Young's Equation for Non-planar Surfaces

Jeng-Rung Jiang, *NTU,* Falin Chen, *NTU,* and Min-Hsing Chang, *TTU*

***Abstract*—In 1805, Young described the so-called Young's equation for a liquid drop resting on a planar surface, which since has been successfully applied for different designs and extended to cover various physical situations. In this paper, we used the method developed by Lubarda to derive a Young's equation for a liquid drop resting on an axisymmetric concave surface, which is considered to account for an extended form of non-planar surfaces. As a result, a new form of Young's equation is derived, which can be reduced into the Young's equation for either a planar boundary or a vertical tube and accordingly can be considered as an extended form of Young's equation for surfaces of various forms.**

***Index Terms*—Young-Laplace equation, contact angle, Young's equation, Boruvka-Neumann equation.**

I. Introduction

THE concept of surface tension was introduced by Thomas Young and almost simultaneously by Pierre-Simon Laplace in 1805. In the work of Laplace [1], he derived an equation to relate the pressure difference between interior and exterior of the liquid drop with the surface tension. In Laplace's essay, he also argued that there shall be an appropriate contact angle between liquid and solid surface to reflect the fact that the drop stays stably on the surface. Later, Young proceeded to discuss Laplace's formulation and clarified the result without any formula [2]. Since then, the equation in terms of contact angles has been called Young's equation and has been employed to investigate resting drops in various situations for more than 150 years. Later, Gibbs [3] introduced the line tension along the triple contact line and then by others [4], [5], [6] obtaining the Young's equation in a more general form, which can predict the contact angle more precisely. Afterwards, Young's equation has been applied successfully in various problems such as coating [7] and electowetting [8], [9], [10]. In the works stated above, the liquid drop was considered to rest on a planar solid substrate. Recently, a liquid drop resting on an inclined plane [11], [12] have also been considered.

In this paper, we consider that the liquid drop is placed on a non-planar surface, which is accounted for by an axisymmetric concave surface, as shown in Fig. 1. For such a configuration, we follow the method developed by Lubarda [13] to recover Young's law and then the new equations. The results show that, firstly, not only the liquid-vapor interface but also the solid-liquid interface satisfies Young-Laplace equation when adhesion is neglected, and secondly, a new equation is derived to describe the relation between the tensions and the contact angles if the adhesive force at the contact line is considered. Also shown by the new equation, as obtained by Lubarda [13], is that the contact angle is independent of gravity as the line tension is ignored. This equation can also be reduced into those applying for the cases involving a planar surface and a vertical tube. The details of the derivation are shown in the following.

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II. Framework

We consider that a uniform liquid rests on an axisymmetric concave surface and is exposed to a uniform vapor atmosphere above. As shown in Fig. 1, the surface is symmetric with respect to the z-axis, the cylindrical coordinate system (ρ, θ, z) is employed. $z = B(\rho)$ accounts for the solid-liquid interface and $z = H(\rho)$ for the liquid-vapor interface. For convenience, we denote the liquid by sub-index l, the vapor atmosphere by v, and the solid surface by s. Without loss of generality, we suppose $H(\rho)$ is a convex surface. For a concave $H(\rho)$, the present derivation can be done by a similar way. The liquid is partitioned into two regions by a horizontal surface, I for the upper part and II for the lower part.

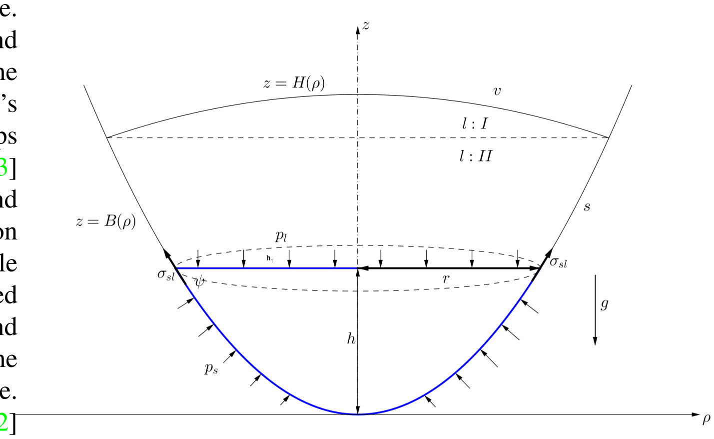

Fig. 1. The free-body diagram of the force balance of the liquid resting on a concave surface. The indicated lengths r and h satisfy $h = B(r)$, ψ is the angle between the concave surface and the horizontal surface at the height h.

III. Young-Laplace equation for the solid-liquid interface

We first consider the force balance for the liquid below a given height h accounted for by $h = B(r)$, where r is the radius of the circular surface on the top of the liquid. As a result, the force balance in z-direction is obtained to be

$$\int_0^r (p_s(B(\rho)) - p_l(h))\, 2\pi\rho d\rho + 2\pi r \sigma_{sl} \sin\psi - \gamma_l \int_0^r (h - B(\rho))\, 2\pi\rho d\rho = 0, \quad (1)$$

where ψ is the angle between the concave surface and the horizontal surface at the height h. In (1), the first term accounts for the net force acted along the concave surface, which is resulted from the difference between the pressure of liquid p_l and the pressure of solid p_s. The second term is the force due to the surface tension σ_{sl} along the circumference of the circular surface on the top. The last term is the gravitational force in which γ_l is the specific weights of the liquid. Here no adhesive force is considered.

Since the liquid rests steadily on the surface, the hydrostatic pressure within the liquid is given as

$$p_l(h) + \gamma_l\,(h - B(\rho)) = p_l(B(\rho)). \tag{2}$$

By using (2), (1) can be reduced into

$$\int_0^r \left(p_s(B(\rho)) - p_l(B(\rho))\right) 2\pi\rho d\rho + 2\pi r \sigma_{sl}\sin\psi = 0. \tag{3}$$

We then take the derivative d/dr on both sides of (3) and yield

$$r\left(p_s(h) - p_l(h)\right) + \sigma_{sl}\left(\sin\psi + r\cos\psi\frac{d\psi}{dr}\right) = 0. \tag{4}$$

By denoting $r' = dr/dh$, the slope of $B(\rho)$ can be expressed by $\tan\psi = dh/dr = 1/r'$. By considering trigonometric identities $\cos\psi = r'/\sqrt{1+r'^2}$ and $\sin\psi = 1/\sqrt{1+r'^2}$ and the differentiation relation $d\psi/dr = -r''/\left(1+r'^2\right)r'$, (4) can be rewritten into

$$p_l(h) - p_s(h) = \sigma_{sl}\left(\frac{1}{r\sqrt{1+r'^2}} - \frac{r''}{\left(1+r'^2\right)^{3/2}}\right). \tag{5}$$

In (5), the two terms in the bracket are exactly the two principal curvatures of the surface $z = B(\rho)$ at the level h. Accordingly, (5) can be expressed as

$$p_l - p_s = \Delta p = 2\sigma_{sl}\kappa_{sl}, \tag{6}$$

where κ_{sl} is the mean curvature of the solid-liquid interface. Equation (6) is in a similar form to the Young-Laplace equation, which applies well on the non-planar surface.

Equation (6) shows that, for a non-planar surface, there is a pressure difference between liquid and solid. More precisely, the pressure difference at the solid-liquid interface is proportional to the mean curvature of the non-planar surface. If the surface is planar, the radius of curvature is infinite so that the right hand side of (6) vanishes, implying that the pressures of liquid and solid are exactly the same. If the surface is a sphere of radius R, that is, $r = \sqrt{R^2 - (h-R)^2} = \sqrt{-h^2 + 2Rh}$, the pressure difference on the solid-liquid interface is a constant no matter what height is considered, namely,

$$p_l(h) - p_s(h) = \sigma_{sl}\cdot\frac{2}{R}. \tag{7}$$

IV. Derivation of the Contact Angle

Now we consider the force balance of the liquid below the liquid-vapor interface, and the free-body diagram of the forces acting on the left half of liquid drop is shown in Fig. 2. Because of the presence of the liquid-vapor interface, the pressure from vapor atmosphere and the line tension τ along the circumference of the liquid-vapor interface shall be considered, and the contact angle between the liquid and the surface shall be involved. The mathematical method of Lubarda [13] is employed again. Since the liquid is considered resting on the concave surface, the net force acting on the liquid in the horizontal direction shall be zero. Because the solid-liquid interface may not be horizontal, the adhesive force between liquid and solid shall be concerned. By using f to account for the adhesive force per unit length at the contact line, we obtain the force balance equation of the liquid of Fig. 2 as follows

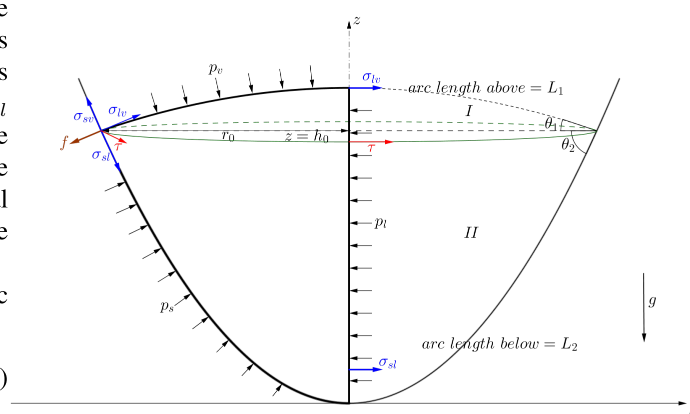

Fig. 2. The free-body diagram of the half portion of liquid in the container with line tension τ and adhesive force f. h_0 is the height of the contact line. L_1 and L_2 are the arc lengths of the upper part and the lower part of the middle cross-section respectively.

$$\begin{aligned}\sigma_{sl}L_2 + \sigma_{lv}L_1 - \sigma_{sv}\cos\theta_2\cdot 2r_0 + 2\tau - f\cdot 2r_0\sin\theta_2\\ + \int_I \left(p_v - p_l\right)dA + \int_{II}\left(p_s - p_l\right)dA = 0,\end{aligned} \tag{8}$$

where σ_{lv} and σ_{sv} are the surface tensions of the liquid-vapor and the solid-vapor interface, respectively. The lengths L_1, L_2, r_0, h_0 and the angles θ_1, θ_2 are shown in Fig. 2.

It is known that, again, on the liquid-vapor interface the force balance in the horizontal direction shall satisfy the Young-Laplace equation

$$p_l - p_v = \Delta p = 2\sigma_{lv}\kappa_{lv}, \tag{9}$$

where κ_{lv} is the mean curvature of the liquid-vapor interface. So that (8) can be rewritten as

$$\begin{aligned}\sigma_{sl}L_2 + \sigma_{lv}L_1 - 2r_0\sigma_{sv}\cos\theta_2 + 2\tau - 2r_0 f\sin\theta_2\\ - \int_{h_0}^{H(0)} 4\sigma_{lv}\kappa_{lv}r dh - \int_0^{h_0} 4\sigma_{sl}\kappa_{sl}r dh = 0,\end{aligned} \tag{10}$$

where (6) is considered and $dA = 2rdh$ is used. Moreover, in the integration $h = B(r)$ when $h \le h_0$ and $h = H(r)$ when $h \ge h_0$. Because, as shown by Lubarda [13], that

$$\int_{h_0}^{H(0)} 2\kappa_{lv}r dh = \frac{L_1}{2} - r_0\cos\theta_1. \tag{11}$$

we similarly can obtain

$$\int_0^{h_0} 2\kappa_{sl}r dh = \frac{L_2}{2} - r_0\cos\theta_2. \tag{12}$$

Consequently, by substituting (11) and (12) into (10), we obtain the Young's equation for a non-planar concave surface as

$$\sigma_{lv}\cos\theta_1+\sigma_{sl}\cos\theta_2-\sigma_{sv}\cos\theta_2+\frac{\tau}{r_0}-f\sin\theta_2=0. \quad (13)$$

Please note that, although the liquid-vapor interface in Fig. 2 is concave, the resultant equation (13) shall be the same if the interface is convex while some terms will turn to be negative in the process of the calculation. Because $r'=dr/dh$ is always well-defined in a local region on the interface, the analysis above is true no matter what kind of the container is, such as the surface of a washbasin having $\theta_2<\pi/2$ or of a jar having $\theta_2>\pi/2$. On the other hand, as the liquid rests on a planar surface, $\theta_2=0$, leading to the original Young's equation

$$\sigma_{lv}\cos\theta_1+\sigma_{sl}-\sigma_{sv}+\frac{\tau}{r_0}=0. \quad (14)$$

This contact angle relation can also be referred as Boruvka-Neumann equation [4], [5], [6], implying that (13) is an extension of Young's equation.

Now we neglect the line tension along the circumference of the liquid-vapor interface. Namely, only surface tension, adhesion and pressure are concerned. Consequently, (13) becomes

$$\sigma_{lv}\cos\theta_1+\sigma_{sl}\cos\theta_2-\sigma_{sv}\cos\theta_2-f\sin\theta_2=0. \quad (15)$$

which can be reduced into the original Young's equation when $\theta_2=0$.

V. For the vertical case

When $\theta_2=\pi/2$, the surface becomes a vertical tube, (13) becomes

$$\sigma_{lv}\cos\theta_1+\frac{\tau}{r_0}-f=0. \quad (16)$$

Note (16) is the force balance in horizontal direction. The force balance in vertical direction has been given by Gennes et al. [14] as

$$\sigma_{sv}+\sigma_{lv}\sin\theta_1-\sigma_{sl}=0. \quad (17)$$

After combining (16) and (17), we can derive

$$\left|f-\frac{\tau}{r_0}\right|=\sqrt{\sigma_{lv}^2-(\sigma_{sl}-\sigma_{sv})^2}. \quad (18)$$

In (18), the quantity in the square root must be non-negative, so the three surface tensions satisfy the inequality

$$\sigma_{lv}\geq|\sigma_{sl}-\sigma_{sv}|. \quad (19)$$

Equation (19) means at the point intersected by vapor, liquid and solid regions [15], the surface tension on the liquid-vapor surface is always larger than or equal to the difference between the surface tensions of the solid-liquid and the solid-vapor interfaces.

VI. Conclusion

In summary, in this paper, by considering the force balance of a liquid drop resting on a concave surface, we are able to derive three equations which can be considered as an extension of Young's equation to depict a new relation between contact angles and surface tensions. The first is (6) showing that the pressure difference at the solid-liquid interface is proportional to the mean curvature of the non-planar surface. The second is (13) showing the relation between the contact angle and surface tension on the free surface of a liquid resting on a non-planar surface, which can be seen as an extension of Young's equation. The third is (19) showing that, at a material element around a triple contact line, the surface tension on the liquid-vapor surface is always larger than or equal to the difference between the surface tensions of the other two interfaces.

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