Simplified Variational Principles for Stationary non-Barotropic Magnetohydrodynamics

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***Abstract*—Variational principles for magnetohydrodynamics were introduced by previous authors both in Lagrangian and Eulerian form. In this paper we introduce simpler Eulerian variational principles from which all the relevant equations of non-barotropic stationary magnetohydrodynamics can be derived for certain field topologies. The variational principle is given in terms of eight independent functions for stationary barotropic flows. This is the same as the eight variables which appear in the standard equations of non-barotropic magnetohydrodynamics which are the magnetic field $\vec{B}$ the velocity field $\vec{v}$, the entropy s and the density ρ.**

***Index Terms*—Magnetohydrodynamics, Variational principles**

I. INTRODUCTION

Variational principles for magnetohydrodynamics were introduced by previous authors both in Lagrangian and Eulerian form. Sturrock [1] has discussed in his book a Lagrangian variational formalism for magnetohydrodynamics. Vladimirov and Moffatt [2] in a series of papers have discussed an Eulerian variational principle for incompressible magnetohydrodynamics. However, their variational principle contained three more functions in addition to the seven variables which appear in the standard equations of incompressible magnetohydrodynamics which are the magnetic field $\vec{B}$ the velocity field $\vec{v}$ and the pressure P. Kats [3] has generalized Moffatt's work for compressible non barotropic flows but without reducing the number of functions and the computational load. Moreover, Kats has shown that the variables he suggested can be utilized to describe the motion of arbitrary discontinuity surfaces [4], [5]. Sakurai [6] has introduced a two function Eulerian variational principle for force-free magnetohydrodynamics and used it as a basis of a numerical scheme, his method is discussed in a book by Sturrock [1]. A method of solving the equations for those two variables was introduced by Yang, Sturrock & Antiochos [8]. Yahalom & Lynden-Bell [9] combined the Lagrangian of Sturrock [1] with the Lagrangian of Sakurai [6] to obtain an **Eulerian** Lagrangian principle for barotropic magnetohydrodynamics which will depend on only six functions. The variational derivative of this Lagrangian produced all the equations needed to describe barotropic magnetohydrodynamics without any additional constraints. The equations obtained resembled the equations of Frenkel, Levich & Stilman [12] (see also [13]). Yahalom [10] have shown that for the barotropic case four functions will suffice. Moreover, it was shown that the cuts of some of those functions [11] are topological local conserved quantities.

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Previous work was concerned only with barotropic magnetohydrodynamics. Variational principles of non barotropic magnetohydrodynamics can be found in the work of Bekenstein & Oron [14] in terms of 15 functions and V.A. Kats [3] in terms of 20 functions. The author of this paper suspect that this number can be somewhat reduced. Moreover, A. V. Kats in a remarkable paper [15] (section IV,E) has shown that there is a large symmetry group (gauge freedom) associated with the choice of those functions, this implies that the number of degrees of freedom can be reduced. Yahalom [16] have shown that only five functions will suffice to describe non barotropic magnetohydrodynamics in the case that we enforce a Sakurai [6] representation for the magnetic field. Morrison [7] has suggested a Hamiltonian approach but this also depends on 8 canonical variables (see table 2 [7]). The work of Yahalom [16] was concerned with general non-stationary flows. Here we shall concentrate on the particular but important stationary flow case and study how the assumptions of stationarity effect the variational formalism.

We anticipate applications of this study both to linear and non-linear stability analysis of known non barotropic magnetohydrodynamic configurations [22], [24] and for designing efficient numerical schemes for integrating the equations of fluid dynamics and magnetohydrodynamics [30], [31], [32], [33]. Another possible application is connected to obtaining new analytic solutions in terms of the variational variables [34].

The plan of this paper is as follows: First we introduce the standard notations and equations of non-barotropic magnetohydrodynamics for the stationary and non-stationary cases. Next we introduce a generalization of the barotropic variational principle suitable for the non-barotropic case. Later we simplify the Eulerian variational principle and formulate it in terms of eight functions. We conclude by writing down the appropriate variational principle for the stationary case.

II. STANDARD FORMULATION OF NON-BAROTROPIC MAGNETOHYDRODYNAMICS

The standard set of equations solved for non-barotropic magnetohydrodynamics are given below:

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B}), \tag{1}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{2}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \tag{3}$$

$$\rho \frac{d\vec{v}}{dt} = \rho (\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v}) = -\vec{\nabla} p(\rho, s) + \frac{(\vec{\nabla} \times \vec{B}) \times \vec{B}}{4\pi}. \tag{4}$$

$$\frac{ds}{dt} = 0. \quad (5)$$

The following notations are utilized: $\frac{\partial}{\partial t}$ is the temporal derivative, $\frac{d}{dt}$ is the temporal material derivative and $\vec{\nabla}$ has its standard meaning in vector calculus. $\vec{B}$ is the magnetic field vector, $\vec{v}$ is the velocity field vector, ρ is the fluid density and s is the specific entropy. Finally $p(\rho, s)$ is the pressure which depends on the density and entropy (the non-barotropic case).

The justification for those equations and the conditions under which they apply can be found in standard books on magnetohydrodynamics (see for example [1]). The above applies to a collision-dominated plasma in local thermodynamic equilibrium. Such conditions are seldom satisfied by physical plasmas, certainly not in astrophysics or in fusion-relevant magnetic confinement experiments. Never the less it is believed that the fastest macroscopic instabilities in those systems obey the above equations [11], while instabilities associated with viscous or finite conductivity terms are slower. It should be noted that due to a theorem by Bateman [35] every physical system can be described by a variational principle (including viscous plasma) the trick is to find an elegant variational principle usually depending on a small amount of variational variables. The current work will discuss only ideal magnetohydrodynamics while viscous magnetohydrodynamics will be left for future endeavors.

Equation (1) describes the fact that the magnetic field lines are moving with the fluid elements ("frozen" magnetic field lines), equation (2) describes the fact that the magnetic field is solenoidal, equation (3) describes the conservation of mass and equation (4) is the Euler equation for a fluid in which both pressure and Lorentz magnetic forces apply. The term:

$$\vec{J} = \frac{\vec{\nabla} \times \vec{B}}{4\pi}, \quad (6)$$

is the electric current density which is not connected to any mass flow. Equation (5) describes the fact that heat is not created (zero viscosity, zero resistivity) in ideal non-barotropic magnetohydrodynamics and is not conducted, thus only convection occurs. The number of independent variables for which one needs to solve is eight ($\vec{v}, \vec{B}, \rho, s$) and the number of equations (1,3,4,5) is also eight. Notice that equation (2) is a condition on the initial $\vec{B}$ field and is satisfied automatically for any other time due to equation (1). For the stationary case in which the physical fields do not depend on time we obtain the following set of stationary equations:

$$\vec{\nabla} \times (\vec{v} \times \vec{B}) = 0, \quad (7)$$

$$\vec{\nabla} \cdot \vec{B} = 0, \quad (8)$$

$$\vec{\nabla} \cdot (\rho \vec{v}) = 0, \quad (9)$$

$$\rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} p(\rho, s) + \frac{(\vec{\nabla} \times \vec{B}) \times \vec{B}}{4\pi}. \quad (10)$$

$$\vec{v} \cdot \vec{\nabla} s = 0. \quad (11)$$

III. Variational principle of non-barotropic magnetohydrodynamics

In the following section we will generalize the approach of [9] for the non-barotropic case. Consider the action:

$$\begin{aligned}
A &\equiv \int \mathcal{L} d^3x dt, \\
\mathcal{L} &\equiv \mathcal{L}_1 + \mathcal{L}_2, \\
\mathcal{L}_1 &\equiv \rho(\frac{1}{2}\vec{v}^2 - \varepsilon(\rho, s)) + \frac{\vec{B}^2}{8\pi}, \\
\mathcal{L}_2 &\equiv \nu[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v})] - \rho\alpha \frac{d\chi}{dt} - \rho\beta \frac{d\eta}{dt} - \rho\sigma \frac{ds}{dt} \\
&- \frac{\vec{B}}{4\pi} \cdot \vec{\nabla}\chi \times \vec{\nabla}\eta. \quad (12)
\end{aligned}$$

In the above ε is the specific internal energy (internal energy per unit of mass). The reader is reminded of the following thermodynamic relations which will become useful later:

$$\begin{aligned}
d\varepsilon &= T ds - P d\frac{1}{\rho} = T ds + \frac{P}{\rho^2} d\rho \\
&\frac{\partial \varepsilon}{\partial s} = T, \qquad \frac{\partial \varepsilon}{\partial \rho} = \frac{P}{\rho^2} \\
w &= \varepsilon + \frac{P}{\rho} = \varepsilon + \frac{\partial \varepsilon}{\partial \rho}\rho = \frac{\partial(\rho\varepsilon)}{\partial \rho} \\
dw &= d\varepsilon + d(\frac{P}{\rho}) = T ds + \frac{1}{\rho} dP \quad (13)
\end{aligned}$$

in the above T is the temperature and w is the specific enthalpy. Obviously $\nu, \alpha, \beta, \sigma$ are Lagrange multipliers which were inserted in such a way that the variational principle will yield the following equations:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \\
&\rho \frac{d\chi}{dt} = 0, \\
&\rho \frac{d\eta}{dt} = 0. \\
&\rho \frac{ds}{dt} = 0. \quad (14)
\end{aligned}$$

It **is not** assumed that $\nu, \alpha, \beta, \sigma$ are single valued. Provided ρ is not null those are just the continuity equation (3), entropy conservation and the conditions that Sakurai's functions are comoving. Taking the variational derivative with respect to $\vec{B}$ we see that

$$\vec{B} = \hat{\vec{B}} \equiv \vec{\nabla}\chi \times \vec{\nabla}\eta. \quad (15)$$

Hence $\vec{B}$ is in Sakurai's form and satisfies equation (2). It can be easily shown that provided that $\vec{B}$ is in the form given in equation (15), and equations (14) are satisfied, then also equation (1) is satisfied.

For the time being we have showed that all the equations of non-barotropic magnetohydrodynamics can be obtained from the above variational principle except Euler's equations. We will now show that Euler's equations can be derived from the above variational principle as well. Let us take an arbitrary variational derivative of the above action with respect to $\vec{v}$, this will result in:

$$\delta_{\vec{v}} A = \int dt \{ \int d^3x dt \rho \delta \vec{v} \cdot [\vec{v} - \vec{\nabla} \nu - \alpha \vec{\nabla} \chi - \beta \vec{\nabla} \eta - \sigma \vec{\nabla} s] + \oint d\vec{S} \cdot \delta \vec{v} \rho \nu + \int d\vec{\Sigma} \cdot \delta \vec{v} \rho [\nu] \}. \quad (16)$$

The integral $\oint d\vec{S} \cdot \delta \vec{v} \rho \nu$ vanishes in many physical scenarios. In the case of astrophysical flows this integral will vanish since $\rho = 0$ on the flow boundary, in the case of a fluid contained in a vessel no flux boundary conditions $\delta \vec{v} \cdot \hat{n} = 0$ are induced ($\hat{n}$ is a unit vector normal to the boundary). The surface integral $\int d\vec{\Sigma}$ on the cut of ν vanishes in the case that ν is single valued and $[\nu] = 0$ as is the case for some flow topologies. In the case that ν is not single valued only a Kutta type velocity perturbation [32] in which the velocity perturbation is parallel to the cut will cause the cut integral to vanish. An arbitrary velocity perturbation on the cut will indicate that $\rho = 0$ on this surface which is contradictory to the fact that a cut surface is to some degree arbitrary as is the case for the zero line of an azimuthal angle. We will show later that the "cut" surface is co-moving with the flow hence it may become quite complicated. This uneasy situation may be somewhat be less restrictive when the flow has some symmetry properties.

Provided that the surface integrals do vanish and that $\delta_{\vec{v}} A = 0$ for an arbitrary velocity perturbation we see that $\vec{v}$ must have the following form:

$$\vec{v} = \hat{\vec{v}} \equiv \vec{\nabla} \nu + \alpha \vec{\nabla} \chi + \beta \vec{\nabla} \eta + \sigma \vec{\nabla} s. \quad (17)$$

The above equation is reminiscent of Clebsch representation in non magnetic fluids [36], [37]. Let us now take the variational derivative with respect to the density ρ we obtain:

$$\begin{aligned} \delta_\rho A &= \int d^3x dt \delta \rho [\frac{1}{2} \vec{v}^2 - w - \frac{\partial \nu}{\partial t} - \vec{v} \cdot \vec{\nabla} \nu] \\ &+ \int dt \oint d\vec{S} \cdot \vec{v} \delta \rho \nu + \int dt \int d\vec{\Sigma} \cdot \vec{v} \delta \rho [\nu] \\ &+ \int d^3x \nu \delta \rho |_{t_0}^{t_1}. \end{aligned} \quad (18)$$

In which $w = \frac{\partial(\varepsilon \rho)}{\partial \rho}$ is the specific enthalpy. Hence provided that $\oint d\vec{S} \cdot \vec{v} \delta \rho \nu$ vanishes on the boundary of the domain and $\int d\vec{\Sigma} \cdot \vec{v} \delta \rho [\nu]$ vanishes on the cut of ν in the case that ν is not single valued[1] and in initial and final times the following equation must be satisfied:

$$\frac{d\nu}{dt} = \frac{1}{2} \vec{v}^2 - w, \quad (19)$$

Since the right hand side of the above equation is single valued as it is made of physical quantities, we conclude that:

$$\frac{d[\nu]}{dt} = 0. \quad (20)$$

Hence the cut value is co-moving with the flow and thus the cut surface may become arbitrary complicated. This uneasy situation may be somewhat be less restrictive when the flow has some symmetry properties.

Finally we have to calculate the variation with respect to both χ and η this will lead us to the following results:

$$\begin{aligned} \delta_\chi A &= \int d^3x dt \delta \chi [\frac{\partial(\rho \alpha)}{\partial t} + \vec{\nabla} \cdot (\rho \alpha \vec{v}) - \vec{\nabla} \eta \cdot \vec{J}] \\ &+ \int dt \oint d\vec{S} \cdot [\frac{\vec{B}}{4\pi} \times \vec{\nabla} \eta - \vec{v} \rho \alpha] \delta \chi \\ &+ \int dt \int d\vec{\Sigma} \cdot [\frac{\vec{B}}{4\pi} \times \vec{\nabla} \eta - \vec{v} \rho \alpha] [\delta \chi] \\ &- \int d^3x \rho \alpha \delta \chi |_{t_0}^{t_1}, \end{aligned} \quad (21)$$

$$\begin{aligned} \delta_\eta A &= \int d^3x dt \delta \eta [\frac{\partial(\rho \beta)}{\partial t} + \vec{\nabla} \cdot (\rho \beta \vec{v}) + \vec{\nabla} \chi \cdot \vec{J}] \\ &+ \int dt \oint d\vec{S} \cdot [\vec{\nabla} \chi \times \frac{\vec{B}}{4\pi} - \vec{v} \rho \beta] \delta \eta \\ &+ \int dt \int d\vec{\Sigma} \cdot [\vec{\nabla} \chi \times \frac{\vec{B}}{4\pi} - \vec{v} \rho \beta] [\delta \eta] \\ &- \int d^3x \rho \beta \delta \eta |_{t_0}^{t_1}. \end{aligned} \quad (22)$$

Provided that the correct temporal and boundary conditions are met with respect to the variations $\delta \chi$ and $\delta \eta$ on the domain boundary and on the cuts in the case that some (or all) of the relevant functions are non single valued. we obtain the following set of equations:

$$\frac{d\alpha}{dt} = \frac{\vec{\nabla} \eta \cdot \vec{J}}{\rho}, \qquad \frac{d\beta}{dt} = -\frac{\vec{\nabla} \chi \cdot \vec{J}}{\rho}, \quad (23)$$

in which the continuity equation (3) was taken into account. By correct temporal conditions we mean that both $\delta \eta$ and $\delta \chi$ vanish at initial and final times. As for boundary conditions which are sufficient to make the boundary term vanish on can consider the case that the boundary is at infinity and both $\vec{B}$ and ρ vanish. Another possibility is that the boundary is impermeable and perfectly conducting. A sufficient condition for the integral over the "cuts" to vanish is to use variations $\delta \eta$ and $\delta \chi$ which are single valued. It can be shown that χ can always be taken to be single valued, hence taking $\delta \chi$ to be single valued is no restriction at all. In some topologies η is not single valued and in those cases a single valued restriction on $\delta \eta$ is sufficient to make the cut term null.

Finally we take a variational derivative with respect to the entropy s:

$$\begin{aligned} \delta_s A &= \int d^3x dt \delta s [\frac{\partial(\rho \sigma)}{\partial t} + \vec{\nabla} \cdot (\rho \sigma \vec{v}) - \rho T] \\ &+ \int dt \oint d\vec{S} \cdot \rho \sigma \vec{v} \delta s - \int d^3x \rho \sigma \delta s |_{t_0}^{t_1}, \end{aligned} \quad (24)$$

in which the temperature is $T = \frac{\partial \varepsilon}{\partial s}$. We notice that according to equation (17) σ is single valued and hence no cuts are needed. Taking into account the continuity equation (3) we obtain for locations in which the density ρ is not null the result:

$$\frac{d\sigma}{dt} = T, \quad (25)$$

provided that $\delta_s A$ vanished for an arbitrary δs.

[1] Which entails either a Kutta type condition for the velocity in contradiction to the "cut" being an arbitrary surface, or a vanishing density perturbation on the cut.

IV. EULER'S EQUATIONS

We shall now show that a velocity field given by equation (17), such that the equations for $\alpha, \beta, \chi, \eta, \nu, \sigma, s$ satisfy the corresponding equations (14,19,23,25) must satisfy Euler's equations. Let us calculate the material derivative of $\vec{v}$:

$$\begin{aligned} \frac{d\vec{v}}{dt} &= \frac{d\vec{\nabla}\nu}{dt} + \frac{d\alpha}{dt}\vec{\nabla}\chi + \alpha\frac{d\vec{\nabla}\chi}{dt} + \frac{d\beta}{dt}\vec{\nabla}\eta + \beta\frac{d\vec{\nabla}\eta}{dt} \\ &+ \frac{d\sigma}{dt}\vec{\nabla}s + \sigma\frac{d\vec{\nabla}s}{dt}. \end{aligned} \tag{26}$$

It can be easily shown that:

$$\begin{aligned} \frac{d\vec{\nabla}\nu}{dt} &= \vec{\nabla}\frac{d\nu}{dt} - \vec{\nabla}v_k\frac{\partial\nu}{\partial x_k} = \vec{\nabla}(\frac{1}{2}\vec{v}^2 - w) - \vec{\nabla}v_k\frac{\partial\nu}{\partial x_k}, \\ \frac{d\vec{\nabla}\eta}{dt} &= \vec{\nabla}\frac{d\eta}{dt} - \vec{\nabla}v_k\frac{\partial\eta}{\partial x_k} = -\vec{\nabla}v_k\frac{\partial\eta}{\partial x_k}, \\ \frac{d\vec{\nabla}\chi}{dt} &= \vec{\nabla}\frac{d\chi}{dt} - \vec{\nabla}v_k\frac{\partial\chi}{\partial x_k} = -\vec{\nabla}v_k\frac{\partial\chi}{\partial x_k}, \\ \frac{d\vec{\nabla}s}{dt} &= \vec{\nabla}\frac{ds}{dt} - \vec{\nabla}v_k\frac{\partial s}{\partial x_k} = -\vec{\nabla}v_k\frac{\partial s}{\partial x_k}. \end{aligned} \tag{27}$$

In which x_k is a Cartesian coordinate and a summation convention is assumed. Inserting the result from equations (27,14) into equation (26) yields:

$$\begin{aligned} \frac{d\vec{v}}{dt} &= -\vec{\nabla}v_k(\frac{\partial\nu}{\partial x_k} + \alpha\frac{\partial\chi}{\partial x_k} + \beta\frac{\partial\eta}{\partial x_k} + \sigma\frac{\partial s}{\partial x_k}) \\ &+ \vec{\nabla}(\frac{1}{2}\vec{v}^2 - w) + T\vec{\nabla}s \\ &+ \frac{1}{\rho}((\vec{\nabla}\eta\cdot\vec{J})\vec{\nabla}\chi - (\vec{\nabla}\chi\cdot\vec{J})\vec{\nabla}\eta) \\ &= -\vec{\nabla}v_k v_k + \vec{\nabla}(\frac{1}{2}\vec{v}^2 - w) + T\vec{\nabla}s \\ &+ \frac{1}{\rho}\vec{J}\times(\vec{\nabla}\chi\times\vec{\nabla}\eta) \\ &= -\frac{\vec{\nabla}p}{\rho} + \frac{1}{\rho}\vec{J}\times\vec{B}. \end{aligned} \tag{28}$$

In which we have used both equation (17) and equation (15) in the above derivation. This of course proves that the non-barotropic Euler equations can be derived from the action given in equation (12) and hence all the equations of non-barotropic magnetohydrodynamics can be derived from the above action without restricting the variations in any way except on the relevant boundaries and cuts.

V. SIMPLIFIED ACTION

The reader of this paper might argue here that the paper is misleading. The author has declared that he is going to present a simplified action for non-barotropic magnetohydrodynamics instead he added six more functions $\alpha, \beta, \chi, \eta, \nu, \sigma$ to the standard set $\vec{B}, \vec{v}, \rho, s$. In the following I will show that this is not so and the action given in equation (12) in a form suitable for a pedagogic presentation can indeed be simplified. It is easy to show that the Lagrangian density appearing in equation (12) can be written in the form:

$$\begin{aligned} \mathcal{L} &= -\rho[\frac{\partial\nu}{\partial t} + \alpha\frac{\partial\chi}{\partial t} + \beta\frac{\partial\eta}{\partial t} + \sigma\frac{\partial s}{\partial t} + \varepsilon(\rho, s)] \\ &+ \frac{1}{2}\rho[(\vec{v} - \hat{\vec{v}})^2 - (\hat{\vec{v}})^2] \\ &+ \frac{1}{8\pi}[(\vec{B} - \hat{\vec{B}})^2 - (\hat{\vec{B}})^2] + \frac{\partial(\nu\rho)}{\partial t} + \vec{\nabla}\cdot(\nu\rho\vec{v}). \end{aligned} \tag{29}$$

In which $\hat{\vec{v}}$ is a shorthand notation for $\vec{\nabla}\nu + \alpha\vec{\nabla}\chi + \beta\vec{\nabla}\eta + \sigma\vec{\nabla}s$ (see equation (17)) and $\hat{\vec{B}}$ is a shorthand notation for $\vec{\nabla}\chi\times\vec{\nabla}\eta$ (see equation (15)). Thus $\mathcal{L}$ has four contributions:

$$\begin{aligned} \mathcal{L} &= \hat{\mathcal{L}} + \mathcal{L}_{\vec{v}} + \mathcal{L}_{\vec{B}} + \mathcal{L}_{boundary}, \\ \hat{\mathcal{L}} &\equiv -\rho\left[\frac{\partial\nu}{\partial t} + \alpha\frac{\partial\chi}{\partial t} + \beta\frac{\partial\eta}{\partial t} + \sigma\frac{\partial s}{\partial t}\right. \\ &+ \left.\varepsilon(\rho, s) + \frac{1}{2}(\vec{\nabla}\nu + \alpha\vec{\nabla}\chi + \beta\vec{\nabla}\eta + \sigma\vec{\nabla}s)^2\right] \\ &- \frac{1}{8\pi}(\vec{\nabla}\chi\times\vec{\nabla}\eta)^2 \\ \mathcal{L}_{\vec{v}} &\equiv \frac{1}{2}\rho(\vec{v} - \hat{\vec{v}})^2, \\ \mathcal{L}_{\vec{B}} &\equiv \frac{1}{8\pi}(\vec{B} - \hat{\vec{B}})^2, \\ \mathcal{L}_{boundary} &\equiv \frac{\partial(\nu\rho)}{\partial t} + \vec{\nabla}\cdot(\nu\rho\vec{v}). \end{aligned} \tag{30}$$

The only term containing $\vec{v}$ is[2] $\mathcal{L}_{\vec{v}}$, it can easily be seen that this term will lead, after we nullify the variational derivative with respect to $\vec{v}$, to equation (17) but will otherwise have no contribution to other variational derivatives. Similarly the only term containing $\vec{B}$ is $\mathcal{L}_{\vec{B}}$ and it can easily be seen that this term will lead, after we nullify the variational derivative, to equation (15) but will have no contribution to other variational derivatives. Also notice that the term $\mathcal{L}_{boundary}$ contains only complete partial derivatives and thus can not contribute to the equations although it can change the boundary conditions. Hence we see that equations (14), equation (19), equations (23) and equation (25) can be derived using the Lagrangian density:

$$\begin{aligned} \hat{\mathcal{L}}[\alpha, \beta, \chi, \eta, \nu, \rho, \sigma, s] &= -\rho[\frac{\partial\nu}{\partial t} + \alpha\frac{\partial\chi}{\partial t} + \beta\frac{\partial\eta}{\partial t} + \sigma\frac{\partial s}{\partial t} \\ &+ \varepsilon(\rho, s) + \frac{1}{2}(\vec{\nabla}\nu + \alpha\vec{\nabla}\chi + \beta\vec{\nabla}\eta + \sigma\vec{\nabla}s)^2] \\ &- \frac{1}{8\pi}(\vec{\nabla}\chi\times\vec{\nabla}\eta)^2 \end{aligned} \tag{31}$$

in which $\hat{\vec{v}}$ replaces $\vec{v}$ and $\hat{\vec{B}}$ replaces $\vec{B}$ in the relevant equations. Furthermore, after integrating the eight equations (14,19,23,25) we can insert the potentials $\alpha, \beta, \chi, \eta, \nu, \sigma, s$ into equations (17) and (15) to obtain the physical quantities $\vec{v}$ and $\vec{B}$. Hence, the general non-barotropic magnetohydrodynamic problem is reduced from eight equations (1,3,4,5) and the additional constraint (2) to a problem of eight first order (in the temporal derivative) unconstrained equations. Moreover, the entire set of equations can be derived from the Lagrangian density $\hat{\mathcal{L}}$.

[2] $\mathcal{L}_{boundary}$ also depends on $\vec{v}$ but being a boundary term is space and time it does not contribute to the derived equations

VI. Stationary non-barotropic magnetohydrodynamics

Stationary flows are a unique phenomena of Eulerian fluid dynamics which has no counter part in Lagrangian fluid dynamics. The stationary flow is defined by the fact that the physical fields $\vec{v}, \vec{B}, \rho, s$ do not depend on the temporal coordinate. This, however, does not imply that the corresponding potentials $\alpha, \beta, \chi, \eta, \nu, \sigma$ are all functions of spatial coordinates alone. Moreover, it can be shown that choosing the potentials in such a way will lead to erroneous results in the sense that the stationary equations of motion can not be derived from the Lagrangian density $\hat{\mathcal{L}}$ given in equation (30). However, this problem can be amended easily as follows. Let us choose $\alpha, \beta, \chi, \nu, \sigma$ to depend on the spatial coordinates alone. Let us choose η such that:

$$\eta = \bar{\eta} - t, \tag{32}$$

in which $\bar{\eta}$ is a function of the spatial coordinates. The Lagrangian density $\hat{\mathcal{L}}$ given in equation (30) will take the form:

$$\begin{aligned}\hat{\mathcal{L}} &= \rho(\beta - \varepsilon(\rho, s)) - \frac{1}{2}\rho(\vec{\nabla}\nu + \alpha\vec{\nabla}\chi + \beta\vec{\nabla}\bar{\eta} + \sigma\vec{\nabla}s)^2 \\ &- \frac{1}{8\pi}(\vec{\nabla}\chi \times \vec{\nabla}\bar{\eta})^2. \end{aligned}\tag{33}$$

The above functional can be compared with Vladimirov and Moffatt [2] equation 6.12 for incompressible flows in which their I is analogue to our β. Notice however, that while β is not a conserved quantity I is.

Varying the Lagrangian $\hat{L} = \int \hat{\mathcal{L}} d^3x$ with respect to $\nu, \alpha, \beta, \chi, \eta, \rho, \sigma, s$ leads to the following equations:

$$\begin{aligned}
&\vec{\nabla} \cdot (\rho \hat{\vec{v}}) = 0, \\
&\rho \hat{\vec{v}} \cdot \vec{\nabla}\chi = 0, \\
&\rho(\hat{\vec{v}} \cdot \vec{\nabla}\bar{\eta} - 1) = 0, \\
&\hat{\vec{v}} \cdot \vec{\nabla}\alpha = \frac{\vec{\nabla}\bar{\eta} \cdot \hat{\vec{J}}}{\rho}, \\
&\hat{\vec{v}} \cdot \vec{\nabla}\beta = -\frac{\vec{\nabla}\chi \cdot \hat{\vec{J}}}{\rho}, \\
&\beta = \frac{1}{2}\hat{\vec{v}}^2 + w, \\
&\rho \hat{\vec{v}} \cdot \vec{\nabla}s = 0, \\
&\rho \hat{\vec{v}} \cdot \vec{\nabla}\sigma = \rho T.
\end{aligned}\tag{34}$$

Calculations similar to the ones done in previous subsections will show that those equations lead to the stationary non-barotropic magnetohydrodynamic equations:

$$\vec{\nabla} \times (\hat{\vec{v}} \times \hat{\vec{B}}) = 0, \tag{35}$$

$$\rho(\hat{\vec{v}} \cdot \vec{\nabla})\hat{\vec{v}} = -\vec{\nabla}p(\rho, s) + \frac{(\vec{\nabla} \times \hat{\vec{B}}) \times \hat{\vec{B}}}{4\pi}. \tag{36}$$

VII. Conclusion

It is shown that stationary non-barotropic magnetohydrodynamics can be derived from a variational principle of eight functions.

Possible applications include stability analysis of stationary magnetohydrodynamic configurations and its possible utilization for developing efficient numerical schemes for integrating the magnetohydrodynamic equations. It may be more efficient to incorporate the developed formalism in the frame work of an existing code instead of developing a new code from scratch. Possible existing codes are described in [17], [18], [19], [20], [21]. I anticipate applications of this study both to linear and non-linear stability analysis of known barotropic magnetohydrodynamic configurations [22], [23], [24]. I suspect that for achieving this we will need to add additional constants of motion constraints to the action as was done by [25], [26] see also [27], [28], [29]. As for designing efficient numerical schemes for integrating the equations of fluid dynamics and magnetohydrodynamics one may follow the approach described in [30], [31], [32], [33].

Another possible application of the variational method is in deducing new analytic solutions for the magnetohydrodynamic equations. Although the equations are notoriously difficult to solve being both partial differential equations and nonlinear, possible solutions can be found in terms of variational variables. An example for this approach is the self gravitating torus described in [34].

One can use continuous symmetries which appear in the variational Lagrangian to derive through Noether theorem new conservation laws. An example for such derivation which still lacks physical interpretation can be found in [38]. It may be that the Lagrangian derived in [10] has a larger symmetry group. And of course one anticipates a different symmetry structure for the non-barotropic case.

Topological invariants have always been informative, and there are such invariants in MHD flows. For example the two helicities have long been useful in research into the problem of hydrogen fusion, and in various astrophysical scenarios. In previous works [9], [11], [40] connections between helicities with symmetries of the barotropic fluid equations were made. The variables of the current variational principles are helpful for identifying and characterizing new topological invariants in MHD [41], [42].

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