Comparative analysis of two problems of the impact interaction of rigid and viscoelastic spherical shells*

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Abstract – Two problems of the impact interaction of two spherical shells, one of which is rigid while the second one is viscoelastic, are considered. In the first problem the viscoelastic shell impacts with the velocity V_0 against the quiescent rigid shell, while in the second problem, on the contrary, the rigid shell with the velocity V_0 bumps the motionless viscoelastic shell. For both problems, integrodifferential equations for the values of local bearing of the material of the viscoelastic shell have been obtained under the assumption that the volume relaxation in viscoelastic shells is negligible. Approximate solutions of the integrodifferential equations allows one to carry out the comparative analysis of the results obtained in the both cases.

Keywords – Fractional derivative standard linear solid model, impact, viscoelastic spherical shell, volume relaxation.

I. Introduction

The problems connected with the analysis of the shock interaction of thin bodies (rods, beams, plates, and shells) with other bodies have widespread application in various fields of science and technology. The physical phenomena involved in the impact event include structural responses, contact effects and wave propagation. Because these problems belong to the problems of dynamic contact interaction, their solution is connected with severe mathematical and calculation difficulties. To overcome this impediment, a rich variety of approaches and methods have been suggested, and the overview of current results in the field can be found in recent state-of-the-art articles by Abrate [1], Rossikhin and Shitikova [2], and Qatu et al. [3].

In many engineering applications, it is important to understand the transient behaviour of thin-walled shell structures subjected to central impact not only by a small projectile but by another shell as well.

Nowadays fractional calculus is really widely used in different fields of science and technology, including various dynamic problems of mechanics of solids and structures [4], and the problems of impact interaction among them [5].

The only paper considering the analysis of two colliding fractionally damped spherical shells in modelling blunt human head impacts has been recently appeared [6], wherein the contact force is represented using linear approach via fractional derivative standard linear solid model. It was assumed that during impact process the microstructure of the shells' materials changed only in the contact domain, i.e., both shells remain elastic except the parts involved into the contact interaction which exhibit locally viscoelastic properties.

Further Rossikhin et al. [7] investigated the collision of two viscoelastic shells, viscoelastic features of which are described by the standard linear solid model with conventional integer derivatives. During the impact process there occurs decrosslinking within the domain of the contact of the colliding bodies, resulting in more freely displacements of molecules with respect to each other, and finally in the decrease of the shells' material viscosity in the contact zone. This circumstance allows one to describe the behaviour of the materials of the colliding spherical shells within the contact domain by the standard linear solid model involving fractional derivatives, since variation in the fractional parameter (the order of the fractional derivative) enables one to control the viscosity of the shells' material. That is why the fractional parameter could be considered as the structural parameter.

The particular case of the impact interaction of a viscoelastic spherical shell with an infinite rigid plate was studied in [8].

In the present paper, we will consider two problems of the impact interaction of two spherical shells, when the nonrelaxed elastic modulus of one shell is much larger than those of the second shell, what allows one to consider one shell as a rigid and another one as viscoelastic. In the first problem the viscoelastic shell impacts with the velocity V_0

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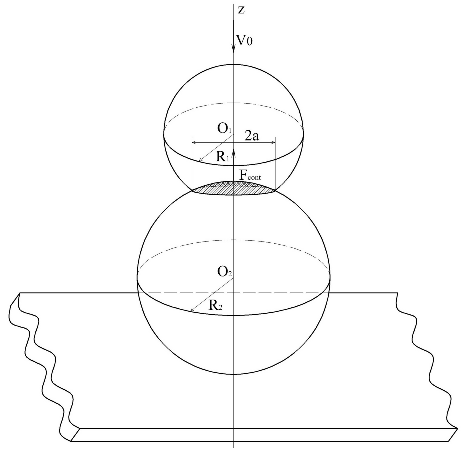

Figure 1: Scheme of the normal impact of a spherical viscoelastic shell against a rigid spherical shell

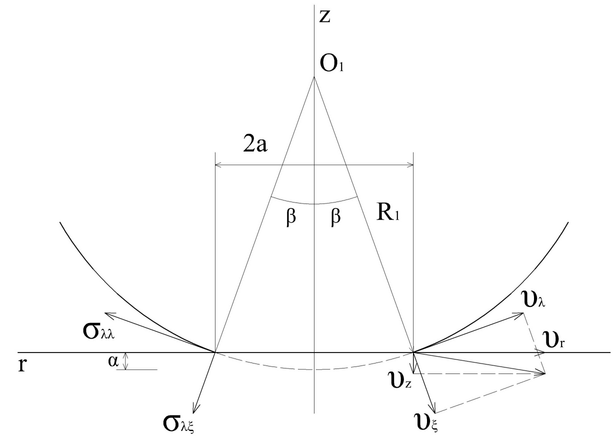

Figure 2: Scheme of velocities and stresses in the shell's element on the boundary of the contact domain [7]

against the quiescent rigid shell, while in the second problem, on the contrary, the rigid shell with the velocity V_0 bumps the motionless viscoelastic shell. For both problems, integrodifferential equations for the values of local bearing of the material of the viscoelastic shell have been obtained under the assumption that the volume relaxation in viscoelastic shells is negligible.

II. Impact Interaction of a Viscoelastic Spherical Shell against a Quiescent Rigid Spherical Shell

Let us consider the problem on a normal impact of a viscoelastic spherical shell with the initial velocity V_0 against a rigid spherical shell resting on a rigid plate (Fig. 1), when the viscoelastic features of the impactor are described by the standard linear solid model with conventional derivatives of integer order.

For this purpose we will proceed from equations of motion of two colliding viscoelastic spherical shells derived recently in [7], wherein we tend the Young's modulus of the second shell to infinity. As a result we obtain the following equation of motion of the contact domain

$$\rho\pi a^2 h\dot{v}_z = 2\pi a h\sigma_{rz}|_{r=a} + F_{\text{cont}} \tag{1}$$

under the action of the transverse force $2\pi ah\widetilde{\sigma_{rz}}|_{r=a}$ and the contact force F_{cont}, which is defined via the generalized Hertzian contact law

$$F_{\text{cont}} = \tilde{k}\alpha^{3/2}, \tag{2}$$

where α is the local bearing of the impactor's material (Fig. 2), $\tilde{k}$ is the operator involving the geometry of colliding bodies, i.e. the radii of the viscoelastic R_1 and rigid R_2 spherical shells, respectively, and viscoelastic features of the impactor defined by the time-dependent functions $\tilde{E}$ and $\tilde{\nu}$

$$\tilde{k} = \frac{4}{3}\,\frac{\sqrt{R'}\tilde{E}}{1-\tilde{\nu}^2}, \tag{3}$$

$$\frac{1}{R'} = \frac{1}{R_1} + \frac{1}{R_2}, \tag{4}$$

ρ and h are the density and thickness of the viscoelastic shell, respectively, a is the radius of the contact domain (Fig. 1), and an overdot denotes the time-derivative.

The following equation

$$V_0 - v_z|_{r=a} = \dot{\alpha} \tag{5}$$

should be added to equations (1) and (2).

In [7] it has been shown that considering $v_r|_{r=a} = \dot{a}$, the value $\sigma_{rz}|_{r=a}$ could be calculated in the following form according to the dynamic condition of compatibility:

$$\sigma_{rz}|_{r=a} = \rho\,(G_1 - G_2)\,\frac{(a^2)^{\cdot}}{2R_1} - \rho\left(G_1\frac{a^2}{R_1^2} + G_2\right) v_z|_{r=a}, \tag{6}$$

where G_1 and G_2 are the velocities of the quasi-longitudinal and quasi-transverse waves (surfaces of strong discontinuity), respectively, which are generated at the moment of impact at the point of tangency (or the point of contact) of the impactor with the target, which then propagate in the form of diverging circles along spherical surface, and are defined as

$$G_1 = \sqrt{\frac{E_\infty}{\rho(1-\nu_\infty^2)}}, \tag{7}$$

$$G_2 = \sqrt{\frac{\mu_\infty}{\rho}}, \tag{8}$$

where E_∞, μ_∞ and ν_∞ are non-relaxed elastic and shear moduli and Poisson's ratios, respectively.

Considering that $a/R_1 \ll 1$, equation (6) is reduced to

$$\sigma_{rz}|_{r=a} = -\rho G_2 v_z|_{r=a}, \tag{9}$$

Now substituting (2) and (9) in (1) and considering that $a^2 = R_1\alpha$ [9] yield

$$\rho\pi R_1 \alpha h \dot{v}_z|_{r=a} = -2\pi (R_1\alpha)^{1/2} h\rho G_2 v_z|_{r=a} + \tilde{k}\alpha^{3/2}. \tag{10}$$

In order to solve equation (10), we should define the operator $\tilde{k}$, resulting in decoding the operator $\widetilde{E}/(1-\widetilde{\nu}^2)$.

For the majority of viscoelastic materials, the bulk modulus K remains constant during the process of mechanical loading of this material [10], resulting in [5]

$$\frac{\widetilde{E}_1}{1-2\widetilde{\nu}} = \frac{E_\infty}{1-2\nu_\infty}. \tag{11}$$

Recently it has been proposed in [7] that during the impact process there could occur decrosslinking within the domain of the contact between the impactor and target, resulting in more freely displacements of molecules with respect to each other, and finally in the decrease of the shell's material viscosity in the contact zone. This circumstance allows one to describe the behaviour of the material of the impacting spherical shell within the contact domain by the standard linear solid model involving fractional derivatives

$$\sigma + \tau_\varepsilon^\gamma D^\gamma \sigma = E_0\left(\varepsilon + \tau_\sigma^\gamma D^\gamma \varepsilon\right), \tag{12}$$

where σ is the stress, ε is the strain, E_0 is the relaxed modulus, τ_ε and τ_σ are the relaxation and creep times, respectively,

$$D^\gamma x(t) = \frac{d}{dt}\int_0^t \frac{(t-t')^{-\gamma}}{\Gamma(1-\gamma)}\, x(t')dt' \tag{13}$$

is the Riemann-Liouville fractional derivative, $\Gamma(1-\gamma)$ is the Gamma-function, γ $(0 < \gamma \le 1)$ is the fractional parameter, and $x(t)$ is a certain function.

Utilizing the model (12), it could be found [5] that

$$\frac{\widetilde{E}}{1-\widetilde{\nu}^2} = \frac{E_\infty}{1-\nu_\infty^2}\left[1 - m_1 \ni_\gamma^* (t_1^\gamma) - m_2 \ni_\gamma^* (t_2^\gamma)\right], \tag{14}$$

where $\ni_\gamma^* (t_i^\gamma)$ $(i = 1, 2)$ is the dimensionless Rabotnov operator [5]

$$\ni_\gamma^* (t_i^\gamma) = \frac{1}{1 + t_i^\gamma D^\gamma}, \tag{15}$$

and

$$t_1^\gamma = \frac{2(1+\nu_\infty)\tau_\varepsilon^\gamma}{2(1+\nu_\infty) + \nu_\varepsilon(1-2\nu_\infty)},$$

$$t_2^\gamma = \frac{2(1-\nu_\infty)\tau_\varepsilon^\gamma}{2(1-\nu_\infty) - \nu_\varepsilon(1-2\nu_\infty)},$$

$$m_1 = \frac{3}{2}\,\frac{(1-\nu_\infty)\nu_\varepsilon}{2(1+\nu_\infty) + (1-2\nu_\infty)\nu_\varepsilon},$$

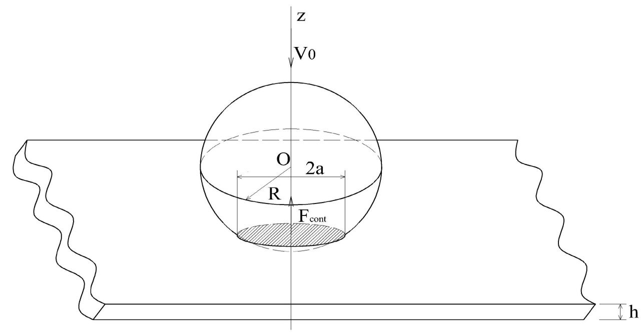

Figure 3: Scheme of the normal impact of a viscoelastic spherical shell against an infinite rigid plate [8]

$$m_2 = \frac{1}{2}\,\frac{(1+\nu_\infty)\nu_\varepsilon}{2(1-\nu_\infty) - (1-2\nu_\infty)\nu_\varepsilon},$$

$$\nu_\varepsilon = \frac{E_\infty - E_0}{E_\infty}.$$

Equation (10) with due account for (5) and (14), as well as the initial conditions

$$\alpha|_{t=0} = 0, \quad \dot{\alpha}|_{t=0} = V_0, \tag{16}$$

is reduced to

$$\ddot{\alpha} + æ\left[\alpha^{1/2}(t) - \Delta_\gamma \alpha^{-1}\int_0^t (t-t')^{\gamma-1} \times \alpha^{3/2}(t')dt'\right] = 0, \tag{17}$$

where

$$æ = \frac{4\sqrt{R'}E_\infty}{3\pi R_1 \rho h(1-\nu_\infty^2)}, \tag{18}$$

$$\Delta_\gamma = \frac{1}{\Gamma(\gamma)}\sum_{j=1}^{2}\frac{m_j}{t_j^\gamma}. \tag{19}$$

Note that a particular case of the normal impact of a viscoelastic spherical shell against an infinite rigid plate (Fig. 3) was considered recently in [8]. In this case, $R_2 \to \infty$ in (4), and therefore $R' = R_1$.

Thus, Eq. (3) is reduced to

$$\tilde{k} = \frac{4}{3}\,\frac{\sqrt{R}\tilde{E}}{1-\tilde{\nu}^2}, \tag{20}$$

where R is the radius of the impactor, i.e. the radius of the viscoelastic shell.

Then the solution presented above is valid for this case as well if R_1 is substituted by R in Eqs. (6) and (10), resulting in the governing Eq. (17), where the coefficient æ takes the form

$$æ = \frac{4E_\infty}{3\pi\sqrt{R}\rho h(1-\nu_\infty^2)}. \tag{21}$$

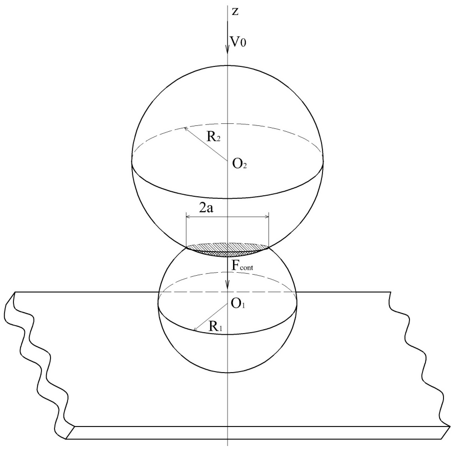

Figure 4: Scheme of the normal impact of a rigid spherical shell against a motionless viscoelastic spherical shell

III. Impact Interaction of a Rigid Spherical Shell against a Quiescent Viscoelastic Spherical Shell

Now we consider the second problem when a rigid spherical shell bumps a viscoelastic spherical shell resting on a rigid plate (Fig. 4). In this case, the equation of motion of the rigid sphere

$$m\ddot{z} = -\tilde{k}\alpha^{3/2} \tag{22}$$

should be added to Eq. (10), where m is the mass of the rigid sphere, and

$$\dot{z} = v_z|_{r=a} + \dot{\alpha}. \tag{23}$$

From Eqs. (22) and (23) we find

$$m\dot{v}_z|_{r=a} = -m\ddot{\alpha} - \tilde{k}\alpha^{3/2}. \tag{24}$$

Adding Eqs. (10) and (24), we have

$$\begin{aligned}(\rho\pi R_1 h\alpha + m)\,\dot{v}_z|_{r=a} &= -m\ddot{\alpha} \\ &-2\pi(R_1\alpha)^{1/2}h\rho G_2 v_z|_{r=a}.\end{aligned} \tag{25}$$

To find the solution at the first approximation, it is possible to ignore the second term in the right hand part of Eq. (25), as we similarly have done above in the first problem. As a result we obtain

$$\dot{v}_z|_{r=a} = -\frac{m}{m+\rho\pi R_1 h\alpha}\,\ddot{\alpha}. \tag{26}$$

Substituting (26) in Eq. (24) yields

$$m\left(1-\frac{m}{m+\rho\pi R_1 h\alpha}\right)\ddot{\alpha} = -\tilde{k}\alpha^{3/2}. \tag{27}$$

Considering the smallness of the value α, from (27) we have

$$\rho\pi R_1 h\alpha\ddot{\alpha} + \tilde{k}\alpha^{3/2} = 0. \tag{28}$$

If we introduce the coefficient æ defined in (18) in Eq. (28) and consider the expression for the contact force, then we are led to Eq. (17).

Thus, the both problems, which are initially described by different sets of integrodifferential equations, have similar solutions. This fact indicates that these problems are reciprocal.

IV. Approximate Solutions

If we consider

$$\alpha \approx V_0 t \tag{29}$$

as a first approximation, then Eq. (17) with due account for

$$\int_0^t (t-t')^{\gamma-1}\, t'^{3/2} dt' = \frac{3}{\gamma}\left(\frac{1}{3}-\frac{1}{5}\gamma\right)t^{3/2+\gamma} \tag{30}$$

takes the form

$$\ddot{\alpha} = -æ V_0^{1/2}\left[t^{1/2} - \Delta_\gamma \frac{3}{\gamma}\left(\frac{1}{3}-\frac{1}{5}\gamma\right)t^{1/2+\gamma}\right]. \tag{31}$$

Integrating (31) yields

$$\begin{aligned}\dot{\alpha} &= V_0 - \frac{2}{3}æ V_0^{1/2} t^{3/2} \\ &+ æ V_0^{1/2}\Delta_\gamma \frac{3}{\gamma}\left(\frac{1}{3}-\frac{1}{5}\gamma\right)\frac{t^{3/2+\gamma}}{3/2+\gamma},\end{aligned} \tag{32}$$

and

$$\begin{aligned}\alpha &= V_0 t - \frac{4}{15}æ V_0^{1/2} t^{5/2} \\ &+ æ V_0^{1/2}\Delta_\gamma \frac{3}{\gamma}\left(\frac{1}{3}-\frac{1}{5}\gamma\right)\frac{t^{5/2+\gamma}}{(3/2+\gamma)(5/2+\gamma)}\end{aligned} \tag{33}$$

A. The Case $\gamma = 0$

In a particular case, when $\gamma = 0$, and therefore

$$\sum_{j=1}^{2} m_j = 0,$$

relationships (32) and (33) take the form

$$\dot{\alpha} = V_0\left(1 - \frac{2}{3}æ V_0^{-1/2} t^{3/2}\right), \tag{34}$$

$$\alpha = V_0 t\left(1 - \frac{4}{15}æ V_0^{-1/2} t^{3/2}\right), \tag{35}$$

from which the contact duration $t_{\rm cont}^{(0)}$ and the time $t_{\rm max}^{(0)}$ at which the maximal local indentation $\alpha_{\rm max}^{(0)}$ takes place could be found

$$t_{\rm cont}^{(0)} \approx \left(\frac{15}{4}\frac{V_0^{1/2}}{æ}\right)^{2/3}, \tag{36}$$

$$t_{\max}^{(0)} \approx \left(\frac{3}{2} \frac{V_0^{1/2}}{æ} \right)^{2/3}, \tag{37}$$

$$\alpha_{\max}^{(0)} \approx \frac{3}{5} V_0 t_{\max}^{0}. \tag{38}$$

The Case $\gamma \neq 0$ *or* 1

When the fractional parameter takes on the magnitudes within the interval $0 < \gamma < 1$, then the duration of contact $t_{\rm cont}^{(\gamma)}$ could be determined as follows

$$t_{\rm cont}^{(\gamma)} \approx t_{\rm cont}^{(0)} + \epsilon, \tag{39}$$

where ϵ is a small value.

Substituting (39) in Eq. (33) and tending $\alpha \to 0$, we obtain

$$V_0 \left(t_{\rm cont}^{(0)} + \epsilon \right) - \frac{4}{15} æ V_0^{1/2} \left(t_{\rm cont}^{(0)} + \epsilon \right)^{5/2}$$
$$+ æ V_0^{1/2} \Delta_\gamma \frac{3}{\gamma} \left(\frac{1}{3} - \frac{1}{5}\gamma \right) \frac{\left(t_{\rm cont}^{(0)} + \epsilon \right)^{5/2+\gamma}}{(3/2+\gamma)(5/2+\gamma)} = 0, \tag{40}$$

whence it follows that

$$\epsilon = \frac{5}{2} \Delta_\gamma \frac{3}{\gamma} \left(\frac{1}{3} - \frac{1}{5}\gamma \right) \frac{\left(t_{\rm cont}^{(0)} \right)^{1+\gamma}}{(3/2+\gamma)(5/2+\gamma)}.$$

Supposing that

$$t_{\max}^{(\gamma)} \approx t_{\max}^{(0)} + \epsilon_1, \tag{41}$$

where ϵ_1 is a small value, and substituting (41) in Eq. (32) with $\dot{\alpha} = 0$

$$V_0 - \frac{2}{3} æ V_0^{1/2} \left(t_{\max}^{(0)} + \epsilon_1 \right)^{3/2}$$
$$+ æ V_0^{1/2} \Delta_\gamma \frac{3}{\gamma} \left(\frac{1}{3} - \frac{1}{5}\gamma \right) \frac{\left(t_{\max}^{(0)} + \epsilon_1 \right)^{3/2+\gamma}}{3/2+\gamma}, \tag{42}$$

we obtain

$$\epsilon_1 = \Delta_\gamma \frac{3}{\gamma} \left(\frac{1}{3} - \frac{1}{5}\gamma \right) \frac{\left(t_{\max}^{(0)} \right)^{1+\gamma}}{(3/2+\gamma)}.$$

Now substituting (41) in (33) we could define

$$\alpha_{\max}^{(\gamma)} = \alpha_{\max}^{(0)} + \frac{9}{2} V_0 \Delta_\gamma \frac{1}{\gamma} \left(\frac{1}{3} - \frac{1}{5}\gamma \right)$$
$$\times \frac{\left(t_{\max}^{(0)} \right)^{1+\gamma}}{(3/2+\gamma)(5/2+\gamma)}. \tag{43}$$

C. The Case $\gamma = 1$

In the particular case $\gamma = 1$, the characteristic values take the form

$$t_{\rm cont}^{(1)} = t_{\rm cont}^{(0)} + \frac{4}{35} \Delta_1 \left(t_{\rm cont}^{(0)} \right)^2, \tag{44}$$

$$t_{\max}^{(1)} = t_{\max}^{(0)} + \frac{4}{25} \Delta_1 \left(t_{\max}^{(0)} \right)^2, \tag{45}$$

$$\alpha_{\max}^{(1)} = \alpha_{\max}^{(0)} + \frac{12}{175} \Delta_1 \left(t_{\max}^{(0)} \right)^2, \tag{46}$$

where $\Delta_1 = \Delta_\gamma|_{\gamma=1}$.

VII. Conclusion

Two problems of the impact interaction of two spherical shells, one of which is rigid while the second one is viscoelastic, have been considered. In the first problem the viscoelastic shell impacts with the velocity V_0 against the quiescent rigid shell, while in the second problem, on the contrary, the rigid shell with the velocity V_0 bumps the motionless viscoelastic shell.

The damping features of the viscoelastic body are described by the standard linear solid model with conventional derivatives of integer order. During the impact process there could occur decrosslinking within the domain of the contact between the impactor and target, resulting in more freely displacements of molecules with respect to each other, and finally in the decrease of the shell's material viscosity in the contact zone. This circumstance allows one to describe the behaviour of the material of the viscoelastic spherical shell within the contact domain by the standard linear solid model involving fractional derivatives. The fractional parameter as an additional parameter could control the changes in the shell's viscosity within the impact domain.

For both problems, integrodifferential equations for the values of local bearing of the material of the viscoelastic shell have been obtained under the assumption that the volume relaxation in viscoelastic shells is negligible. It has been shown that the problems under consideration are reciprocal ones.

The approximate analysis carried out on the base of the suggested model allows us to make the following conclusion: maximal viscosity increases all values characterizing the process of shells interaction, $t_{\rm cont}$, $t_{\max}$, and $\alpha_{\max}$, since with the increase in the fractional parameter from zero to unit the viscosity enhances, resulting in the increment of the characteristic values from $t_{\rm cont}^{(0)}$, $t_{\max}^{(0)}$, and $\alpha_{\max}^{(0)}$ to $t_{\rm cont}^{(1)}$, $t_{\max}^{(1)}$, and $\alpha_{\max}^{(1)}$, respectively.

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