot \ddot{\varphi}_1 + m_2 \cdot \frac{\ell_2^2}{3}\cdot \ddot{\varphi}_2 + \frac{1}{2} m_2 \cdot \ell_1 \cdot \ell_2 \cdot \sin(\varphi_2 - \varphi_1)\cdot \dot{\varphi}_1^2 + \\ + \frac{1}{2} m_2 \cdot \ell_2 \cdot \cos\varphi_2 \cdot (\ddot{y} + g) = 0\end{gathered} \tag{11}$$

Natural frequencies of model linearization in down stable position neighborhood are

$$\omega_{min,max} = \sqrt{3\cdot g \cdot \frac{\left\{\begin{aligned}&[(2\cdot m_2 + m_1)\cdot \ell_2 + (3\cdot m_2 + m_1)\cdot \ell_1] \mp \\ &\mp \sqrt{\ell_2^2 (2\cdot m_2 + m_1)^2 + \ell_1 \cdot \ell_2 (6\cdot m_2^2 - m_1 \cdot m_2 - 2\cdot m_1^2) + \ell_1^2 (3\cdot m_2 + m_1)^2}\end{aligned}\right\}}{\ell_1 \cdot \ell_2 (3\cdot m_2 + 4\cdot m_1)}} \tag{12}$$

For $\ell_1 = \ell_2 = \ell;\ m_1 = 1.08\text{kg},\ m_2 = 2\text{kg}$ we can obtain from [3]

$$\begin{gathered} A < 0.707\cdot \ell \\ \frac{5.726\cdot \sqrt{g\cdot \ell}}{A} < \omega \end{gathered} \tag{13}$$

On Fig.13 is setting of ordering from Fig. 12 to the Dynast simulation space.

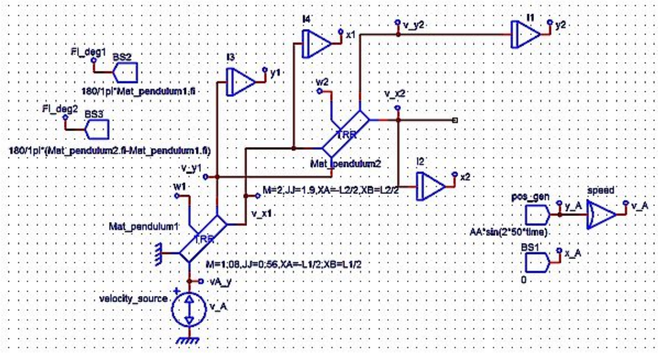


Fig.13 double physical pendulum stabilization by means of proximal joint vertical movement setting as physical model to the Dynast system

Fig.14 shows that both physical pendulums turn, at sufficient proximal join movement frequency, around 1st and 2nd joint against each other.

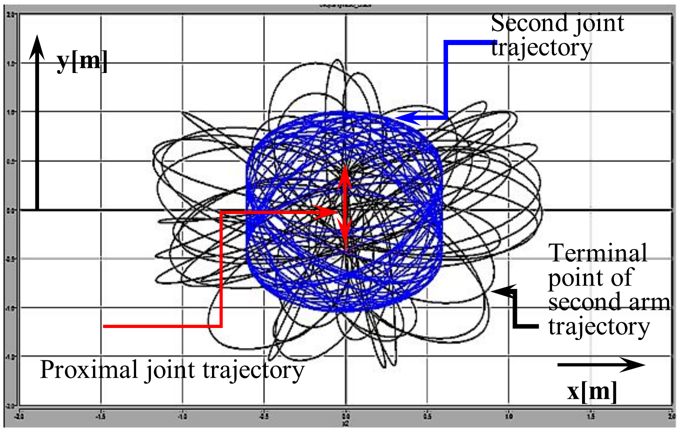


Fig.14 double physical pendulum behavior trajectory in 2D space at proximal joint high oscillation frequency

At well select vertical oscillations' amplitude and frequency of the first joint it is possible to stabilize the both arms in inverted (upper) position -Fig. 15.

On Fig. 16 is practical experiment demonstration taken over from [8].

In fine we state, that mentioned problem is solvable also for continuously distributed mass. Mentioned problem is described in [6] and Fig.15 taken over [6] shows achieved experimental results.

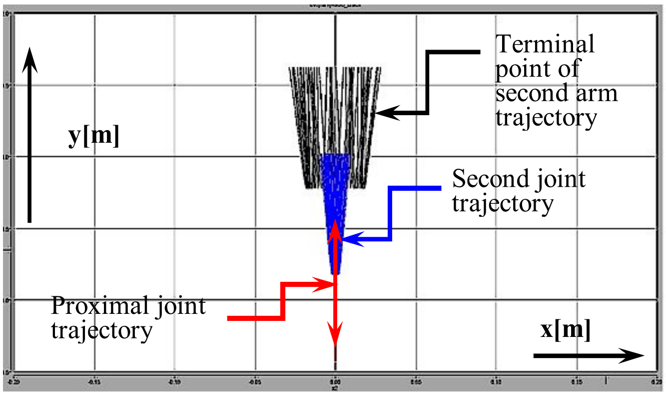


Fig.15 double physical pendulum behavior trajectory in 2D space at oscillation frequency and amplitude of proximal joint ensuring stabilization of both arms

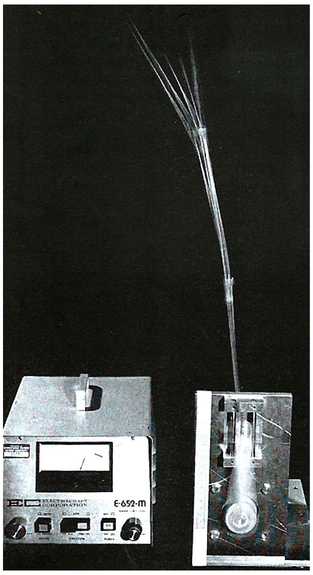
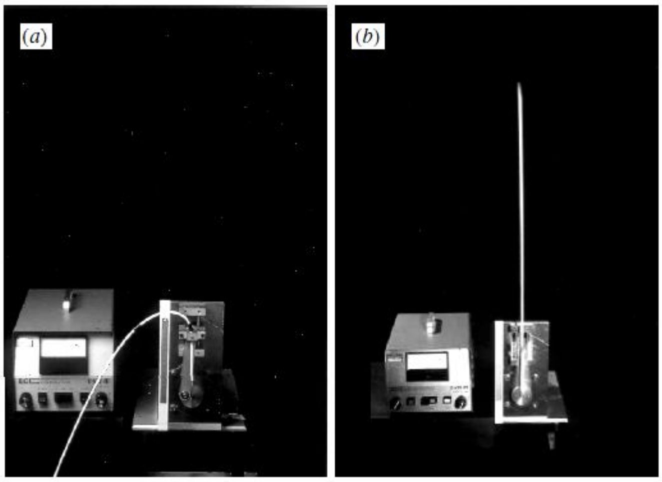


Fig.16 triple physical pendulum practical experiment at oscillation frequency and amplitude of proximal joint ensuring stabilization of arms



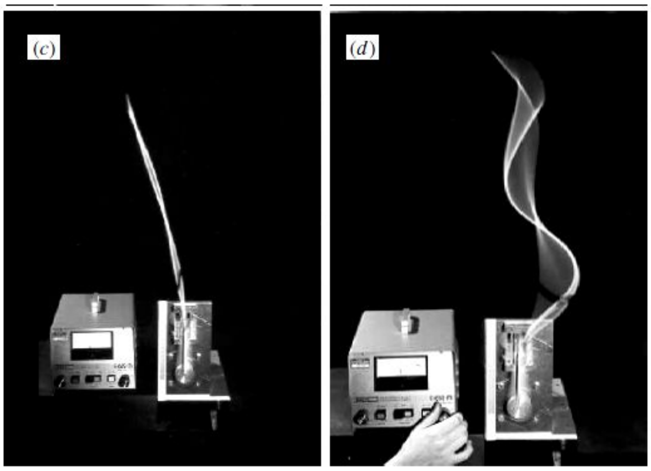


Fig.17 the experimental demonstration: (a) the sagging buckled wire; (b) the stabilized vertical wire; (c) leaning motion near the lower-frequency `falling-over' instability; and (d) motion near the higher-frequency dynamic instability.

Kindly note the [6] title: Indian trick with rope. Perhaps is concerned one of the most famous tricks in all of magic history: Rope ejected to the air doesn't fall down, but it will stay „stand still" in vertical position. The reference about this „trick" reaches until fourteenth century. And it is cross-eyed. Is it really cross - eyed?

VI. Conclusion

Stated contribution shows mathematical and double physical inverse pendulum stabilization possibilities. In the first parts is described and simulated as physical model generally known inverse mathematical pendulum case stabilization by force of proximal joint horizontal movement, which is documented by cascade control of pendulum angular speed and position by means of cart with pendulum proximal joint.

In the second parts is stated mathematical, physical and double physical pendulum stabilization by vertical oscillations of proximal joint. Example is simulated by means of physical model in Dynast system and so forms suitable starting point to the next detailed analysis on chaos theory principle and its verification by simulation experiments.

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