Impact of a Viscoelastic Sphere against an Elastic Kirchhoff-Love Plate Embedded into a Fractional Derivative Kelvin-Voigt Medium*

Yury A. Rossikhin, Marina V. Shitikova and Phan Thanh Trung

Abstract – In the present paper, we consider the problem on a low-velocity transverse impact of a viscoelastic sphere upon an elastic Kirchhoff-Love plate in a viscoelastic medium, the viscoelastic features of which are described by the fractional derivative Kelvin-Voigt model. Within the contact domain the contact force is defined by the generalized Hertzian contact. The Green function for the target was constructed, what allows us to obtain the integral equation for the contact force and local indentation using the algebra of Rabotnov's fractional operators. An approximate analytical solution has been found.

Keywords – Low-velocity transverse impact, Kirchhoff-Love plate, generalized Hertzian contact, contact force, fractional derivative models of viscoelasticity, Rabotnov fractional operator

I. Introduction

Nowadays fractional calculus is widely used in different fields of science and technology, including various dynamic problems of mechanics of solids and structures [1].

Usually in the papers relating to the dynamic response of such viscoelastic bodies as beams, plates and membranes the utilization of the Kelvin-Voigt model with fractional derivatives is carried out [2]-[6]. As this takes place, it is supposed that Poisson's ratio is time-independent during the process of deformation and as a preassigned operator it is selected Young's operator $\widetilde{E}$ defined by

$$\widetilde{E} = E_1 \left(1 + \tau_{\sigma_1}^{\gamma_1} D^{\gamma}\right), \tag{1}$$

where E_1 is relaxed elastic modulus, τ_{σ_1} is the retardation time, γ_1 $(0 < \gamma_1 \leq 1)$ is the fractional parameter, D^{γ_1} is the Riemann-Liouville fractional derivative [1]

$$D^{\gamma_1} x(t) = \frac{d}{dt} \int_0^t \frac{x(t')dt'}{\Gamma(1-\gamma_1)(t-t')^{\gamma_1}}, \tag{2}$$

$\Gamma(1-\gamma_1)$ is the Gamma function, $x(t)$ is an arbitrary function, and when its Poisson's operator $\tilde{\nu}$ is considered as the time-independent value, then this case coincides with the case of the dynamic behaviour of elastic bodies in a viscoelastic medium.

However as experimental data have shown [7, 8], Poisson's ratio is always an operator $\tilde{\nu}$, and only the bulk extension-compression operator $\widetilde{K}$ may be expressed as the time-independent value, which for the most viscoelastic materials weakly varies during deformation.

On the other hand, as it is shown in [9], the viscoelastic model (1) with a constant bulk extension-compression operator is completely inapplicable for description of the dynamic response of viscoelastic bodies, and the Kelvin-Voigt model itself is only acceptable for the description of the dynamic behaviour of elastic bodies in a viscoelastic medium.

It has been recently shown for viscoelastic beams [10] and plates [11] that when operator $\tilde{E}$ is defined by Eq. (1) and the Poisson's operator $\tilde{\nu}$ is considered as the time-independent value, then this case coincides with the case of the dynamic behaviour of elastic bodies in a viscoelastic medium.

Thus, the authors of such papers as [2]–[6], consciously or not, replace one problem with another, namely: a problem of the dynamic response of viscoelastic bodies in a conventional medium with a problem of dynamic response of elastic bodies in a viscoelastic medium.

In the present paper, we generalize the approach presented in [11] and consider the problem on a transverse impact of a viscoelastic sphere upon an elastic Kirchhoff-Love plate in a fractional derivative Kelvin-Voigt medium. As this takes place, the viscoelastic features of the impactor are described by the fractional derivative standard linear solid model. Within the contact domain the contact force will be defined by the generalized Hertzian contact law, in contrast to linear approach utilized in [12] allowing one to apply Laplace transform technique. The functional equation for determining the contact force will be obtained using the algebra of Rabotnov's fractional operators.

*The authors are with the Voronezh State Technical University, Research Center on Dynamics of Solids and Structures, 20-letija Oktjabrja Street 84, 394006 Voronezh, Russia, and Phan Thanh Trung is on leave from HUTECH, HCMC University of Technology and Education, Vietnam. This research was supported by the Ministry of Education and Science of the Russian Federation, Project No.9.5138.2017/Basic Part.

II. Problem Formulation

Let us consider the problem on a transverse impact of a viscoelastic sphere upon an elastic Kirchhoff-Love plate, when the viscoelastic features of the surrounding medium are described by a fractional derivative Kelvin-Voigt model. In this case, the equations of motion of a spherical impactor of radius R and mass m and the elastic rectangular plate with the dimensions a and b and of thickness h have, respectively, the form

$$m\ddot{w}_2 = -P(t), \tag{3}$$

$$\frac{D}{\rho h}\nabla^2 w_1 + \frac{\mu}{\rho h} D^{\gamma_1} w_1 + \ddot{w}_1 = \frac{1}{\rho h} P(t) \times \delta\left(x - \frac{a}{2}\right)\delta\left(y - \frac{b}{2}\right), \tag{4}$$

where $P(t)$ is the contact force, $\nabla^2 = (\partial/\partial x + \partial/\partial y)^2$, $D = E_1 h^3/12(1-\nu_1^2)$ is the cylindrical rigidity, $w_1(x,y,t)$ is the plate deflection, E_1, ν_1 and ρ are its Young's modulus, Poisson's ratio and density, respectively, x and y are Cartesian coordinates, $\delta(x - \frac{a}{2})$ is the Dirac delta-function, overdots denote time-derivatives, μ is the coefficient of viscosity, and $D^{\gamma_1} w_1$ is the Riemann-Liouville fractional derivative defined in (2).

The second term in (4) represents the action of external frictional forces initiated in the surrounding fractional derivative Kelvin-Voigt medium during vibrations of the plate under the action of the contact force $P(t)$.

Equations (3) and (4) are subjected to the following initial conditions:

$$w_1(x,y,0) = 0, \quad \dot{w}_1(x,y,0) = 0, \\ w_2(0) = 0, \quad \dot{w}_2(0) = V_0, \tag{5}$$

where V_0 is the initial velocity of the impactor at the moment of impact.

Integrating twice Eq. (3) yields

$$w_2(t) = -\frac{1}{m}\int_0^t P(t')(t-t')dt' + V_0 t. \tag{6}$$

Expanding displacement $w_1(x,y,t)$ for a simply-supported Kirchhoff-Love plate in terms of eigenfunctions

$$w_1(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} x_{mn}(t) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}, \tag{7}$$

and substituting (7) in (4) with due account for orthogonality of sines on the segments $0 \le x \le a$, $0 \le y \le b$, we are led to the infinite set of uncoupled equations

$$\ddot{x}_{mn}(t) + \frac{\mu_{mn}}{\rho h} D^{\gamma_1} x_{mn}(t) + \Omega_{mn}^2 x_{mn}(t) \\ = F_{mn} P(t), \quad (m, n = 1, 2, ...) \tag{8}$$

where μ_{mn} is the coefficient of viscosity of the harmonic with indices m and n, $x_{mn}(t)$ are generalized displacements, and

$$F_{mn} = \frac{1}{\rho h}\sin\frac{n\pi}{2}\sin\frac{m\pi}{2},$$

$$\Omega_{mn}^2 = \frac{D}{\rho h}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2.$$

Considering the Rayleigh hypothesis of proportionality between the elastic and viscous matrices, i.e.,

$$\frac{\mu_{mn}}{\rho h} = \Omega_{mn}^2 \tau_{\sigma_1}^{\gamma_1}, \tag{9}$$

where $\tau_{\sigma_1}^{\gamma_1}$ is the coefficient of proportionality, Eq. (8) is reduced to

$$\ddot{x}_{1mn}(t) + \Omega_{mn}^2(1 + \tau_{\sigma_1}^{\gamma_1} D^{\gamma_1}) x_{mn}(t) = F_{mn} P(t). \tag{10}$$

Equation (10) describes the vibrations of the force driven fractional derivative Kelvin-Voigt oscillator [1].

III. Green Function for the Fractional Derivative Kelvin-Voigt Model

In order to find the solution of Eq. (4), it is necessary to find the Green function $G_{mn}(t)$ for each oscillator from (10)

$$G_{mn}(t) = A_{0mn}(t) + A_{mn}e^{-\alpha_{mn}t}\sin(\omega_{mn}t - \varphi_{mn}), \tag{11}$$

where the indices mn indicate the ordinal number of the oscillator, and all values entering in (11) have the same structure and the same physical meaning as the corresponding values discussed in [1], i.e. A_{mn} is the amplitude, α_{mn} is the damping coefficient, and ω_{mn} and φ_{mn} are the frequency and phase, respectively.

Reference to Eq. (11) shows that the Green function possesses two terms, one of which, $A_{0mn}(t)$, describes the drift of the equilibrium position and is represented by the integral involving the distribution function of dynamic and rheological parameters, while the other term is the product of two time-dependent functions, exponent and sine, and it describes damped vibrations around the drifting equilibrium position.

Now let us write Eq. (10) in terms of the Green function $G_{mn}(t)$

$$\ddot{G}_{mn}(t) + \Omega_{mn}^2 \tau_{\sigma_1}^{\gamma_1} D^{\gamma_1} G_{mn}(t) + \Omega_{mn}^2 G_{mn}(t) \\ = F_{mn}\delta(t) \quad (m, n = 1, 2, ...). \tag{12}$$

Applying the Laplace transform to Eq. (12) yields

$$\bar{G}_{mn} = \frac{F_{mn}}{p^2 + \kappa_{mn} p^{\gamma_1} + \Omega_{mn}^2}, \tag{13}$$

where an overbar denotes the Laplace transform of the corresponding function, p is the transform parameter, and $\kappa_{mn} = \Omega_{mn}^2 \tau_{\sigma_1}^{\gamma_1}$.

If we omit the numbers mn in (13), then it will coincide with formula (2.2.1) in Sect. 2.2 [13] devoted to the vibrations of the fractional derivative Kelvin-Voigt oscillator. All further formulas of this Section, (2.2.2)–(2.2.6), refer to the analysis of the roots of the characteristic equation

$$p^2 + \kappa_{mn} p^{\gamma_1} + \Omega_{mn}^2 = 0, \tag{14}$$

which at each pair of m and n possesses two complex conjugate roots $(p_{mn})_{1,2} = r_{mn} e^{\pm i\psi_{mn}} = -\alpha_{mn} \pm i\omega_{mn}$ (see the root locus at $m = 1,\ n = 1$ in Fig. 19 of [13]), and the inversion of the expression (13) on the first sheet of the Riemannian surface. If we insert the indices m and n in these formulas, then we obtain the desired relationship (11), where the function $A_{0mn}(t)$ describes the drift of the equilibrium position

$$A_{0mn}(t) = \int_0^\infty \tau^{-1} B_{mn}(\tau, \kappa_{mn}) e^{-t/\tau} d\tau, \tag{15}$$

the function $B_{mn}(\tau, \kappa_{mn})$

$$B_{mn}(\tau, \kappa_{mn}) = \frac{\sin \pi\gamma_1}{\pi} \, F_{mn} \tau [\theta_{mn}(\tau)]^{-1}$$
$$\times \left\{ [\theta_{mn}(\tau)]^{-1} \kappa_{mn}^{-1} \tau^{\gamma_1 - 2} + \theta_{mn}(\tau) \kappa_{mn} \tau^{2-\gamma_1} + 2\cos\pi\gamma_1 \right\}$$

gives us the distribution of the creep (retardation) parameters of the dynamic system,

$$\theta_{mn}(\tau) = \tau^2 \Omega_{mn}^2 + 1,$$

and the amplitude A_{mn} and phase φ_{mn} of vibrations are defined, respectively, as

$$A_{mn} = 2F_{mn} \left[4r_{mn}^2 + \gamma_1^2 \kappa_{mn}^2 r_{mn}^{2(\gamma_1 - 1)} \right.$$
$$\left. + 4\gamma_1 \kappa_{mn} r_{mn}^{\gamma_1} \cos(2-\gamma_1)\psi_{mn} \right]^{-1/2},$$
$$\tan\varphi_{mn} = -\frac{2r_{mn}\cos\psi_{mn} + \gamma_1 \kappa_{mn} r_{mn}^{\gamma_1 - 1} \cos(1-\gamma_1)\psi_{mn}}{2r_{mn}\sin\psi_{mn} - \gamma_1 \kappa_{mn} r_{mn}^{\gamma_1 - 1} \sin(1-\gamma_1)\psi_{mn}}$$

IV. Determination of the Contact Force

Knowing the Green functions, the solution of Eq. (2) takes the form

$$w_1(x,y,t) = \sum_{m=1}^\infty \sum_{n=1}^\infty \sin\left(\frac{m\pi}{a}\, x\right) \sin\left(\frac{n\pi}{b}\, y\right)$$
$$\times \int_0^t G_{mn}(t-t') P(t') dt'. \tag{16}$$

Let us introduce the value characterizing the relative approach of the sphere and plate, i.e., penetration of the elastic plate by the elastic sphere, is

$$y(t) = w_2(t) - w_1\left(\frac{a}{2}, \frac{b}{2}, t\right), \tag{17}$$

which is connected with the contact force by the generalized Hertzian law

$$P(t) = \widetilde{k} y^{3/2}, \tag{18}$$

where

$$\widetilde{k} = \frac{4}{3}\sqrt{R} E' \tag{19}$$

is the operator involving the geometry and viscoelastic features of the impactor and elastic features of the target, which are described due to the Volterra corresponding principle by the operator E'

$$\frac{1}{E'} = J' = \frac{1-\nu_1^2}{E_1} + \frac{1-\widetilde{\nu}_2^2}{\widetilde{E}_2}, \tag{20}$$

and $\widetilde{\nu}_2$ and $\widetilde{E}_2$ are the operators for the viscoelastic sphere (impactor).

Further in order to obtain the integro-differential equation for the values $y(t)$ and $P(t)$, it is necessary to assign the form of the operator $\widetilde{E}_2$.

Assume that viscoelastic features of the impactor's material are described by the fractional derivative standard linear solid model, i.e. the operator $\widetilde{E}_2$ has the form

$$\sigma + \tau_{\varepsilon_2}^{\gamma_2} D^{\gamma_2} \sigma = E_0 \left(\varepsilon + \tau_{\sigma_2}^{\gamma_2} D^{\gamma_2} \varepsilon \right), \tag{21}$$

where γ_2 is impactor's fractional parameter, τ_{ε_2} and τ_{σ_2} are its relaxation and retardation times, respectively, and E_0 is the relaxed magnitude of the impactor's material elastic modulus.

Following Rabotnov [7] and Rossikhin and Shitikova [14], assume that the bulk modulus of the impactor's material is a constant value, i.e.,

$$\frac{\widetilde{E}_2}{1 - 2\widetilde{\nu}_2} = \frac{E_\infty}{1 - 2\nu_\infty}, \tag{22}$$

where E_∞ and ν_∞ are the nonrelaxed magnitudes of impactor's material elastic modulus and Poisson's ratio, respectively.

From (22) it could be found [14] that Poisson's time-dependent operator could be written in the form

$$\widetilde{\nu}_2 = \nu_\infty + \frac{1}{2}\left(1 - 2\nu_\infty\right) \nu_\varepsilon \ni_\gamma^* \left(\tau_{\varepsilon_2}^{\gamma_2}\right), \tag{23}$$

where $\ni_\gamma^* \left(\tau_{\varepsilon_2}^{\gamma_2}\right)$ is the dimensionless Rabotnov's fractional operator [9]

$$\ni_\gamma^* \left(\tau_{i2}^{\gamma_2}\right) = \frac{1}{1 + \tau_{i2}^{\gamma_2} D^{\gamma_2}} \quad (i = \epsilon, \sigma) \tag{24}$$

and

$$E_2^* = E_\infty \left[1 - \nu_\varepsilon \ni_\gamma^* \left(\tau_{\varepsilon_2}^{\gamma_2}\right) \right] \tag{25}$$

with

$$\nu_\sigma = \frac{J_0 - J_\infty}{J_\infty} = \frac{E_\infty - E_0}{E_0},$$
$$\nu_\varepsilon = \frac{J_0 - J_\infty}{J_0} = \frac{E_\infty - E_0}{E_\infty}, \tag{26}$$
$$\frac{\nu_\varepsilon}{\nu_\sigma} = \frac{J_\infty}{J_0} = \frac{E_0}{E_\infty} = \frac{\tau_{\varepsilon_2}^{\gamma_2}}{\tau_{\sigma_2}^{\gamma_2}},$$

where J_0 and J_∞ are, respectively, relaxed and nonrelaxed compliances.

Using the algebra of dimensionless Rabotnov's fractional operators recently developed in [9, 14], it is possible to decode the operator $(1-\widetilde{\nu}_2^2)\widetilde{E}_2^{-1}$, resulting in

$$\frac{1-\widetilde{\nu}_2^2}{\widetilde{E}_2} = \frac{1-\nu_\infty^2}{E_\infty}\left[1+\frac{(1-2\nu_\infty)^2\nu_\varepsilon}{4(1-\nu_\infty^2)}\ \ni_\gamma^*\left(\tau_{\varepsilon_2}^{\gamma_2}\right)\right. \left. +\frac{3\nu_\sigma}{4(1-\nu_\infty^2)}\ \ni_\gamma^*\left(\tau_{\sigma_2}^{\gamma_2}\right)\right]. \quad (27)$$

Substituting operator (27) in (20) yields

$$\frac{1}{E'} = \frac{1-\nu_1^2}{E_1}+\frac{1-\nu_\infty^2}{E_\infty}+\frac{(1-2\nu_\infty)^2\nu_\varepsilon}{4E_\infty}\ \ni_\gamma^*\left(\tau_{\varepsilon_2}^{\gamma_2}\right) +\frac{3\nu_\sigma}{4E_\infty}\ \ni_\gamma^*\left(\tau_{\sigma_2}^{\gamma_2}\right). \quad (28)$$

Now substituting (28) in the Hertzian contact law (18) with due account for Eqs. (6), (16) and (17), we are led to the integral equation for defining the contact force

$$\begin{aligned}&\left(\frac{3}{4\sqrt{R}}\right)^{2/3}\left[\left(\frac{1-\nu_1^2}{E_1}+\frac{1-\nu_\infty^2}{E_\infty}\right)P(t)\right.\\ &+\frac{(1-2\nu_\infty)^2}{4E_\infty}\int_0^t \ni_\gamma\left(-\frac{t-t'}{\tau_{\varepsilon_2}}\right)P(t')dt'\\ &\left.+\frac{3\nu_\sigma}{4E_\infty}\int_0^t \ni_\gamma\left(-\frac{t-t'}{\tau_{\sigma_2}}\right)P(t')dt'\right]^{2/3}\\ &=-\frac{1}{m}\int_0^t P(t')(t-t')dt'+V_0t\\ &-\sum_{m=1}^\infty\sum_{n=1}^\infty\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{n\pi}{2}\right)\int_0^t G_{mn}(t-t')P(t')dt' \quad (29)\end{aligned}$$

where

$$\ni_\gamma\left(-\frac{t}{\tau_{i2}}\right) = \frac{t^{\gamma_2-1}}{\tau_{i2}^{\gamma_2}}\sum_{n=0}^\infty\frac{(-1)^n(t/\tau_{i2})^{\gamma_2 n}}{\Gamma[\gamma_2(n+1)]}. \quad (30)$$

V. Defining the Local Indentation

In order to find the equation in terms of $y(t)$, it is necessary to utilize relationship (17) with due account for (6), (16), (18) and (19). Since formula (19) involves the operator $\widetilde{k}$, then for its construction it is needed to inverse the operator $\widetilde{k}^{-1}=\frac{3}{4\sqrt{R}}\,E'^{-1}$, where operator E'^{-1} is defined in (28). As it has been shown in [14], the operator $\widetilde{k}$ entering in (17) has the form

$$\begin{aligned}\widetilde{k} &= \frac{4\sqrt{R}}{3}\left(\frac{1-\nu_1^2}{E_1}+\frac{1-\widetilde{\nu}_2^2}{\widetilde{E}_2}\right)^{-1}\\ &=\frac{4\sqrt{R}}{3d}\left[1-e_1\ni_\gamma^*\left(t_1^{\gamma_2}\right)-e_2\ni_\gamma^*\left(t_2^{\gamma_2}\right)\right], \quad (31)\end{aligned}$$

where

$$\begin{aligned}t_{1,2}^{-\gamma} &= \frac{1}{2}\left[\frac{1+g_1}{\tau_{\varepsilon_2}^{\gamma_2}}+\frac{1+g_2}{\tau_{\sigma_2}^{\gamma_2}}\right]\\ &\pm\frac{1}{2}\sqrt{\left[\frac{1+g_1}{\tau_{\varepsilon_2}^{\gamma_2}}-\frac{1+g_2}{\tau_{\sigma_2}^{\gamma_2}}\right]^2+\frac{4g_1g_2}{\tau_{\varepsilon_2}^{\gamma_2}\tau_{\sigma_2}^{\gamma_2}}},\end{aligned}$$

$$d=\frac{1-\nu_1^2}{E_1}+\frac{1-\nu_\infty^2}{E_\infty},$$

$$g_1=\frac{(1-2\nu_\infty)^2\nu_\varepsilon}{4E_\infty}, \qquad g_2=\frac{3\nu_\sigma}{4E_\infty},$$

$$e_1=\frac{b_2-a_2}{a_1b_2-a_2b_1}>0, \qquad e_2=\frac{a_1-b_1}{a_1b_2-a_2b_1}>0,$$

$$a_1=\frac{t_1^{-\gamma_2}}{t_1^{-\gamma_2}-\tau_{\varepsilon_2}^{-\gamma_2}}>0, \qquad a_2=\frac{t_2^{-\gamma_2}}{t_2^{-\gamma_2}-\tau_{\varepsilon_2}^{-\gamma_2}}>0,$$

$$b_1=\frac{t_1^{-\gamma_2}}{t_1^{-\gamma_2}-\tau_{\sigma_2}^{-\gamma_2}}<0, \qquad b_2=\frac{t_2^{-\gamma_2}}{t_2^{-\gamma_2}-\tau_{\sigma_2}^{-\gamma_2}}>0.$$

Now considering (31) a nonlinear integral equation for determining the value $y(t)$ takes the form

$$\begin{aligned}y(t) &= V_0t-\frac{4\sqrt{R}}{3dm}\int_0^t\left[y^{3/2}(t')\right.\\ &\left.-\sum_{j=1}^2 e_j\int_0^{t'}\ni_\gamma\left(-\frac{t'-t''}{t_j}\right)y^{3/2}(t'')dt''\right](t-t')dt'\\ &-\frac{4\sqrt{R}}{3d}\sum_{m=1}^\infty\sum_{n=1}^\infty\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{n\pi}{2}\right)\\ &\times\int_0^t G_{mn}(t-t')\left[y^{3/2}(t')\right.\\ &\left.-\sum_{j=1}^2 e_j\int_0^{t'}\ni_\gamma\left(-\frac{t'-t''}{t_j}\right)y^{3/2}(t'')dt''\right]dt'. \quad (32)\end{aligned}$$

Since the impact process is of short duration, then

$$\ni_\gamma\left(-\frac{t}{t_j}\right)\approx\frac{t^{\gamma_2-1}}{t_j^{\gamma_2}\Gamma(\gamma_2)} \quad (j=1,2), \quad (33)$$

while the Green function $G_{mn}(t)$, which vanishes to zero at $t=0$ according to the limiting theorem

$$\lim_{p\to\infty}\bar{G}_{mn}(p)p=G(0)=0 \quad (34)$$

is represented in the form

$$G_{mn}(t)\approx tA_{mn}\omega_{mn}\cos\varphi_{mn}. \quad (35)$$

Considering (33)-(35), Eq. (32) is reduced to

$$y(t)=V_0t-\frac{4\sqrt{R}}{3d}\left(\frac{1}{m}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\omega_{mn}\cos\varphi_{mn}\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}\right)\times\int_0^t\left[y^{3/2}(t')-\sum_{j=1}^{2}\frac{e_j}{t_j^{\gamma_2}\Gamma(\gamma_2)}\int_0^{t'}(t'-t'')^{\gamma_2-1}y^{3/2}(t'')dt''\right](t-t')dt'. \tag{36}$$

A. Approximate Solution

As a first approximation for the function $y(t)$ the expression

$$y=V_0t \tag{37}$$

could be utilized. Considering (37) and relationship

$$\left(1-\frac{t''}{t'}\right)^{\gamma_2}\approx 1-\gamma_2\,\frac{t''}{t'},$$

we could calculate the integral

$$\int_0^{t'}(t'-t'')^{\gamma_2-1}y^{3/2}(t'')dt''=-\frac{V_0^{3/2}}{\gamma_2}\int_0^{t'}(t'')^{3/2}d(t'-t'')^{\gamma_2}=\frac{3V_0^{3/2}}{\gamma_2}\left(\frac{1}{3}-\frac{1}{5}\,\gamma_2\right)(t')^{3/2+\gamma_2}. \tag{38}$$

Now substituting (37) and (38) in the right-hand side of (36) yields

$$y(t)=V_0t-\frac{4}{35}\Delta_{\gamma_1}V_0^{3/2}t^{7/2}+3\Delta_{\gamma_1}\delta_{\gamma_2}V_0^{3/2}\,\frac{(1/3-1/5\gamma_2)t^{7/2+\gamma_2}}{\gamma_2(5/2+\gamma_2)(7/2+\gamma_2)}, \tag{39}$$

where

$$\Delta_{\gamma_1}=\frac{4\sqrt{R}}{3d}\left(\frac{1}{m}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\omega_{mn}\cos\varphi_{mn}\times\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}\right),$$

$$\delta_{\gamma_2}=\frac{1}{\Gamma(\gamma_2)}\sum_{j=1}^{2}\frac{e_j}{t_j^{\gamma_2}}.$$

B. The Case $\gamma_2\to 0$

Since at $\gamma_2\to 0$ the value $\sum\limits_{j=1}^{2}e_j=0$ [14], then relationship (39) is reduced to

$$y(t)=V_0t-\frac{4}{35}\,\Delta_{\gamma_1}V_0^{3/2}t^{7/2}. \tag{40}$$

Formula (40) is valid for the case of a low-velocity transverse impact of an elastic sphere upon an elastic plate in a viscoelastic medium, i.e. considering external friction of the surrounding medium. From (40) it is possible to find the contact duration by vanishing $y(t)$ to zero. As a result we obtain

$$t_{\rm cont}^{(0)}=\left(\frac{35}{4}\,\frac{1}{\Delta_{\gamma_1}\sqrt{V_0}}\right)^{2/5}. \tag{41}$$

Equating dy/dt to zero, we obtain the magnitude of the value $t=t_{\rm max}^{(0)}$ at which $y(t)$ attains its maximal value $y_{\rm max}^{(0)}$

$$t_{\rm max}^{(0)}=\left(\frac{5}{2}\,\frac{1}{\Delta_{\gamma_1}\sqrt{V_0}}\right)^{2/5}, \tag{42}$$

resulting in

$$y_{\rm max}^{(0)}=\frac{5}{7}\,V_0t_{\rm max}^{(0)}. \tag{43}$$

C. The Case $\gamma_2\neq 0$

Now we consider the case $\gamma_2\neq 0$. Assuming that in this case all characteristic values differ a little from the corresponding values at $\gamma_2=0$ yields

$$t_{\rm cont}^{(\gamma_2)}=t_{\rm cont}^{(0)}\left[1+\frac{15}{2}\,\delta_{\gamma_2}\left(t_{\rm cont}^{(0)}\right)^{\gamma_2}\times\frac{1/3-1/5\gamma_2}{\gamma_2(5/2+\gamma_2)(7/2+\gamma_2)}\right], \tag{44}$$

$$t_{\rm max}^{(\gamma_2)}=t_{\rm max}^{(0)}\left[1+3\delta_{\gamma_2}\left(t_{\rm max}^{(0)}\right)^{\gamma_2}\times\frac{1/3-1/5\gamma_2}{\gamma_2(5/2+\gamma_2)}\right], \tag{45}$$

$$y_{\rm max}^{(\gamma_2)}=y_{\rm max}^{(0)}+\frac{15}{2}\,\delta_{\gamma_2}V_0\left(t_{\rm max}^{(0)}\right)^{1+\gamma_2}\times\frac{1/3-1/5\gamma_2}{\gamma_2(5/2+\gamma_2)(7/2+\gamma_2)}. \tag{46}$$

D. The Case $\gamma_2=1$

At the limiting case $\gamma_2=1$, i.e. in the case of conventional viscosity, formulae (44)-(46) are reduced to

$$t_{\rm cont}^{(1)}=t_{\rm cont}^{(0)}\left(1+\frac{4}{63}\,\delta_1t_{\rm cont}^{(0)}\right), \tag{47}$$

$$t_{\rm max}^{(1)}=t_{\rm max}^{(0)}\left(1+\frac{4}{35}\,\delta_1t_{\rm max}^{(0)}\right), \tag{48}$$

$$y_{\rm max}^{(1)}=y_{\rm max}^{(0)}+\frac{4}{63}\,\delta_1V_0t_{\rm max}^{(0)}, \tag{49}$$

where $\delta_1=\delta_{\gamma_2}|_{\gamma_2=1}$.

Reference to the above formulas shows that with the increase in the fractional parameter γ_2 from 0 to 1 the viscosity of the impactor increases from 0 to its maximal magnitude, resulting in the increase of such characteristic values as the time of contact, the time needed the impactor's penetration to achieve its maximal magnitudes, and the maximal value of the local bearing of impactor and target's materials itself. The enumerated values increase from the magnitudes $t^{(0)}_{\rm cont}$, $t^{(0)}_{\rm max}$, $y^{(0)}_{\rm max}$ to $t^{(1)}_{\rm cont}$, $t^{(1)}_{\rm max}$, $y^{(1)}_{\rm max}$, respectively.

VI. Conclusion

In the present paper, the problem on impact of a viscoelastic spherical impactor upon an elastic Kirchhoff-Love plate in a viscoelastic medium has been formulated for the case, when the viscoelastic features within the contact domain are described by the fractional derivative standard linear solid model, while the damping features of the surrounding medium are modelled by the fractional derivative Kelvin-Voigt model. Thus, two different fractional parameters control the main characteristics of the impact interaction.

The Green function for the target was constructed, what allows us to obtain the integral equation for the contact force and local indentation using the algebra of Rabotnov's fractional operators. An approximate analytical solution has been found.

References

[1] Yu.A. Rossikhin and M.V. Shitikova, "Application of fractional calculus for dynamic problems of solid mechanics: Novel trends and recent results," *Applied Mechanics Reviews*, vol. 63(1), Article ID 010801, 2010.

[2] D. Ingman and J. Suzdalnitsky, "Response of viscoelastic plate to impact," *ASME Journal of Vibration and Acoustics*, vol. 130, Article ID 011010, 2008.

[3] N.G. Babouskos and J.T. Katsikadelis, "Nonlinear vibrations of viscoelastic plates of fractional derivative type: An AEM solution," *The Open Mechanics Journal*, vol. 4, pp. 8–20, 2010.

[4] J.T. Katsikadelis, "Nonlinear dynamic analysis of viscoelastic membranes described with fractional differential models," *Journal of Theoretical and Applied Mechanics*, vol. 50, pp. 743-753, 2012.

[5] S. Di Lorenzo, M. Di Paola, F.P. Pinnola, A. Pirrotta, "Stochastic response of fractionally damped beams," *Probabilistic Engineering Mechanics*, vol. 35, pp. 37–43, 2014.

[6] M.R. Permoon, J. Rashidinia, A. Parsa, H. Haddadpour, and R. Salehi, "Application of radial basis functions and sinc method for solving the forced vibration of fractional viscoelastic beam," *Journal of Mechanical Science and Technology*, vol. 30, pp. 3001–3008, 2016.

[7] Yu.N. Rabotnov, *Elements of Hereditary Solid Mechanics*, Nauka, Moscow 1977 (Engl. transl. by Mir Publishers, Moscow 1980).

[8] H.H. Hilton, Clarifications of certain ambiguities and failings of Poisson's ratios in linear viscoelasticity, *Journal of Elasticity*, vol. 104, pp. 303–318, 2011.

[9] Yu.A. Rossikhin, M.V. Shitikova, Features of fractional operators involving fractional derivatives and their applications to the problems of mechanics of solids, In: *Fractional Calculus: History, Theory and Applications* (R. Daou and M. Xavier, Eds), Chapter 8, Nova Science Publishers, New York, 2015, pp. 165-226.

[10] Yu.A. Rossikhin, M.V. Shitikova, and M.G. Estrada Meza, "Modeling of the impact response of a beam in a viscoelastic medium," *Applied Mathematical Sciences*, vol. 10, pp. 2471–2481, 2016.

[11] Yu.A. Rossikhin, M.V. Shitikova, and P.T. Trung, "Application of the fractional derivative Kelvin-Voigt model for the analysis of impact response of a Kirchhoff-Love plate," *WSEAS Transactions on Mathematics*, vol. 15, pp. 598–501, 2016.

[12] Yu.A. Rossikhin and M.V. Shitikova, "The analysis of the impact response of a thin plate via fractional derivative standard linear solid model", *Journal of Sound and Vibrations*, vol. 330, pp. 1985–2003, 2011.

[13] Yu.A. Rossikhin and M.V. Shitikova, "Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids", *Applied Mechanics Reviews*, vol. 50, 15–67, 1997.

[14] Yu.A. Rossikhin and M.V. Shitikova, "Two approaches for studying the impact response of viscoelastic engineering systems: An overview," *Computers and Mathematics with Applications*, vol. 66, pp. 755–773, 2013.