New results for the A Posteriori estimates of the two dimensional time dependent Navier-Stokes equation

Ghina Nassreddine and Toni Sayah

***Abstract*—**In this paper, we study the two dimensional time dependent Navier-Stockes problem. We introduce the discrete problem which is based on the implicit Euler scheme for the time discretization and the finite element method for the space discretization. We establish, by using the Gronwall lemma, an *a posteriori* error estimation with two types of errors indicators related to the discretization in time and space. The upper bounds is obtained without any restriction to the exact and numerical solutions compared to those obtained by [Bernardi & Sayah (2015)] where the numerical solution must be in a neighborhood of the exact solution providing from the application of Poussin-Rappaz theorem. This is the main advantage of the present work.

***Keywords*—**Navier-Stockes problem, finite element method, *a posteriori* estimation.

I. Introduction

Let Ω be a bounded simply connected open domain in $\mathbb{R}^2$, with a Lipschitz-continuous connected boundary $\Gamma = \partial\Omega$, and let $[0,T]$ denotes an interval in $\mathbb{R}$ where T is a positive constant. Let also $\mathbf{n}$ be the unit outward normal vector to Ω on its boundary Γ. We consider the following time-dependent Navier-Stokes system:

$$(P)\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in}]0,T[\times\Omega, \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in}[0,T]\times\Omega, \\ \mathbf{u} &= 0 \quad \text{on } [0,T]\times\Gamma, \\ \mathbf{u}(0,\mathbf{x}) &= 0 \quad \text{on } \Omega, \end{cases}$$

where $\mathbf{f}$ represents a density of body forces and the viscosity ν is a positive constant. the unknouwns are the velocity $\mathbf{u}$ and the pressure p of the fluid.

In this paper, we establish an *a posteriori* error estimate corresponding to the system (P). The idea of the *a posteriori* error estimates is based on an upper bound of the error between the exact and numerical solutions with a sum of local indicators expressed in each element of the mesh.

To get more precision and to minimize the error, the goal is to decrease this indicators by using the adaptive mesh techniques which consists to refine or coarsen some regions of the mesh.

The *a posteriori* error estimate is optimal if we can make each one of these indicators bounded by the local error of the solution around the corresponding element.

G. Nassreddine and T. Sayah, Laboratoire de Mathématiques et Applications, Unité de recherche Mathématiques et modélisation , Faculté des sciences, Université Saint-Joseph, B.P. 11-514 Riad El Solh Beyrouth 1107 2050, Liban (ghina.nassreddine@net.usj.edu.lb, toni.sayah@usj.edu.lb).

We refer for example to the books Verfürth [16] or Ainsworth and Oden [1]. For the time dependent problems, we have two types of computable error indicators, the first one being linked to the time discretization and the second one to the space discretization. We have to handle the two kinds of indicators, some times: we change the time step and in an other times we adapt the mesh. A large amount of work has been made concerning the *a posteriori* errors. We can cite for example, Ladevèze [11] for constitutive relation error estimators for time-dependent non-linear FE analysis, Verfürth [17] for the heat equation, Bernardi and Verfürth [7] for the time dependent Stokes equations, Bernardi and Süli [6] for the time and space adaptivity for the secondorder wave equation, Bergam, Bernardi and Mghazli [2] for some parabolic equations, Ern and Vohralïk [9] for estimation based on potential and flux reconstruction for the heat equation, and Bernardi and Sayah [4] for the time dependent Stokes equations with mixed boundary conditions.

In [5], Bernardi and Sayah treate the time dependent Navier-Stokes equations with mixed boundary conditions in two and three dimensions. They applicate Poussin-Rappaz Theorem [13] which consists to construct an application F such that $F(\mathbf{u}) = 0$ and $DF(\mathbf{u})$ is an isomorphism of some space X and deduce the upper bound of the error if, at each time iteration, the numerical solution is in a neighborhood of the exact solution. In this paper, we treat the only the two dimensional Navier-Stokes problem and establish a *a posteriori* error estimate without any restriction on the exact and numerical solutions by using the continuous and discrete Gronwall Lemma, and consequently without any condition on the time and mesh steps. The mains idea becomes from [4] and [5] for the Stokes and Navier-Stokes problems but the non-linear term will be treated by using Gronwall Lemma.

The outline of the paper is as follows:

- Section 2 is devoted to the study of the continuous problem.
- In section 3, we introduce the discrete problem and we recall its main properties.
- In sections 4, 5 and 6, we study the *a posteriori* errors and derive quasi-optimal estimates.

II. Analysis pf the model

In this section, we introduce the variational problem corresponding to Problem (P) and we recall the continuous and

discrete Gronwall Lemma. We begin by some notations and definitions.
Let $\alpha = (\alpha_1, \alpha_2)$ be a couple of non negative integers and $|\alpha| = \alpha_1 + \alpha_2$. We define the partial derivative ∂^α by

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_1^{\alpha_2}}.$$

Then, for any positive integer m and number $p \geq 1$, we recall the classical Sobolev space

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega),\, \partial^\alpha v \in L^p(\Omega), \quad \forall \mid \alpha \mid\leq m\}$$

equipped with the following semi-norm and norm :

$$|v|_{m,p,\Omega} = \left\{ \sum_{|\alpha|=m} \int_\Omega |\partial^\alpha v(\mathbf{x})|^p d\mathbf{x} \right\}^{1/p}$$

$$||v||_{m,p,\Omega} = \left\{ \sum_{k \leq m} |v|^p_{k,p,\Omega} \right\}^{1/p}.$$

We denote by $H^m(\Omega) = W^{m,2}(\Omega)$. As usual, we shall omit p when $p = 2$ and denote by $(\cdot, \cdot)$ the scalar product of $L^2(\Omega)$. Let $\mathbf{v}$ be a vector valued function; we set

$$||\mathbf{v}||_{L^p(\Omega)^2} = (\int_\Omega |\mathbf{v}(\mathbf{x})|^p d\mathbf{x})^{\frac{1}{p}}.$$

In view of the boundary conditions in system (P), we thus consider the space

$$H_0^1(\Omega) = \{v \in H^1(\Omega), v\, = 0 \text{ on } \Gamma\}.$$

We denote by $X = H_0^1(\Omega)^2$ and by M the space of functions in $L^2(\Omega)$ with a zero mean-value on Ω. For any $1 \leq p < +\infty$, there exists a constant S_p only depending on Ω such that

$$\forall \mathbf{v} \in X, \quad ||\mathbf{v}||_{L^p(\Omega)^2} \leq S_p ||\mathbf{v}||_X. \tag{1}$$

Furthermore, we have the following inequality for every $\mathbf{v} \in X$

$$||\mathbf{v}||_{L^4(\Omega)^2} \leq 2^{1/4} ||\mathbf{v}||^{1/2}_{L^2(\Omega)^2} ||\mathbf{v}||^{1/2}_X. \tag{2}$$

We introduce the kernel

$$V = \big\{\mathbf{v} \in X;\ \forall q \in M,\ \int_\Omega q(\mathbf{x})\, \mathrm{div}\, \mathbf{v}(\mathbf{x})\, d\mathbf{x} = 0\big\},$$

which is a closed subspace of X and coincides with

$$V = \big\{\mathbf{v} \in X;\ \mathrm{div}\, \mathbf{v} = 0 \text{ in } \Omega\big\}.$$

As usual, for handling time-dependent problems, it is convenient to consider functions defined on a time interval $]a, b[$ with values in a separable functional space, say Y. More precisely, let $|| \cdot ||_Y$ denote the norm of Y; then for any r, $1 \leq r \leq \infty$, we define

$$L^r(a, b; Y) = \Big\{\mathbf{f} \text{ measurable in }]a, b[; \int_a^b |\mathbf{f}(t)|^r_Y dt < \infty\Big\},$$

equipped with the norm

$$||\mathbf{f}||_{L^r(a,b;Y)} = \Big(\int_a^b |\mathbf{f}(t)|^r_Y dt\Big)^{1/r},$$

with the usual modifications if $r = \infty$.

Definition II.1. *We introduce the trilinear form c defined by:*

$$c(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_\Omega (\mathbf{u} \cdot \nabla \mathbf{v})\mathbf{w}\ d\mathbf{x}.$$

Lemma II.2. *For every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in X$ we have*

$$\begin{aligned} |c(\mathbf{u}, \mathbf{v}, \mathbf{w})| &\leq ||\mathbf{u}||_{L^4(\Omega)^2} ||\mathbf{v}||_X |\mathbf{w}|_{L^4(\Omega)^2} \\ &\leq C\ ||\mathbf{u}||_X ||\mathbf{v}||_X ||\mathbf{w}||_X. \end{aligned}$$

Lemma II.3. *We assume that $\mathbf{u}, \mathbf{v} \in X$ and div $\mathbf{u} = 0$, then* $c(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0$.

We assume that $\mathbf{f}$ belongs to $L^2(0, T; X')$ where $X' = H^{-1}(\Omega)^2$ is the dual space of X, and we notice $\mathbf{u}(t) = \mathbf{u}(t, .)$. The corresponding variational formulation in $]0, T[$ to Problem (P) is the following denoted (FV):

$$\begin{cases} \text{find } \mathbf{u} \in X \text{ and } \ p \in M \text{ such that:} \\ \forall \mathbf{v} \in X, \quad \dfrac{d}{dt}(\mathbf{u}, \mathbf{v}) + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (\mathrm{div}\, \mathbf{v}, p) \\ \qquad\qquad\qquad\qquad\qquad\qquad = < \mathbf{f}, \mathbf{v} > \\ \forall q \in M, \quad (\mathrm{div}\, \mathbf{u}, q) = 0 \\ \mathbf{u}(0) = 0. \end{cases}$$

Proposition II.4. *Any solution of Problem (FV) is a solution of Problem (P) where the first two equations are satisfied in the sense of distributions. Furthermore, every solution of (II) verifies the bound*

$$||\mathbf{u}||^2_{L^\infty(0,T;L^2(\Omega)^2)} + \frac{\nu}{2} ||\mathbf{u}||^2_{L^2(0,T;X)} \leq \frac{1}{2\nu} ||\mathbf{f}||^2_{L^2(0,T;X')} \tag{3}$$

Remark II.5. *The spaces M and X satisfy a uniform inf-sup condition: There exists a constant $\beta_* > 0$ such that*

$$\forall q \in M, \quad \sup_{\mathbf{v} \in X} \frac{\displaystyle\int_\Omega q(\mathbf{x})\, div\, \mathbf{v}(\mathbf{x}) d\mathbf{x}}{||\mathbf{v}||_X} \geq \beta_* ||q||_{L^2(\Omega)}.$$

Proposition II.6. *(see [10] or [15]) For any data $\mathbf{f}$ in $L^2(0, T; X')$, Problem (FV) has a solution $(\mathbf{u}, p)$.*
In the following, we will introduce the generalized and discrete Gronwall lemma.

Lemma II.7. *(Gronwall lemma) Let ϕ a positive integrable function, ψ and y two positive piecewise continuous functions on $[a, b]$. We suppose there exists real numbers $t_i \in]a, b[, i = 1, \ldots N$ such that ψ and y are continuous in $]t_i, t_{i+1}[$. If the following bound holds*

$$\forall t \in [a, b], \qquad y(t) \leq \phi(t) + \int_a^t \psi(s) y(s) ds, \tag{4}$$

then we have following inequality

$$y(t) \leq \phi(t) + \int_a^t \phi(s)\psi(s)\ exp\Big(\int_s^t \psi(\tau) d\tau \Big) ds. \tag{5}$$

Lemma II.8. *(Discrete Gronwall Lemma) let $(y_n)_n$, $(f_n)_n$et $(g_n)_n$ be three positive sequences verifying:*

$$\forall n \geq 0, \quad y_n \leq f_n + \sum_{k=0}^{n-1} g_k y_k.$$

Then, we have:

$$\forall n \geq 0, \quad y_n \leq f_n + \sum_{k=0}^{n-1} f_k g_k \prod_{j=k+1}^{n-1} \big(1 + g_j\big) \tag{6}$$

and

$$\forall n \geq 0, \qquad y_n \leq f_n + \sum_{k=0}^{n-1} f_k g_k \; exp\big(\sum_{j=k}^{n-1} g_j\big). \tag{7}$$

III. The discrete problem

From now on, we assume that Ω is a polyhedron and that $\mathbf{f}$ belongs to $\mathcal{C}^0(0,T;X')$. In order to describe the time discretization with an adaptive choice of local time steps, we introduce a partition of the interval $[0,T]$ in two subintervals $[t_{n-1},t_n]$, $1 \leq n \leq N$, such that $0 = t_0 < t_1 < \cdots < t_N = T$. We denote by τ_n the length of $[t_{n-1},t_n]$, by τ the N-tuple $(\tau_1,\ldots,\tau_N)$, by $|\tau|$ the maximum of the τ_n, $1 \leq n \leq N$, and finally by σ_τ the regularity parameter

$$\sigma_\tau = \max_{2 \leq n \leq N} \frac{\tau_n}{\tau_{n-1}}. \tag{8}$$

In what follows, we work with a regular family of partitions, i.e. we assume that σ_τ is bounded independently of τ.
We introduce the operator π_τ (resp. $\pi_{l,\tau}$): For any Banach space Y and any function g continuous from $]0,T]$ (resp. $[0,T[$) into Y, $\pi_\tau g$ (resp. $\pi_{l,\tau} g$) denotes the step function which is constant and equal to $g(t_n)$ (resp. $g(t_{n-1})$) on each interval $]t_{n-1},t_n]$, $1 \leq n \leq N$. Similarly, with any sequence $(\phi_n)_{0 \leq n \leq N}$ in Y, we associate the step function $\pi_\tau \phi_\tau$ (resp. $\pi_{l,\tau}\phi_\tau$) which is constant and equal to ϕ_n (resp. ϕ_{n-1}) on each interval $]t_{n-1},t_n]$, $1 \leq n \leq N$.
Furthermore, for each family $(\mathbf{v}_n)_{0 \leq n \leq N}$ in Y^{N+1}, we agree to associate the function $\mathbf{v}_\tau$ on $[0,T]$ which is affine on each interval $[t_{n-1},t_n]$, $1 \leq n \leq N$, and equal to $\mathbf{v}_n$ at t_n, $0 \leq n \leq N$. More precisely, this function is equal on the interval $[t_{n-1},t_n]$ to

$$\mathbf{v}_\tau(t) = \frac{t - t_{n-1}}{\tau_n}(\mathbf{v}_n - \mathbf{v}_{n-1}) + \mathbf{v}_{n-1} = -\frac{t_n - t}{\tau_n}(\mathbf{v}_n - \mathbf{v}_{n-1}) + \mathbf{v}_n.$$

We now describe the space discretization. For each n, $0 \leq n \leq N$, let $(\mathcal{T}_{nh})_h$ be a regular family of triangulations of Ω by triangles, in the usual sense that:

- for each h, $\bar{\Omega}$ is the union of all elements of $\mathcal{T}_{nh}$;
- the intersection of two different elements of $\mathcal{T}_{nh}$, if not empty, is a vertex or a whole edge of both of them;
- the ratio of the diameter of an element κ in $\mathcal{T}_{nh}$ to the diameter of its inscribed circle is bounded by a constant independent of n and h.

As usual, h denotes the maximal diameter of the elements of all $\mathcal{T}_{nh}$, $0 \leq n \leq N$, while for each n, h_n denotes the maximal diameter of the elements of $\mathcal{T}_{nh}$. For each κ in $\mathcal{T}_{nh}$ and each nonnegative integer k, we denote by $P_k(\kappa)$ the space of restrictions to κ of polynomials with 2 variables and total degree at most k.

In what follows, $c, c', C, C', c_1, \ldots$ stand for generic constants which may vary from line to line but are always independent of h and n. From now on, we call finite element space associated with $\mathcal{T}_{nh}$ a space of functions such that their restrictions to any element κ of $\mathcal{T}_{nh}$ belong to a space of polynomials of fixed degree.

For each n and h, we associate with $\mathcal{T}_{nh}$ two finite element spaces X_{nh} and M_{nh} which are contained in X and M, respectively, and such that the following inf-sup condition holds for a constant $\beta > 0$, which is usually independent of n and h,

$$\forall q_h \in M_{nh}, \quad \sup_{\mathbf{v}_h \in X_{nh}} \frac{\displaystyle\int_\Omega q_h(\mathbf{x}) \,\mathrm{div}\, \mathbf{v}_h(\mathbf{x})\, d\mathbf{x}}{\|\mathbf{v}_h\|_X} \geq \beta ||q_h||_{L^2(\Omega)}. \tag{9}$$

Indeed, there exist many examples of finite element spaces satisfying these conditions. We give one example of them dealing with continuous discrete pressures. The velocity is discretized with the "Mini-Element"

$$X_{nh} = \big\{\mathbf{v}_h \in X;\ \forall \kappa \in \mathcal{T}_{nh},\ \mathbf{v}_h|_\kappa \in P_b(\kappa)^2\big\},$$

where the space $P_b(\kappa)$ is spanned by functions in $P_1(\kappa)$ and the bubble function on κ (for each element κ, the bubble function is equal to the product of the barycentric coordinates associated with the vertices of κ). The pressure is discretized with classical continuous finite elements of order one

$$M_{nh} = \big\{q_h \in M \cap H^1(\Omega);\ \forall \kappa \in \mathcal{T}_{nh},\ q_h|_\kappa \in P_1(\kappa)\big\}.$$

As usual, we denote by V_{nh} the kernel

$$V_{nh} = \big\{\mathbf{v}_h \in X_{nh};\ \forall q_h \in M_{nh},\ \int_\Omega q_h(\mathbf{x})\,\mathrm{div}\ \mathbf{v}_h(\mathbf{x})\, d\mathbf{x} = 0\big\}.$$

Definition III.1. *We introduce the trilinear form* d *on* X^3 *by*

$$d(\mathbf{u},\mathbf{v},\mathbf{w}) = c(\mathbf{u},\mathbf{v},\mathbf{w}) + \frac{1}{2}\int_\Omega div\, \mathbf{u}\, \mathbf{v}\mathbf{w}.$$

Remark III.2. *We have:* $d(\mathbf{u},\mathbf{v},\mathbf{v}) = 0, \forall \mathbf{u},\mathbf{v} \in X$.

The discrete problem associated with problem (FV), denoted $(FV_{n,h})$, is: Having $\mathbf{u}_h^{n-1} \in X_{n-1h}$, find $(\mathbf{u}_h^n, p_h^n) \in X_{nh} \times M_{nh}$ solution of:

$$\begin{cases} \forall \mathbf{v}_h \in X_{nh}, \dfrac{1}{\tau_n}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h^n, \nabla \mathbf{v}_h) \\ \qquad + d(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \mathbf{v}_h) - (p_h^n, \mathrm{div}\, \mathbf{v}_h) = \langle \mathbf{f}^n, \mathbf{v}_h \rangle, \\ \forall q_h \in M_{nh}, \quad (\mathrm{div}\, \mathbf{u}_h^n, q_h) = 0, \end{cases} \tag{10}$$

by assuming that $\mathbf{u}_h^0 = 0$ and taking $\mathbf{f}^n(\mathbf{x}) = \mathbf{f}(\mathbf{x}, t_n)$ for all $\mathbf{x} \in \Omega$.

We begin by showing a bound for the solution $\mathbf{u}_h^n$ of Problem $(FV_{n,h})$.

Theorem III.3. *At each time step, knowing* $\mathbf{u}_h^{n-1} \in X_{n-1h}$*, Problem* $(FV_{n,h})$ *admits a unique solution* $(\mathbf{u}_h^n, p_h^n)$ *with values in* $X_{nh} \times M_{nh}$*. This solution satisfies, for* $m = 1, \ldots, N$,

$$\begin{aligned} \frac{1}{2}||\mathbf{u}_h^m||^2_{L^2(\Omega)^2} + \frac{\nu}{2}\sum_{n=1}^m \tau_n \|\mathbf{u}_h^n\|_X^2 &\leq \frac{c^2}{\nu}||\pi_\tau \mathbf{f}||^2_{L^2(0,T;X')} \\ &\leq \frac{c'^2}{\nu}||\mathbf{f}||^2_{L^\infty(0,T;X')}. \end{aligned} \tag{11}$$

Proof. For $\mathbf{u}_h^{n-1} \in X_{n-1h}$, it is clear that Problem$(FV_{n,h})$ has a unique solution $(\mathbf{u}_h^n, p_h^n)$ as a consequence of the coerciveness of the corresponding bilinear form on $X_{nh} \times X_{nh}$ and the inf-sup condition (9). Therefore, we take $\mathbf{v}_h = \mathbf{u}_h^n$ in $(FV_{n,h})$, and we use the relation $a(a-b) = \frac{1}{2}a^2 + \frac{1}{2}(a-b)^2 - \frac{1}{2}b^2$ and inequality (1) to obtain :

$$\begin{aligned} \frac{1}{2}||\mathbf{u}_h^n||^2_{L^2(\Omega)^2} - \frac{1}{2}||\mathbf{u}_h^{n-1}||^2_{L^2(\Omega)^2} + \nu\tau_n||\mathbf{u}_h^n||^2_X \\ \leq \frac{\tau_n \varepsilon}{2}||\mathbf{f}^n||^2_{X'} + \frac{\tau_n c^2}{2\varepsilon}||\mathbf{u}_h^n||^2_X. \end{aligned}$$

We choose $\varepsilon = \dfrac{c^2}{\nu}$ and sum over $n = 1, \dots m$. We obtain :

$$\frac{1}{2}||\mathbf{u}_h^m||^2_{L^2(\Omega)^2} + \frac{\nu}{2}\sum_{n=1}^{m}\tau_n\|\mathbf{u}_h^n\|^2_X \le \sum_{n=1}^{m}\frac{\tau_n c^2}{2\nu}||\mathbf{f}^n||^2_{X'}.$$

This implies the estimates. □

IV. A POSTERIORI ERROR ANALYSIS

We now intend to prove *a posteriori* error estimates between the exact solution $(\mathbf{u}, p)$ of Problem (FV) and the numerical solution of Problem $(FV_{n,h})$. Several steps are needed for that.

A. Construction of the error indicators

We first introduce the space

$$Z_{nh} = \{\mathbf{g}_h \in L^2(\Omega)^2;\ \forall\kappa \in \mathcal{T}_{nh},\ \mathbf{g}_h|_\kappa \in P_\ell(\kappa)^2\},$$

where ℓ is usually lower than the maximal degree of polynomials in X_{nh}, and, for $1 \le n \le N$, we fix an approximation $\mathbf{f}_h^n$ of the data $\mathbf{f}^n$ in Z_{nh}.

Next, for every element κ in $\mathcal{T}_{nh}$, we denote by

- ε_κ the set of edges of κ that are not contained in $\partial\Omega$,
- Δ_κ the union of elements of $\mathcal{T}_{nh}$ that intersect κ,
- Δ_e the union of elements of $\mathcal{T}_{nh}$ that intersect the edges e,
- h_κ the diameter of κ and h_e the diameter of the edge e,
- $[\cdot]_e$ the jump through e for each edge e in an ε_κ (making its sign precise is not necessary).
- $\mathbf{n}_\kappa$ the unit outward normal vector to κ on $\partial\kappa$.

For the demonstration of the next theorems, we introduce for an element κ of $\mathcal{T}_{nh}$, the bubble function ψ_κ (resp. ψ_e for the edge e) which is equal to the product of the 3 barycentric coordinates associated with the vertices of κ (resp. of the 2 barycentric coordinates associated with the vertices of e). We also consider a lifting operator $\mathcal{L}_e$ defined on polynomials on e vanishing on ∂e into polynomials on the at most two elements κ containing e and vanishing on $\partial\kappa \setminus e$, which is constructed by affine transformation from a fixed operator on the reference element. We recall the next results from [16, Lemma 3.3].

Property IV.1. *Denoting by $P_r(\kappa)$ the space of polynomials of degree smaller than r on κ, we have*

$$\forall v \in P_r(\kappa), \qquad \begin{cases} c||v||_{0,\kappa} \le ||v\psi_\kappa^{1/2}||_{0,\kappa} \le c'||v||_{0,\kappa}, \\ |v|_{1,\kappa} \le c h_\kappa^{-1}||v||_{0,\kappa}. \end{cases}$$

Property IV.2. *Denoting by $P_r(e)$ the space of polynomials of degree smaller than r on e, we have*

$$\forall\ v \in P_r(e), \qquad c\|v\|_{0,e} \le \|v\psi_e^{1/2}\|_{0,e} \le c'\|v\|_{0,e},$$

and, for all polynomials v in $P_r(e)$ vanishing on ∂e, if κ is an element which contains e,

$$\|\mathcal{L}_e v\|_{0,\kappa} + h_e \mid \mathcal{L}_e v \mid_{1,\kappa} \le c h_e^{1/2}\|v\|_{0,e}.$$

We also introduce a Clément type regularization operator $\mathcal{C}_{nh}$ [8] which has the following properties, see [3, section IX.3]: For any function $\mathbf{w}$ in $H^1(\Omega)^2$, $\mathcal{C}_{nh}\mathbf{w}$ belongs to the continuous affine finite element space and satisfies for any κ in $\mathcal{T}_{nh}$ and e in ε_κ,

$$\begin{aligned} ||\mathbf{w} - \mathcal{C}_{nh}\mathbf{w}||_{0,\kappa} &\le c h_\kappa||\mathbf{w}||_{1,\Delta_\kappa}, \\ ||\mathbf{w} - \mathcal{C}_{nh}\mathbf{w}||_{0,e} &\le c h_e^{1/2}||\mathbf{w}||_{1,\Delta_e}. \end{aligned} \tag{12}$$

Furthermore, we introduce the Scott-Zhang operator $\mathcal{F}_h$ [14] which has the following properties : For any function $\mathbf{v} \in H^1(\Omega)^2$, we have

$$|\mathbf{v} - \mathcal{F}_h\mathbf{v}|_{m,\Omega} \le C h_n^{l-m}|\mathbf{v}|_{l,\Omega}, \tag{13}$$

where C is a constant independent of h_n, m and l are integers such that: $m = 0, 1$ and $0 \le m \le l \le 2$.

For the *a posteriori* error studies, we consider the piecewise affine function $\mathbf{u}_h$ which takes in the interval $[t_{n-1}, t_n]$ the values

$$\mathbf{u}_h(t) = \frac{t - t_{n-1}}{\tau_n}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}) + \mathbf{u}_h^{n-1}, \tag{14}$$

and the piecewise constant function p_h equal to p_h^n on the interval $]t_{n-1}, t_n]$. We prove optimal *a posteriori* error estimates by using the norm:

$$\begin{aligned} [[\mathbf{u} - \mathbf{u}_h]](t_n) = \Big(&||\mathbf{u}(t_n) - \mathbf{u}_h(t_n)||^2_{L^2(\Omega)^2} \\ &+\nu\max\Big(\int_0^{t_n}||\mathbf{u}(t) - \mathbf{u}_h(t)||^2_X dt, \\ &\sum_{m=1}^{n}\int_{t_{m-1}}^{t_m}||\mathbf{u}(t) - \pi_\tau\mathbf{u}_h(t)||^2_X dt\Big)\Big)^{1/2}. \end{aligned} \tag{15}$$

An easy calculation leads to the following lemma.

Lemma IV.3. *The solutions of Problems (FV) and $(FV_{n,h})$ verify for $t \in]t_{n-1}, t_n]$ and for all $\mathbf{v}(t)$ in X,*

$$\begin{aligned} &(\frac{\partial}{\partial t}(\mathbf{u} - \mathbf{u}_h)(t), \mathbf{v}(t)) + \nu(\nabla(\mathbf{u}(t) - \pi_\tau\mathbf{u}_h(t)), \nabla\mathbf{v}(t)) \\ &-(div\,\mathbf{v}(t), p(t) - p_h(t)) - \frac{1}{2}(div\,\pi_{l,\tau}\mathbf{u}_h(t)\pi_\tau\mathbf{u}_h(t), \mathbf{v}(t)) \\ &+(\mathbf{u}(t)\nabla\mathbf{u}(t) - \pi_{l,\tau}\mathbf{u}_h(t)\nabla\pi_\tau\mathbf{u}_h(t), \mathbf{v}(t)) \\ &= \langle\mathbf{f}(t), \mathbf{v}(t)\rangle - \frac{1}{\tau_n}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{v}(t)) \\ &-\nu(\nabla\,\pi_\tau\mathbf{u}_h(t), \nabla\,\mathbf{v}(t)) + (div\,\mathbf{v}\,(t), p_h(t)) \\ &-(\pi_{l,\tau}\mathbf{u}_h(t)\nabla\pi_\tau\mathbf{u}_h(t), \mathbf{v}(t)) - \frac{1}{2}(div\,\pi_{l,\tau}\mathbf{u}_h(t)\pi_\tau\mathbf{u}_h(t), \mathbf{v}(t)), \end{aligned} \tag{16}$$

and for all $q(t)$ in M,

$$\begin{aligned} &\int_\Omega q(t, \mathbf{x})\,div\,(\mathbf{u}(t, \mathbf{x}) - \mathbf{u}_h(t, \mathbf{x}))\,d\mathbf{x} \\ &\qquad = -\int_\Omega q(t, \mathbf{x})\,div\,\mathbf{u}_h(t, \mathbf{x})\,d\mathbf{x}. \end{aligned} \tag{17}$$

We introduce the residual $R(\mathbf{u}_h) \in L^2(0, T, X')$ given by: for t in $[t_{n-1}, t_n]$ and for all $\mathbf{v}$(t) in X

$$\begin{aligned} &\langle R(\mathbf{u}_h)(t), \mathbf{v}(t)\rangle = \langle\mathbf{f}(t), \mathbf{v}(t)\rangle - \big(\frac{\partial\mathbf{u}_h}{\partial t}(t), \mathbf{v}(t)\big) \\ &-\nu(\nabla\,\pi_\tau\mathbf{u}_h(t), \nabla\,\mathbf{v}(t)) + (\mathrm{div}\,\mathbf{v}(t), p_h(t)) \\ &-(\pi_{l,\tau}\mathbf{u}_h(t)\nabla\pi_\tau\mathbf{u}_h(t), \mathbf{v}(t)) - \frac{1}{2}(\mathrm{div}\,\pi_{l,\tau}\mathbf{u}_h(t)\pi_\tau\mathbf{u}_h(t), \mathbf{v}(t)). \end{aligned} \tag{18}$$

Using $(FV_{n,h})$, we introduce the space residual $R^h \in L^2(0, T; X')$ and the time residual $R^\tau \in L^2(0, T; X')$ such that, for all $t \in$

$[t_{n-1}, t_n]$, all $\mathbf{v}(t) \in X$ and every approximation $\mathbf{v}_h(t) \in X_{nh}$ of $\mathbf{v}(t)$, we have:

$$\begin{aligned}&\langle R(\mathbf{u}_h)(t), \mathbf{v}(t)\rangle = \langle \mathbf{f}(t) - \mathbf{f}^n, \mathbf{v}(t)\rangle \\ &+\langle \mathbf{f}^n - \mathbf{f}_h^n + R^h(\mathbf{u}_h)(t), (\mathbf{v} - \mathbf{v}_h)(t)\rangle + \langle R^\tau(\mathbf{u}_h)(t), \mathbf{v}(t)\rangle,\end{aligned} \tag{19}$$

where

$$\begin{aligned}&\langle R^h(\mathbf{u}_h)(t), \mathbf{v}(t) - \mathbf{v}_h(t)\rangle = \\ &(\mathbf{f}_h^n - \frac{1}{\tau_n}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}), \mathbf{v}(t) - \mathbf{v}_h(t)) + (\mathrm{div}\,(\ v(t) - \mathbf{v}_h(t)), p_h^n) \\ &\quad -(\mathbf{u}_h^{n-1}\nabla \mathbf{u}_h^n, \mathbf{v}(t) - \mathbf{v}_h(t)) - \nu(\nabla \mathbf{u}_h^n, \nabla(\mathbf{v}(t) - \mathbf{v}_h(t))) \\ &\quad -\frac{1}{2}(\mathrm{div}\,\mathbf{u}_h^{n-1}(t)\mathbf{u}_h^n(t), \mathbf{v}(t) - \mathbf{v}_h(t)),\end{aligned} \tag{20}$$

$$\begin{aligned}&\langle R^\tau(\mathbf{u}_h)(t), \mathbf{v}\rangle = -\nu(\nabla\,(\mathbf{u}_h(t) - \pi_\tau \mathbf{u}_h(t)), \nabla\,\mathbf{v}(t)) \\ &\quad -(\mathbf{u}_h(t)\nabla \mathbf{u}_h(t) - \pi_{l,\tau}\mathbf{u}_h(t)\nabla \pi_\tau \mathbf{u}_h(t), \mathbf{v}(t)) \\ &\quad -\frac{1}{2}(\mathrm{div}\,\mathbf{u}_h(t)\mathbf{u}_h(t) - \mathrm{div}\,\pi_{l,\tau}\mathbf{u}_h(t)\pi_\tau\mathbf{u}_h(t), \mathbf{v}(t)).\end{aligned} \tag{21}$$

Lemma IV.4. *The system (16)-(17) can be written in the following form:* $\forall(\mathbf{v}, q) \in X \times M$,

$$\begin{aligned}&(\frac{\partial}{\partial t}(\mathbf{u} - \mathbf{u}_h)(t), \mathbf{v}(t)) + \nu(\nabla\,(\mathbf{u}(t) - \mathbf{u}_h(t)), \nabla\,\mathbf{v}(t)) \\ &+(\mathbf{u}(t)\nabla \mathbf{u}(t) - \mathbf{u}_h(t)\nabla \mathbf{u}_h(t), \mathbf{v}(t)) \\ &-\frac{1}{2}(div\,\mathbf{u}_h(t)\ \mathbf{u}_h(t), \mathbf{v}(t)) - (div\,\mathbf{v}(t), p(t) - p_h(t)) \\ &\quad = \langle \mathbf{f} - \mathbf{f}^n, \mathbf{v}\rangle + \langle \mathbf{f}^n - \mathbf{f}_h^n, \mathbf{v} - \mathbf{v}_h\rangle \\ &\quad\quad +\langle R^h(\mathbf{u}_h), \mathbf{v} - \mathbf{v}_h\rangle + \langle R^\tau(\mathbf{u}_h)(t), \mathbf{v}\rangle\end{aligned} \tag{22}$$

and

$$\begin{aligned}&\int_\Omega q(t,\mathbf{x})\,div(\mathbf{u}(t,\mathbf{x}) - \mathbf{u}_h(t,\mathbf{x}))\,d\mathbf{x} = \\ &\quad -\int_\Omega q(t,\mathbf{x})\,div\,\mathbf{u}_h(t,\mathbf{x})\,d\mathbf{x}.\end{aligned} \tag{23}$$

In order to derive the upper bounds corresponding to Systems (22)-(23), we use the integration by parts formula to rewrite the residual operators $R^h(\mathbf{u}_h)(t)$ and $R^\tau(\mathbf{u}_h)(t)$ in the following forms:

$$\begin{aligned}&\langle R^h(\mathbf{u}_h)(t), \mathbf{v}(t) - \mathbf{v}_h(t)\rangle \\ &= \sum_{\kappa\in\mathcal{T}_{nh}} \Big\{ \int_\kappa (\mathbf{f}_h^n - \frac{1}{\tau_n}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}) + \nu\Delta \mathbf{u}_h^n - \mathbf{u}_h^{n-1}\nabla \mathbf{u}_h^n \\ &\quad -\frac{1}{2}\mathrm{div}\,\mathbf{u}_h^{n-1}\mathbf{u}_h^n - \nabla p_h^n)(\mathbf{x}) \times \big(\mathbf{v}(t,\mathbf{x}) - \mathbf{v}_h(t,\mathbf{x})\big)d\mathbf{x} \\ &\quad -\sum_{e\in\varepsilon_\kappa}\int_e (\nu\nabla \mathbf{u}_h^n\cdot\mathbf{n} - p_h^n\mathbf{n})(\sigma)\cdot(\mathbf{v}(t,\sigma) - \mathbf{v}_h(t,\sigma))\,d\sigma\Big\},\end{aligned} \tag{24}$$

$$\begin{aligned}&\langle R^\tau(\mathbf{u}_h)(t), \mathbf{v}(t)\rangle \\ &= \frac{t_n - t}{\tau_n}\sum_{\kappa\in\mathcal{T}_{nh}}\Big\{\nu\int_\kappa \nabla(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})(\mathbf{x})\cdot\ \nabla\,\mathbf{v}(t,\mathbf{x})\,d\mathbf{x} \\ &\quad + \int_\kappa (\mathbf{u}_h^{n-1}\nabla(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\,\mathbf{v}(t,\mathbf{x})\,d\mathbf{x} \\ &\quad + \int_\kappa \mathrm{div}\,\mathbf{u}_h^{n-1}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\,\mathbf{v}(t,\mathbf{x})\,d\mathbf{x}\Big\} \\ &-\frac{t - t_{n-1}}{\tau_n}\sum_{\kappa\in\mathcal{T}_{nh}}\Big\{\int_\kappa (\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\nabla \mathbf{u}_h(t,\mathbf{x})\,\mathbf{v}(t,\mathbf{x})\,d\mathbf{x} \\ &\quad + \int_\kappa \mathrm{div}\,(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\mathbf{u}_h(t,\mathbf{x})\,\mathbf{v}(t,\mathbf{x})\,d\mathbf{x}\Big\},\end{aligned} \tag{25}$$

where σ stands the tangential coordinates on e .
All these lead to the following definition of the error indicators:

Definition IV.5. *For each* κ *in* $\mathcal{T}_{nh}$,

$$(\eta_{n,\kappa}^\tau)^2 = \tau_n||\mathbf{u}_h^n - \mathbf{u}_h^{n-1}||_{X(\kappa)}^2, \tag{26}$$

$$\begin{aligned}(\eta_{n,\kappa}^h)^2 = h_\kappa^2||f_h^n - \frac{1}{\tau_n}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}) + \nu\Delta \mathbf{u}_h^n - \mathbf{u}_h^{n-1}\nabla \mathbf{u}_h^n \\ -\frac{1}{2}div\,\mathbf{u}_h^{n-1}\mathbf{u}_h^n - \nabla p_h^n||_{0,\kappa}^2 \\ +||div\,\mathbf{u}_h^n||_{0,\kappa}^2 + \sum_{e\in\varepsilon_\kappa} h_e||\,\nu\nabla \mathbf{u}_h^n\cdot\mathbf{n} - p_h^n\mathbf{n}||_{0,e}^2.\end{aligned} \tag{27}$$

Remark IV.6. *Even if these indicators are a little complex, each term in them is easy to compute since it only depends on the discrete solution and involves (usually low degree) polynomials.*

Lemma IV.7. *The following estimates hold for* $1 \le n \le N$,

1) *For all* $\mathbf{v}$ *in X and* $\mathbf{v}_h = \mathcal{C}_{nh}\mathbf{v}$*:*

$$\langle R^h(\mathbf{u}_h), \mathbf{v} - \mathbf{v}_h\rangle \le C\Big(\sum_{\kappa\in\mathcal{T}_{nh}}(\eta_{n,\kappa}^h)^2\Big)^{1/2}||\mathbf{v}||_X. \tag{28}$$

2) *For all* $\mathbf{v}$ *in X and* $t \in]t_{n-1}, t_n]$,

$$\begin{aligned}&\langle R^\tau(\mathbf{u}_h)(t), \mathbf{v}\rangle \le \\ &C\frac{\max\{t - t_{n-1}, t_n - t\}}{{\tau_n}^2}\Big(\sum_{\kappa\in\mathcal{T}_{nh}}(\eta_{n,\kappa}^\tau)^2\Big)^{1/2}||\mathbf{v}||_X.\end{aligned} \tag{29}$$

Proof. We proceed in two steps, one for each estimate.
1)We derive the result from formula (24) with $\mathbf{v}_h = \mathcal{C}_{nh}\mathbf{v}$, by using the Cauchy–Schwarz inequality, the proprties of $\mathcal{C}_{nh}$ and the inequality: $(ab + cd) \le (a + c)(b + d)$.
2)By considering Equation (25), using a Cauchy–Schwarz inequality and noting that both $\mathbf{u}_h^{n-1}$ and $\mathbf{u}_h$ are bounded in appropriate norms (see the proof of Theorem $III.3$), we derive (29). □

Remark IV.8. *: We note that we can not establish directly the upper bound corresponding to the system (22) and (23) due to the non-linear terms inside. Thus, we will use in the next section the Gronwall Lemma to avoid this difficulties.*

V. UPPER BOUNDS OF THE ERROR

In this section, we derive an upper bound corresponding to Problems (FV) and $(FV_{n,h})$. In fact, in [5], *a posteriori* error estimates are shoed for the time dependent Navier-Stokes equations in two and three dimensions under restrictive conditions where the discrete solution $\mathbf{u}_h$ corresponding to Problem $(FV_{n,h})$ is in a neighborhood of the exact solution $\mathbf{u}$ of Problem (FV). This restriction can be traduced by a condition on τ_n and h_n. However, in this work we show in the two dimensions, the same upper bound but without any restrictions on $\mathbf{u}_h$. The main idea is the application of Gronwall continuous and discrete lemma and requires the application of Gronwall lemma.

To prove the upper bound, we follow the idea used by Bernardi and Verfurth [7] or Bernardi and Sayah [4] for the Stokes problem in order to uncouple time and space errors. But in this work, the non linear term providing from the Navier-Stokes system requires more sophisticated calculations.

We introduce an auxiliary problem corresponding to the time discretization and calculate upper bounds for the errors between the corresponding solution and the exact solution firstly and the discrete solution secondly. Finally, we combine the obtained errors to derive the desired upper bound for the *a posteriori* error estimation. We introduce the following time semi-discrete problem: Knowing $\mathbf{u}^{n-1} \in X$, find $(\mathbf{u}^n, p^n) \in X \times M$ solution of

$$(P_{aux}) \begin{cases} \forall v \in X, \quad \frac{1}{\tau_n}(\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{v}) + \nu(\nabla \mathbf{u}^n, \nabla \mathbf{v}) \\ \qquad\qquad + (\mathbf{u}^{n-1} \nabla \mathbf{u}^n, \mathbf{v}) - (\operatorname{div} \mathbf{v}, p^n) = \langle \mathbf{f}^n, \mathbf{v} \rangle, \\ \forall q \in M, \ (\operatorname{div} \mathbf{u}^n, q) = 0. \end{cases}$$

Lemma V.1. *By assuming that* $\mathbf{u}^0 = 0$. *It is clear that Problem* (P_{aux}) *has a unique solution owing to the ellipticity of the bilinear form and the infsup condition on the form for the divergence. Furthermore, we have:*

$$\frac{1}{2}||\mathbf{u}^m||^2_{L^2(\Omega)^2} + \sum_{n=1}^{m} \tau_n \|\mathbf{u}^n\|^2_X \le \frac{c'^2}{\nu} ||\mathbf{f}||^2_{L^\infty(0,T;X')}. \tag{30}$$

We recall the definition of the piecewise affine function $\mathbf{u}_\tau$ which take in the interval $[t_{n-1}, t_n]$ the values

$$\mathbf{u}_\tau(t) = \frac{t - t_{n-1}}{\tau_n}(\mathbf{u}^n - \mathbf{u}^{n-1}) + \mathbf{u}^{n-1} = \frac{t - t_n}{\tau_n}(\mathbf{u}^n - \mathbf{u}^{n-1}) + \mathbf{u}^n, \tag{31}$$

and we define p_τ as the piecewise constant function equal to p^n on the interval $]t_{n-1}, t_n]$.

Theorem V.2. *The following a posteriori error estimate holds between the velocity* $\mathbf{u}$ *of Problem* (FV) *and the velocity* $\mathbf{u}_\tau$ *associated with the solutions* $(\mathbf{u}^n)_{0 \le n \le N}$ *of Problem* (P_{aux}): *For* $1 \le m \le N$,

$$||\mathbf{u}(t_m) - \mathbf{u}_\tau(t_m)||^2_{L^2(\Omega)^2} + \nu \int_0^{t_m} ||\mathbf{u}(s) - \mathbf{u}_\tau(s)||^2_X ds$$
$$\le C\Big(||\mathbf{f} - \pi_\tau \mathbf{f}||^2_{L^2(0,t_m;X')} + \sum_{n=1}^{m} \tau_n ||\mathbf{u}^n - \mathbf{u}^n_h||^2_X$$
$$+ \sum_{n=1}^{m} \sum_{\kappa \in \mathcal{T}_{nh}} \big(\eta^\tau_{n,\kappa})^2 \Big).$$

Proof. By combining Problems (FV) and (P_{aux}) , we observe that the pair $(\mathbf{u} - \mathbf{u}_\tau, p - p_\tau)$ satifies $(\mathbf{u} - \mathbf{u}_\tau)(0)$, and, for $t \in]t_{n-1}, t_n]$, $1 \le n \le N$ and for $(\mathbf{v}(t), q) \in X \times M$,

$$\begin{cases} (\frac{\partial}{\partial t}(\mathbf{u} - \mathbf{u}_\tau)(t), \mathbf{v}(t)) + \nu(\nabla(\mathbf{u}(t) - \mathbf{u}_\tau(t)), \nabla \mathbf{v}(t)) \\ +(\mathbf{u}(t)\nabla \mathbf{u}(t) - \mathbf{u}_\tau(t)\nabla \mathbf{u}_\tau(t), \mathbf{v}(t)) - (\operatorname{div} \mathbf{v}(t), p(t) - p_\tau(t)) \\ = (\mathbf{f}(t) - \mathbf{f}^n(t), \mathbf{v}(t)) + \langle R^\tau(\mathbf{u}_\tau)(t), \mathbf{v} \rangle. \\ \\ \int_\Omega q(t, \mathbf{x}) \operatorname{div}(\mathbf{u}(t, \mathbf{x}) - \mathbf{u}_\tau(t, \mathbf{x}))\, d\mathbf{x} = 0. \end{cases} \tag{32}$$

By taking $\mathbf{v} = \mathbf{u} - \mathbf{u}_\tau$ and $q = p - p_\tau$ in (32), we obtain:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}||\mathbf{v}(t)||^2_{L^2(\Omega)^2} + \nu||\mathbf{v}(t)||^2_X = (\mathbf{f}(t) - \mathbf{f}^n(t), \mathbf{v}(t)) \\ &+\langle R^\tau(\mathbf{u}_\tau)(t), \mathbf{v}(t)\rangle - (\mathbf{v}(t)\nabla \mathbf{u}(t), \mathbf{v}(t)). \end{aligned} \tag{33}$$

Let us check and bound the right side of equation (33). The last term can be bounded by using (2) as following:

$$\begin{aligned} (\mathbf{v}(t)\nabla \mathbf{u}(t), \mathbf{v}(t)) &\le ||\mathbf{u}(t)||_X ||\mathbf{v}(t)||^2_{L^4(\Omega)^2} \\ &\le \sqrt{2}||\mathbf{u}(t)||_X ||\mathbf{v}(t)||_{L^2(\Omega)^2} ||\mathbf{v}(t)||_X \\ &\le \frac{2}{\nu}||\mathbf{u}(t)||^2_X ||\mathbf{v}(t)||^2_{L^2(\Omega)^2} + \frac{\nu}{4}||\mathbf{v}(t)||^2_X. \end{aligned}$$

Furthermore, the residual term in the right hand side of Equation (33) can be bounded exactly as (29) and we get:

$$\begin{aligned} &\langle R^\tau(\mathbf{u}_\tau)(t), \mathbf{v}(t)\rangle \le C\frac{\max\{t - t_{n-1}, t_n - t\}}{\tau_n^2} \\ &\Big(\sum_{\kappa \in \mathcal{T}_{nh}} \tau_n ||\mathbf{u}^n - \mathbf{u}^{n-1}||^2_{X(\kappa)}\Big)^{1/2} ||\mathbf{v}(t)||_X. \end{aligned} \tag{34}$$

Thus, we integrate Equation (33) between t_{n-1} and t_n, use the above bounds and summing over n to get

$$||\mathbf{v}(t_m)||^2_{L^2(\Omega)^2} + \nu \int_0^{t_m} ||\mathbf{v}(s)||^2_X ds \le$$
$$C_1\Big(||\mathbf{f} - \pi_\tau \mathbf{f}||^2_{L^2(0,t_m;X')} + \sum_{n=1}^{m} \tau_n ||\mathbf{u}^n - \mathbf{u}^{n-1}||^2_X \Big)$$
$$+ \int_0^{t_m} (\frac{2}{\nu}||\mathbf{u}(s)||^2_X) ||\mathbf{v}(s)||^2_{L^2(\Omega)^2} ds.$$

We apply the Gronwall Lemma (II.8) with the functions given in each interval $]t_{n-1}, t_n]$ by the following form:

$$y(t_m) = ||\mathbf{v}(t_m)||^2_{L^2(\Omega)^2} + \nu \int_0^{t_m} ||\mathbf{v}(s)||^2_X ds,$$
$$\psi(s) = \frac{2}{\nu}||\mathbf{u}(s)||^2_X,$$
$$\phi(t_m) = C_1\Big(||\mathbf{f} - \pi_\tau \mathbf{f}||^2_{L^2(0,t_m;X')} + \sum_{n=1}^{m} \tau_n ||\mathbf{u}^n - \mathbf{u}^{n-1}||^2_X \Big).$$

We obtain the following bound:

$$||\mathbf{u}(t_m) - \mathbf{u}_\tau(t_m)||^2_{L^2(\Omega)^2} + \nu \int_0^{t_m} ||\mathbf{u}(s) - \mathbf{u}_\tau(s)||^2_X ds$$
$$\le C_1\Big(||\mathbf{f} - \pi_\tau \mathbf{f}||^2_{L^2(0,t_m;X')} + \sum_{n=1}^{m} \tau_n ||\mathbf{u}^n - \mathbf{u}^{n-1}||^2_X \Big)$$
$$+ \int_0^{t_m} \phi(s) \frac{2}{\nu}||\mathbf{u}(s)||^2_X \ exp\Big(\int_s^{t_m} \frac{2}{\nu}||\mathbf{u}(\tau)||^2_X d\tau \Big) ds.$$

By remarking that for every $s \le t_m$, $\phi(s) \le \phi(t_m)$, we get

$$\int_0^{t_m} \phi(s)\frac{2}{\nu}||\mathbf{u}(s)||_X^2 \; exp\Big(\int_s^{t_m} \frac{2}{\nu}||\mathbf{u}(\tau)||_X^2 d\tau\Big) ds \le \frac{2}{\nu}\phi(t_m) exp\;\Big(\int_0^{t_m} ||\mathbf{u}(\tau)||_X^2 d\tau\Big) \int_0^{t_m} ||\mathbf{u}(s)||_X^2 ds,$$

and finally, Proposition (II.4) gives us the following bound

$$||\mathbf{u}(t_m) - \mathbf{u}_\tau(t_m)||^2_{L^2(\Omega)^2} + \nu \int_0^{t_m} ||\mathbf{u}(s) - \mathbf{u}_\tau(s)||_X^2 ds \le C_2\Big(||\mathbf{f} - \pi_\tau \mathbf{f}||^2_{L^2(0,t_m;X')} + \sum_{n=1}^m \tau_n ||\mathbf{u}^n - \mathbf{u}^{n-1}||_X^2\Big).$$

By using the following triangle inequality:

$$||\mathbf{u}^n - \mathbf{u}^{n-1}||_X \le ||\mathbf{u}^n - \mathbf{u}_h^n||_X + ||\mathbf{u}_h^n - \mathbf{u}_h^{n-1}||_X + ||\mathbf{u}^{n-1} - \mathbf{u}_h^{n-1}||_X,$$

we conclude the result by summing over $n = 1, ..., m$. □

To derive an *a posteriori* estimate between the solution $\mathbf{u}$ of problem (FV) and the solution $\mathbf{u}_h$ corresponding to the solutions $\mathbf{u}_h^n$ of $(FV_{n,h})$, it suffices to get an *a posteriori* estimate between the solution $\mathbf{u}_\tau$ of Problem (P_{aux}) and $\mathbf{u}_h$ and to apply the triangle inequality using the previous theorem.
By taking the difference between the first equations of Problems (P_{aux}) and $(FV_{n,h})$, we derive the following lemma.

Lemma V.3. *For any* $\mathbf{v}$ *in* X *and* v_h *in* X_{nh},

$$\begin{aligned}&\frac{1}{\tau_n}\Big((\mathbf{u}^n - \mathbf{u}^{n-1}) - (\mathbf{u}_h^n - \mathbf{u}_h^{n-1}), \mathbf{v}\Big) + \nu\Big(\nabla(\mathbf{u}^n - \mathbf{u}_h^n), \nabla\mathbf{v}\Big) \\ &+\Big(\mathbf{u}^{n-1}\nabla\mathbf{u}^n - \mathbf{u}_h^{n-1}\nabla\mathbf{u}_h^n, \mathbf{v}\Big) - \frac{1}{2}(div\,\mathbf{u}_h^{n-1}\mathbf{u}_h^n, \mathbf{v}) \\ &-(div\,\mathbf{v}, p^n - p_h^n) = \langle \mathbf{f}^n - \mathbf{f}_h^n + R^h(\mathbf{u}_h), \mathbf{v} - \mathbf{v}_h\rangle\end{aligned} \tag{35}$$

and

$$\int_\Omega q(t,\mathbf{x}) div\,(\mathbf{u}^n - \mathbf{u}_h^n)(\mathbf{x}) d\mathbf{x} = -\int_\Omega q(t,\mathbf{x}) div\,(\mathbf{u}_h^n)(\mathbf{x}) d\mathbf{x}.$$

In order to get an *a posteriori error* estimate between the solutions $\mathbf{u}$ and $\mathbf{u}_\tau$, we introduce the operator Π (see [7] ou [4]) defined from X into itself as follow: For each $\mathbf{v}$ in X, $\Pi\mathbf{v}$ denotes the velocity $\mathbf{w}$ of the unique weak solution $(\mathbf{w}, r)$ in $X \times M$ of the Stokes problem:

$$\begin{aligned}&\forall \mathbf{t} \in X, \quad (\nabla\mathbf{w}, \nabla\mathbf{t}) - (\text{div}\,\mathbf{t}, r) = 0,\\ &\forall q \in M, \quad (\text{div}\,\mathbf{w}, q) = (\text{div}\,\mathbf{v}, q).\end{aligned} \tag{36}$$

The next lemma states some properties of the operator Π.

Lemma V.4. *The operator* Π *has the following properties:*

1) *For all* $\mathbf{v}$ *in* V, $\Pi\mathbf{v}$ *is zero,*
2) *The following estimates hold for all* $\mathbf{v}$ *in* X,

$$||\mathbf{v} - \Pi\mathbf{v}||_X \le ||\mathbf{v}||_X \quad and \quad ||\Pi\mathbf{v}||_X \le \frac{1}{\beta_*}||div\,\mathbf{v}||_{L^2(\Omega)}.$$

3) $\forall \mathbf{v}_h \in V_{nh}$ *and* $1 \le n \le N$,

$$||\Pi\mathbf{v}_h||_{L^2(\Omega)^2} \le c h_n^{1/2} ||div\,\mathbf{v}_h||_{L^2(\Omega)}.$$

Proof. The proofs of (1) and (2) can be found in ([7] or [4]).
To find the last estimate, for every $\mathbf{v}_h \in V_{nh}$, we introduce the duality argument:

$$\begin{aligned}\Delta\Phi + \nabla\rho = \Pi\mathbf{v}_h \quad &\text{in } \Omega,\\ \text{div}\,\Phi = 0 \quad &\text{in } \Omega,\\ \Phi = 0 \quad &\text{on } \partial\Omega.\end{aligned} \tag{37}$$

This problem has a unique solution $(\Phi, \rho) \in H^{3/2}(\Omega)^2 \times H^{1/2}(\Omega)$ (see [7] ou [4]). Moreover, this estimate satisfies the following relation:

$$||\Phi||_{H^{3/2}(\Omega)^2} + ||\rho||_{H^{1/2}(\Omega)} \le c||\Pi\mathbf{v}_h||_{L^2(\Omega)^2}. \tag{38}$$

By combining the last two problems , we have:

$$\begin{aligned}||\Pi\mathbf{v}_h||^2_{L^2(\Omega)^2} &= (\Pi\mathbf{v}_h, \Pi\mathbf{v}_h) = (\nabla\Phi, \nabla\Pi\mathbf{v}_h) - (\text{div}\,\Pi\mathbf{v}_h, \rho)\\ &= (\text{div}\,\Phi, r) - (\text{div}\,\Pi\mathbf{v}_h, \rho).\end{aligned}$$

As $\text{div}\,\Phi$ vanishes and $\text{div}\,\Pi\mathbf{v}_h = \text{div}\,\mathbf{v}_h$, we obtain:

$$||\Pi\mathbf{v}_h||^2_{L^2(\Omega)^2} = -(\text{div}\,\mathbf{v}_h, \rho).$$

By using the definition of V_{nh} and for all $\rho_h \in M_{nh}$,

$$||\Pi\mathbf{v}_h||^2_{L^2(\Omega)^2} = (\text{div}\,\mathbf{v}_h, \rho_h - \rho) \le ||\text{div}\,\mathbf{v}_h||_{L^2(\Omega)} ||\rho - \rho_h||_{L^2(\Omega)}.$$

By taking $\rho_h = F_h\rho, m = 0$ and $l = \frac{1}{2}$ in (13), we get:

$$\forall \rho \in H^{1/2}(\Omega), ||\rho - \rho_h||_{L^2(\Omega)} \le C h_n^{1/2} |\rho|_{H^{1/2}(\Omega)}.$$

Finally, by using the relation (38) we get the result. □

We are now in a position to prove *a posteriori* estimate between the solution $\mathbf{u}_\tau$ of Problem (P_{aux}) and the solution $\mathbf{u}_h$ of $(FV_{n,h})$.

Theorem V.5. *Suppose there exists a positive constant* C_s *such that for all* $1 \le n \le N$ *we have* $h_n \le C_s\tau_n$. *The following a posteriori error estimate holds between the solutions* $\mathbf{u}^m$ *and* $\mathbf{u}_h^m$ *of Problems* (P_{aux}) *and* $(FV_{n,h})$.

$$\begin{aligned}&||\mathbf{u}^m - \mathbf{u}_h^m||^2_{L^2(\Omega)^2} + \sum_{n=1}^m \tau_n ||\mathbf{u}^n - \mathbf{u}_h^n||_X^2 \\ &\le c\sum_{n=1}^m \tau_n \left(\sum_{\kappa \in \mathcal{T}_{nh}} \Big(h_\kappa^2 ||\mathbf{f}^n - \mathbf{f}_h^n||^2_{0,\kappa} + (\eta^h_{n,\kappa})^2\Big)\right).\end{aligned} \tag{39}$$

Proof. For abbreviation we set $\mathbf{e}^n = \mathbf{u}^n - \mathbf{u}_h^n$ and $\varepsilon^n = p^n - p_h^n, 0 \le n \le N$. For any $1 \le n \le N$, we have

$$\begin{aligned}&\frac{1}{2}||\mathbf{e}^n||^2_{L^2(\Omega)^2} - \frac{1}{2}||\mathbf{e}^{n-1}||^2_{L^2(\Omega)^2} + \frac{1}{2}||\mathbf{e}^n - \mathbf{e}^{n-1}||^2_{L^2(\Omega)}\\ &+\nu\tau_n||\mathbf{e}^n||_X^2 = (\mathbf{e}^n - \mathbf{e}^{n-1}, \mathbf{e}^n) + \nu\tau_n(\nabla\mathbf{e}^n, \nabla\mathbf{e}^n).\end{aligned}$$

By intercalating $\Pi\mathbf{e}^n$ in the both terms of the second member and noting that $\text{div}\,(\mathbf{e}^n - \Pi\mathbf{e}^n) = 0$, we obtain:

$$\begin{aligned}&(\mathbf{e}^n - \mathbf{e}^{n-1}, \mathbf{e}^n) + \nu\tau_n(\nabla\mathbf{e}^n, \nabla\mathbf{e}^n) = (\mathbf{e}^n - \mathbf{e}^{n-1}, \Pi\mathbf{e}^n)\\ &+\nu\tau_n(\nabla\mathbf{e}^n, \nabla\Pi\mathbf{e}^n) + (\mathbf{e}^n - \mathbf{e}^{n-1}, \mathbf{e}^n - \Pi\mathbf{e}^n)\\ &+\nu\tau_n(\nabla\mathbf{e}^n, \nabla(\mathbf{e}^n - \Pi\mathbf{e}^n)) - \tau_n(\text{div}\,(\mathbf{e}^n - \Pi\mathbf{e}^n), \varepsilon^n).\end{aligned} \tag{40}$$

By taking $\mathbf{v} = \mathbf{e}^n - \Pi\mathbf{e}^n$ in (35), we have for every $\mathbf{v}_h \in X_{nh}$

$$\begin{aligned}&(\mathbf{e}^n - \mathbf{e}^{n-1}, \mathbf{e}^n) + \nu\tau_n(\nabla\mathbf{e}^n, \nabla\mathbf{e}^n)\\ &= (\mathbf{e}^n - \mathbf{e}^{n-1}, \Pi\mathbf{e}^n) + \nu\tau_n(\nabla\mathbf{e}^n, \nabla\Pi\mathbf{e}^n)\\ &+\tau_n\langle\mathbf{f}^n - \mathbf{f}_h^n, \mathbf{v} - \mathbf{v}_h\rangle + \tau_n\langle R^h(\mathbf{u}_h), \mathbf{v} - \mathbf{v}_h\rangle\\ &-\tau_n(\mathbf{u}^{n-1}\nabla\mathbf{u}^n - \mathbf{u}_h^{n-1}\nabla\mathbf{u}_h^n, \mathbf{v}) + \frac{1}{2}\tau_n(\text{div}\,\mathbf{u}_h^{n-1}\mathbf{u}_h^n, \mathbf{v}).\end{aligned} \tag{41}$$

Next, we evaluate all the terms on the right-hand side separately by using the inequality $ab \le \frac{1}{4}a^2 + b^2$. Taking into account that $\Pi\mathbf{e}^n = -\Pi\mathbf{u}_h^n$ and using lemma (V.4), the first and second terms can be bounded as:

$$(\mathbf{e}^n - \mathbf{e}^{n-1}, \Pi\mathbf{e}^n) \le \frac{1}{4}||\mathbf{e}^n - \mathbf{e}^{n-1}||^2_{L^2(\Omega)^2} + c_1||\text{div}\,\mathbf{u}_h^n||^2_{L^2(\Omega)^2}$$

and

$$\nu\tau_n(\nabla\mathbf{e}^n, \nabla\Pi\mathbf{e}^n) \le \frac{\nu\tau_n}{4}||\mathbf{e}^n||_X^2 + c_1\tau_n||\text{div}\,\mathbf{u}_h^n||^2_{L^2(\Omega)^2}.$$

To estimate the third and fourth terms of (41), we take $\mathbf{v}_h = \mathcal{C}_{nh}\mathbf{v}$, use the definition of R^h and Lemma (V.4) to derive:

$$\begin{aligned}&\tau_n\langle f^n - f_h^n, \mathbf{v} - \mathcal{C}_{nh}\mathbf{v}\rangle\\ &\leq c\tau_n \sum_{\kappa\in\mathcal{T}_{nh}} h_\kappa ||\mathbf{f}^n - \mathbf{f}_h^n||_{L^2(\kappa)^2}||\mathbf{v}||_{H^1(\Delta_\kappa)^2}\\ &\leq c_2\tau_n\Big(\sum_{\kappa\in\mathcal{T}_{nh}} h_\kappa^2||\mathbf{f}^n - \mathbf{f}_h^n||^2_{L^2(\kappa)^2}\Big)^{1/2}||\mathbf{v}||_{H^1(\Omega)^2}\\ &\leq \frac{2c_3^2\tau_n}{\nu}\Big(\sum_{\kappa\in\mathcal{T}_{nh}} h_\kappa^2||\mathbf{f}^n - \mathbf{f}_h^n||^2_{L^2(\kappa)^2}\Big) + \frac{\nu\tau_n}{8}||\mathbf{e}^n||^2_X,\end{aligned}$$

and

$$\begin{aligned}\tau_n\langle R^h(\mathbf{u}_h), \mathbf{v} - \mathbf{v}_h\rangle &\leq C\tau_n\Big(\sum_{\kappa\in\mathcal{T}_{nh}}(\eta^h_{n,\kappa})^2\Big)^{1/2}||\mathbf{v}||_X\\ &\leq \frac{C^2}{\nu}\sum_{\kappa\in\mathcal{T}_{nh}}\tau_n(\eta^h_{n,\kappa})^2 + \frac{\nu\tau_n}{4}||\mathbf{e}^n||^2_X.\end{aligned}$$

Finaly, we bound the last two terms of the equation (41). We have the relation:

$$\begin{aligned}&\tau_n(\mathbf{u}^{n-1}\nabla\mathbf{u}^n - \mathbf{u}_h^{n-1}\nabla\mathbf{u}_h^n, \mathbf{v}) + \frac{\tau_n}{2}(\operatorname{div}\mathbf{u}_h^{n-1}\mathbf{u}_h^n, \mathbf{v})\\ &= \tau_n(\mathbf{e}^{n-1}\nabla\mathbf{u}^n, \mathbf{v}) + \frac{\tau_n}{2}(\operatorname{div}\mathbf{e}^{n-1}\mathbf{u}^n, \mathbf{v})\\ &\quad+\tau_n(\mathbf{u}_h^{n-1}\nabla\mathbf{e}^n, \mathbf{v}) + \frac{\tau_n}{2}(\operatorname{div}\mathbf{u}_h^{n-1}\mathbf{e}^n, \mathbf{v}).\end{aligned}$$

We denote by $A = A_1 + A_2$ where $A_1 = \tau_n(\mathbf{e}^{n-1}\nabla\mathbf{u}^n, \mathbf{v})$ and $A_2 = \frac{\tau_n}{2}(\operatorname{div}\mathbf{e}^{n-1}\mathbf{u}^n, \mathbf{v})$, and $B = \tau_n(\mathbf{u}_h^{n-1}\nabla\mathbf{e}^n, \mathbf{v}) + \frac{\tau_n}{2}(\operatorname{div}\mathbf{u}_h^{n-1}\mathbf{e}^n, \mathbf{v})$.
We bound first the term A_1 by using (2):

$$\begin{aligned}&A_1 \leq c_4\tau_n||\mathbf{e}^{n-1}||_{L^4(\Omega)}||\mathbf{u}^n||_X||\mathbf{v}||_{L^4(\Omega)}\\ &\leq c_5\tau_n||\mathbf{e}^{n-1}||^{1/2}_{L^2(\Omega)}||\mathbf{e}^{n-1}||^{1/2}_X||\mathbf{u}^n||_X\\ &\quad(||\mathbf{e}^n||_{L^4(\Omega)} + ||\Pi\mathbf{e}^n||_{L^4(\Omega)})\\ &\leq c_5\tau_n||\mathbf{e}^{n-1}||^{1/2}_{L^2(\Omega)}||\mathbf{e}^{n-1}||^{1/2}_X||\mathbf{u}^n||_X\\ &\quad\big(||\mathbf{e}^n||^{1/2}_{L^2(\Omega)}||\mathbf{e}^n||^{1/2}_X + c_6(h_n^{1/2}||\operatorname{div}\mathbf{u}_h^n||_{L^2(\Omega)})^{1/2}||\mathbf{e}^n||^{1/2}_X\big).\end{aligned}$$

We bound separately the two termes of the right hand side of last inequality.
By using the inequality $ab \leq \frac{\varepsilon}{2}a^2 + \frac{2}{\varepsilon}b^2$, intercalating $\mathbf{e}^n$, and using (30) we get:

$$\begin{aligned}&c_5\tau_n||\mathbf{e}^{n-1}||^{1/2}_{L^2(\Omega)}||\mathbf{e}^{n-1}||^{1/2}_X||\mathbf{u}^n||_X||\mathbf{e}^n||^{1/2}_{L^2(\Omega)}||\mathbf{e}^n||^{1/2}_X\\ &\leq c_5\tau_n||\mathbf{u}^n||_X\Big(\frac{\varepsilon_1}{2}||\mathbf{e}^{n-1}||_{L^2(\Omega)}||\mathbf{e}^{n-1}||_X\\ &\quad+\frac{1}{2\varepsilon_1}||\mathbf{e}^n||_{L^2(\Omega)}||\mathbf{e}^n||_X\Big)\\ &\leq c_5\tau_n\frac{\varepsilon_1}{2}\Big(\frac{\varepsilon_2}{2}||\mathbf{e}^{n-1}||^2_X + \frac{1}{2\varepsilon_2}||\mathbf{e}^{n-1}||^2_{L^2(\Omega)}||\mathbf{u}^n||^2_X\Big)\\ &\quad+c_5\frac{\tau_n}{2\varepsilon_1}\Big(\frac{\varepsilon_3}{2}||\mathbf{e}^n||^2_X + \frac{1}{2\varepsilon_3}||\mathbf{e}^n||^2_{L^2(\Omega)}||\mathbf{u}^n||^2_X\Big)\\ &\leq c_5\tau_n\frac{\varepsilon_1\varepsilon_2}{4}||\mathbf{e}^{n-1}||^2_X + c_5\tau_n\frac{\varepsilon_1}{4\varepsilon_2}||\mathbf{e}^{n-1}||^2_{L^2(\Omega)}||\mathbf{u}^n||^2_X\\ &\quad+c_5\frac{\tau_n\varepsilon_3}{4\varepsilon_1}||\mathbf{e}^n||^2_X + c_5\tau_n\frac{1}{4\varepsilon_1\varepsilon_3}||\mathbf{e}^{n-1}||^2_{L^2(\Omega)}||\mathbf{u}^n||^2_X\\ &\quad+c_7\tfrac{1}{4\varepsilon_1\varepsilon_3}||\mathbf{e}^n - \mathbf{e}^{n-1}||^2_{L^2(\Omega)}.\end{aligned}$$

Furthermore, by using the relation $h_n \leq C_s\tau_n$ and (30), we have:

$$\begin{aligned}&\tau_n||\mathbf{e}^{n-1}||^{1/2}_{L^2(\Omega)}||\mathbf{e}^{n-1}||^{1/2}_X||\mathbf{u}^n||_X(h_n^{1/2}||\operatorname{div}\mathbf{u}_h^n||_{L^2(\Omega)})^{1/2}||\mathbf{e}^n||^{1/2}_X\\ &\leq \tau_n\frac{c_8}{2\varepsilon_4}||\operatorname{div}\mathbf{u}_h^n||_{L^2(\Omega)}||\mathbf{e}^n||_X\\ &\quad+\tau_n\frac{\varepsilon_4}{2}h_n^{1/2}||\mathbf{u}^n||^2_X||\mathbf{e}^{n-1}||_{L^2(\Omega)^2}||\mathbf{e}^{n-1}||_X\\ &\leq \tau_n\frac{c_8}{2\varepsilon_4}||\operatorname{div}\mathbf{u}_h^n||_{L^2(\Omega)}||\mathbf{e}^n||_X + \tau_n\frac{\varepsilon_4}{2}(\frac{\varepsilon_5}{2}||\mathbf{e}^{n-1}||^2_X h_n||\mathbf{u}^n||^2_X\\ &\quad+\frac{1}{\varepsilon_5}||\mathbf{u}^n||^2_X||\mathbf{e}^{n-1}||^2_{L^2(\Omega)^2})\\ &\leq \tau_n\frac{c_8}{2\varepsilon_4}(\frac{1}{2\varepsilon_6}||\operatorname{div}\mathbf{u}_h^n||^2_{L^2(\Omega)} + \frac{\varepsilon_6}{2}||\mathbf{e}^n||^2_X) + \tau_n c_9\frac{\varepsilon_4\varepsilon_5}{4}||\mathbf{e}^{n-1}||^2_X\\ &\quad+\tau_n\frac{\varepsilon_4}{2\varepsilon_5}||\mathbf{u}^n||^2_X||\mathbf{e}^{n-1}||^2_{L^2(\Omega)^2}).\end{aligned}$$

Now, by using (30), we bound the term A_2 as follow:

$$\begin{aligned}&A_2 \leq c_{10}\frac{\tau_n}{2}||\mathbf{e}^{n-1}||_X||\mathbf{u}^n||_{L^4(\Omega)}||\mathbf{v}||_{L^4(\Omega)}\\ &\leq c_{10}\frac{\tau_n}{2}\Big(\frac{\varepsilon_7}{2}||\mathbf{e}^{n-1}||^2_X + c_{11}\frac{1}{2\varepsilon_7}||\mathbf{u}^n||_{L^2(\Omega)}||\mathbf{u}^n||_X(||\mathbf{e}^n||^2_{L^4(\Omega)}\\ &\qquad+||\Pi\mathbf{e}^n||^2_{L^4(\Omega)})\Big)\\ &\leq c_{10}\tau_n\frac{\varepsilon_7}{4}||\mathbf{e}^{n-1}||^2_X + c_{12}\frac{\tau_n}{4\varepsilon_7}||\mathbf{u}^n||_{L^2(\Omega)}||\mathbf{u}^n||_X||\mathbf{e}^n||_{L^2(\Omega)}||\mathbf{e}^n||_X\\ &\quad+c_{13}h_n^{1/2}\frac{\tau_n}{4\varepsilon_7}||\mathbf{u}^n||_{L^2(\Omega)}||\mathbf{u}^n||_X||\operatorname{div}\mathbf{u}_h^n||_{L^2(\Omega)}||\mathbf{e}^n||_X\\ &\leq c_{10}\tau_n\frac{\varepsilon_7}{4}||\mathbf{e}^{n-1}||^2_X + c_{14}\frac{\tau_n}{4\varepsilon_7}\Big(\frac{\varepsilon_8}{2}||\mathbf{e}^n||^2_X\\ &\quad+\frac{1}{2\varepsilon_8}||\mathbf{e}^n - \mathbf{e}^{n-1}||^2_{L^2(\Omega)}||\mathbf{u}^n||^2_X + \frac{1}{2\varepsilon_8}||\mathbf{e}^{n-1}||^2_{L^2(\Omega)}||\mathbf{u}^n||^2_X\Big)\\ &\quad+c_{13}\frac{\tau_n}{4\varepsilon_7}\Big(\frac{\varepsilon_9}{2}||\mathbf{e}^n||^2_X + \frac{h_n}{2\varepsilon_8}||\mathbf{u}^n||^2_X||\operatorname{div}\mathbf{u}_h^n||^2_{L^2(\Omega)}\Big)\\ &\leq c_{10}\tau_n\frac{\varepsilon_7}{4}||\mathbf{e}^{n-1}||^2_X + c_{14}\frac{\tau_n}{4\varepsilon_7}\Big(\frac{\varepsilon_8}{2}||\mathbf{e}^n||^2_X\\ &\quad+\frac{1}{2\varepsilon_8}||\mathbf{e}^{n-1}||^2_{L^2(\Omega)}||\mathbf{u}^n||^2_X\Big) + c_{15}\frac{1}{8\varepsilon_7\varepsilon_8}||\mathbf{e}^n - \mathbf{e}^{n-1}||^2_{L^2(\Omega)}\\ &\quad+\bar{c}_{16}\frac{\tau_n}{4\varepsilon_7}\Big(\frac{\varepsilon_9}{2}||\mathbf{e}^n||^2_X + \frac{1}{2\varepsilon_8}||\operatorname{div}\mathbf{u}_h^n||^2_{L^2(\Omega)}\Big).\end{aligned}$$

Finally, A can be bounded as:

$$\begin{aligned}A &\leq \tau_n(\frac{c_8}{4\varepsilon_4\varepsilon_6} + \frac{\bar{c_16}}{8\varepsilon_7\varepsilon_8})||\operatorname{div}\mathbf{u}_h^n||^2_{L^2(\Omega)}\\ &\quad+\tau_n(\frac{c_8\varepsilon_6}{4\varepsilon_4} + \frac{c_{14}\varepsilon_8}{8\varepsilon_7} + \frac{\bar{c_16}\varepsilon_9}{8\varepsilon_7} + \frac{c_5\varepsilon_3}{4\varepsilon_1})||\mathbf{e}^n||^2_X\\ &+\tau_n(\frac{c_9\varepsilon_4\varepsilon_5}{4} + \frac{c_{10}\varepsilon_7}{4} + \frac{c_5\varepsilon_1\varepsilon_2}{4})||\mathbf{e}^{n-1}||^2_X\\ &\quad+(\frac{c_{15}}{8\varepsilon_7\varepsilon_8} + \frac{c_7}{4\varepsilon_1\varepsilon_3})||\mathbf{e}^{n-1} - \mathbf{e}^n||^2_{L^2(\Omega)^2}\\ &+\tau_n(\frac{\varepsilon_4}{2\varepsilon_5} + \frac{c_{14}}{8\varepsilon_7\varepsilon_8} + \frac{c_5\varepsilon_1}{4\varepsilon_2} + \frac{c_5}{4\varepsilon_1\varepsilon_3})||\mathbf{u}^n||^2_X||\mathbf{e}^{n-1}||^2_{L^2(\Omega)^2}.\end{aligned} \tag{42}$$

Let us now bound the term B. It can be written as following:

$$\begin{aligned}B &= \tau_n(\mathbf{u}_h^{n-1}\nabla(\mathbf{e}^n - \Pi\mathbf{e}^n + \Pi\mathbf{e}^n), \mathbf{e}^n - \Pi\mathbf{e}^n)\\ &\quad+\frac{\tau_n}{2}(\operatorname{div}\mathbf{u}_h^{n-1}(\mathbf{e}^n - \Pi\mathbf{e}^n + \Pi\mathbf{e}^n), \mathbf{e}^n - \Pi\mathbf{e}^n)\\ &\leq \tau_n(\mathbf{u}_h^{n-1}\nabla\Pi\mathbf{e}^n, \mathbf{e}^n - \Pi\mathbf{e}^n) + \frac{\tau_n}{2}(\operatorname{div}\mathbf{u}_h^{n-1}\Pi\mathbf{e}^n, \mathbf{e}^n - \Pi\mathbf{e}^n)\\ &\leq \tau_n(\mathbf{u}_h^{n-1}\nabla\Pi\mathbf{e}^n, \mathbf{e}^n) + \frac{\tau_n}{2}(\operatorname{div}\mathbf{u}_h^{n-1}\Pi\mathbf{e}^n, \mathbf{e}^n).\end{aligned}$$

Or by using (2), (8) and (11), we have the bounds

$$\tau_n(\mathbf{u}_h^{n-1}\nabla\Pi\mathbf{e}^n,\mathbf{e}^n) \le c_{16}\tau_n||\mathbf{u}_h^{n-1}||_{L^4(\Omega)}||\text{div}\,\mathbf{u}_h^n||_{L^2(\Omega)}||\mathbf{e}^n||_{L^4(\Omega)}$$
$$\le \frac{c_{16}^2}{2\varepsilon_{10}}\tau_n||\text{div}\,\mathbf{u}_h^n||_{L^2(\Omega)}^2 + \frac{\varepsilon_{10}}{2}\tau_n||\mathbf{e}^n||_{L^4(\Omega)}^2||\mathbf{u}_h^{n-1}||_{L^4(\Omega)}^2$$
$$\le \frac{c_{16}^2}{2\varepsilon_{10}}\tau_n||\text{div}\,\mathbf{u}_h^n||_{L^2(\Omega)}^2 + c_{17}\frac{\varepsilon_{10}}{2}\tau_n||\mathbf{e}^n||_{L^2(\Omega)}||\mathbf{e}^n||_X||\mathbf{u}_h^{n-1}||_{L^2(\Omega)}||\mathbf{u}_h^{n-1}||_X$$
$$\le \frac{c_{16}^2}{2\varepsilon_{10}}\tau_n||\text{div}\,\mathbf{u}_h^n||_{L^2(\Omega)}^2 + c_{17}\frac{\varepsilon_{10}}{2}\tau_n(\frac{\varepsilon_{11}}{2}||\mathbf{e}^n||_X^2 + \frac{1}{2\varepsilon_{11}}||\mathbf{e}^n||_{L^2(\Omega)}^2||\mathbf{u}_h^{n-1}||_X^2),$$
$$\le \frac{c_{16}^2}{2\varepsilon_{10}}\tau_n||\text{div}\,\mathbf{u}_h^n||_{L^2(\Omega)}^2 + c_{17}\frac{\varepsilon_{10}}{2}\tau_n(\frac{\varepsilon_{11}}{2}||\mathbf{e}^n||_X^2 + \frac{1}{2\varepsilon_{11}}||\mathbf{e}^{n-1}||_{L^2(\Omega)}^2||\mathbf{u}_h^{n-1}||_X^2) + \bar{c}_{17}\frac{\varepsilon_{10}}{4\varepsilon_{11}}||\mathbf{e}^n - \mathbf{e}^{n-1}||_{L^2(\Omega)}^2,$$

and

$$\frac{\tau_n}{2}(\text{div}\,\mathbf{u}_h^{n-1}\Pi\mathbf{e}^n,\mathbf{e}^n)$$
$$\le \frac{\tau_n}{2}(\frac{1}{2\varepsilon_{12}}||\text{div}\,\mathbf{u}_h^{n-1}||_{L^2(\Omega)}^2 + \frac{\varepsilon_{12}}{2}||\Pi\mathbf{u}_h^n||_{L^4(\Omega)}^2||\mathbf{e}^n||_{L^4(\Omega)}^2)$$
$$\le \frac{\tau_n}{4\varepsilon_{12}}||\text{div}\,\mathbf{u}_h^{n-1}||_{L^2(\Omega)}^2 + c_{18}\tfrac{\varepsilon_{12}}{4}\tau_n||\Pi\mathbf{u}_h^n||_{L^2(\Omega)}||\Pi\mathbf{u}_h^n||_X||\mathbf{e}^n||_{L^2(\Omega)}||\mathbf{e}^n||_X$$
$$\le \frac{\tau_n}{4\varepsilon_{12}}||\text{div}\,\mathbf{u}_h^{n-1}||_{L^2(\Omega)}^2 + c_{19}h_n^{1/2}\tfrac{\varepsilon_{12}}{4}\tau_n||\text{div}\,u_h^n||_{L^2(\Omega)}||\Pi u_h^n||_X||\mathbf{e}^n||_{L^2(\Omega)}||\mathbf{e}^n||_X$$
$$\le \frac{\tau_n}{4\varepsilon_{12}}||\text{div}\,\mathbf{u}_h^{n-1}||_{L^2(\Omega)}^2 + c_{20}\tfrac{\varepsilon_{12}}{4}\tau_n(\tfrac{\varepsilon_{13}}{2}||\mathbf{e}^n|^2||_X + \frac{h_n}{2\varepsilon_{13}}||\text{div}\,\mathbf{u}_h^n||_{L^2(\Omega)}^2||\mathbf{u}_h^n||_X^2||\mathbf{e}^n||_{L^2(\Omega)}^2)$$
$$\le \frac{\tau_n}{4\varepsilon_{12}}||\text{div}\,\mathbf{u}_h^{n-1}||_{L^2(\Omega)}^2 + c_{20}\tfrac{\varepsilon_{12}}{4}\tau_n(\tfrac{\varepsilon_{13}}{2}||\mathbf{e}^n|^2||_X + \tfrac{1}{2\varepsilon_{13}}||\mathbf{u}_h^n||_X^2||\mathbf{e}^n||_{L^2(\Omega)}^2)$$
$$\le \frac{\tau_n}{4\varepsilon_{12}}||\text{div}\,\mathbf{u}_h^{n-1}||_{L^2(\Omega)}^2 + c_{20}\frac{\varepsilon_{12}}{4}\tau_n(\frac{\varepsilon_{13}}{2}||\mathbf{e}^n|^2||_X + \frac{1}{2\varepsilon_{13}}||\mathbf{u}_h^n||_X^2||\mathbf{e}^{n-1}||_{L^2(\Omega)}^2) + \frac{\bar{c}_{20}\varepsilon_{12}}{8\varepsilon_{13}}||\mathbf{e}^n - \mathbf{e}^{n-1}||_{L^2(\Omega)}^2.$$

Finally, we obtain:

$$B \le \frac{c_{16}^2}{2\varepsilon_{10}}\tau_n||\text{div}\,\mathbf{u}_h^n||_{L^2(\Omega)}^2 + \frac{\tau_n}{4\varepsilon_{12}}||\text{div}\,\mathbf{u}_h^{n-1}||_{L^2(\Omega)}^2 + (\bar{c}_{17}\frac{\varepsilon_{10}}{4\varepsilon_{11}} + \frac{\bar{c}_{20}\varepsilon_{12}}{8\varepsilon_{13}})||\mathbf{e}^n - \mathbf{e}^{n-1}||_{L^2(\Omega)}^2 + \tau_n(\frac{c_{17}\varepsilon_{10}\varepsilon_{11}}{4} + c_{20}\frac{\varepsilon_{12}\varepsilon_{13}}{8})||\mathbf{e}^n||_X^2 + \tau_n c_{20}\frac{\varepsilon_{12}}{8\varepsilon_{13}}||\mathbf{u}_h^n||_X^2||\mathbf{e}^{n-1}||_{L^2(\Omega)}^2 + \tau_n c_{17}\frac{\varepsilon_{10}}{4\varepsilon_{11}}||\mathbf{u}_h^{n-1}||_X^2||\mathbf{e}^{n-1}||_{L^2(\Omega)}^2. \quad (43)$$

Thus, by summing (42) and (43), using Equation (41) and the relation $||\text{div}\,\mathbf{u}_h^n||_{0,\kappa}^2 \le (\eta_{n,\kappa}^h)^2$, using the above bounds and (8), summing over n from 1 to m, and taking $\varepsilon_1 = 24\sqrt{c_5c_7}$, $\varepsilon_2 = \frac{1}{12\varepsilon_1 c_{20}}$, $\varepsilon_3 = \frac{\varepsilon_1}{24c_5}$, ε_4 an arbitrary real number, $\varepsilon_5 = \frac{\sigma_\tau}{24c_9\varepsilon_4}$, $\varepsilon_6 = \frac{\varepsilon_4}{24c_8}$, $\varepsilon_7 = \min(\frac{\sigma_\tau}{24c_{10}}, \frac{4\sqrt{c_{14}c_{15}}}{12c_{14}})$, $\varepsilon_8 = 4\sqrt{c_{14}c_{15}}$, $\varepsilon_9 = \frac{\varepsilon_7}{12c_5}$, $\varepsilon_{10} = \frac{1}{6\sqrt{c_{17}\bar{c}_{17}}}$, $\varepsilon_{11} = \frac{1}{4\sqrt{\bar{c}_{17}c_{17}}}$, $\varepsilon_{12} = \frac{1}{8\sqrt{c_{20}\bar{c}_{20}}}$, $\varepsilon_{13} = \sqrt{\frac{\bar{c}_{20}}{16c_{20}}}$, we get

$$||\mathbf{e}^m||_{L^2(\Omega)^2}^2 + \nu\sum_{n=1}^m \tau_n||\mathbf{e}^n||_X^2$$
$$\le C_5\sum_{n=1}^m \tau_n\Big(\sum_{\kappa\in\mathcal{T}_{nh}} \big(h_\kappa^2||\mathbf{f}^n - \mathbf{f}_h^n)||_{L^2(\kappa)^2}^2 + (\eta_{n,\kappa}^h)^2\big)\Big) + C_6\sum_{n=1}^{m-1}\tau_n||\mathbf{e}^n||_{L^2(\Omega)^2}^2\left(C_2||\mathbf{u}^n||_X^2 + C_3||\mathbf{u}_h^{n+1}||_X^2\right).$$

We apply the Gronwall Lemma (II.8) with the following functions:

$$y_m = ||\mathbf{e}^m||_{L^2(\Omega)^2}^2 + \nu\sum_{n=1}^m \tau_n||\mathbf{e}^n||_X^2,$$

$$f_m = C\sum_{n=1}^m \tau_n\Big(\sum_{\kappa\in\mathcal{T}_{nh}} \big(h_\kappa^2||\mathbf{f}^n - \mathbf{f}_h^n)||_{L^2(\kappa)^2}^2 + (\eta_{n,\kappa}^h)^2\big)\Big)$$

and

$$g_n = \tau_n\left(C_2||\mathbf{u}^n||_X^2 + C_3||\mathbf{u}_h^{n+1}||_X^2\right).$$

We obtain:

$$||\mathbf{e}^m||_{L^2(\Omega)^2}^2 + \nu\textstyle\sum_{n=1}^m \tau_n||\mathbf{e}^n||_X^2$$
$$\le C_5\sum_{n=1}^m \tau_n\Big(\sum_{\kappa\in\mathcal{T}_{nh}} \big(h_\kappa^2||\mathbf{f}^n - \mathbf{f}_h^n)||_{L^2(\kappa)^2}^2 + (\eta_{n,\kappa}^h)^2\big)\Big) + C_6\sum_{n=0}^{m-1} f_n\tau_n\left(C_2||\mathbf{u}^n||_X^2 + C_3||\mathbf{u}_h^{n+1}||_X^2\right) exp\Big(\textstyle\sum_{j=n}^{m-1} C_2||\mathbf{u}^j||_X^2 + C_3||\mathbf{u}_h^{j+1}||_X^2\Big).$$

We use the relation $f_n \le f_m, n \le m$, and the bounds (11) and (30) to get:

$$||\mathbf{e}^m||_{L^2(\Omega)^2}^2 + \nu\sum_{n=1}^m \tau_n||\mathbf{e}^n||_X^2 \le c\sum_{n=1}^m \tau_n\Big(\sum_{\kappa\in\mathcal{T}_{nh}} \big(h_\kappa^2||\mathbf{f}^n - \mathbf{f}_h^n)||_{L^2(\kappa)^2}^2 + (\eta_{n,\kappa}^h)^2\big)\Big).$$

□

Lemma V.6. *We have the bound:*

$$\frac{1}{4}\sum_{n=1}^m \tau_n||\mathbf{u}^n - \mathbf{u}_h^n||_X^2 \le \sum_{n=1}^m \int_{t_{n-1}}^{t_n} ||\mathbf{u}_\tau(s) - \mathbf{u}_h(s)||_X^2 ds \le \frac{1+\sigma_\tau}{2}\sum_{n=1}^m \tau_n||\mathbf{u}^n - \mathbf{u}_h^n||_X^2. \quad (44)$$

Proof. For the proof of this lemma, we refer to [4] page 15. □

Corollary V.7. *A posterior error estimate holds between the velocity* $\mathbf{u}$ *solution of problem (FV) and the velocity* $\mathbf{u}_h$ *corresponding to the solutions* $\mathbf{u}_h^n$ *of problem* $(FV_{n,h})$*:*

$$||\mathbf{u}(t_m) - \mathbf{u}_h^m||_{L^2(\Omega)^2}^2 + \int_0^{t_m} ||\mathbf{u}(s) - \mathbf{u}_h(s)||_X^2$$
$$\le C\Big(\textstyle\sum_{n=1}^m \sum_{\kappa\in\mathcal{T}_{nh}}(\tau_n(\eta_{n,\kappa}^h)^2 + (\eta_{n,\kappa}^\tau)^2) + \displaystyle\sum_{n=1}^m \tau_n\sum_{\kappa\in\mathcal{T}_{nh}} h_\kappa^2||\mathbf{f}^n - \mathbf{f}_h^n||_{0,\kappa}^2 + ||\mathbf{f} - \pi_\tau\mathbf{f}||_{L^2(0,t_m,X')}^2\Big). \quad (45)$$

Proof. The proof is a direct consequence of Theorems (V.2) and (V.5). First, we use the triangle inequality:

$$||\mathbf{u}(t_m) - \mathbf{u}_h^m||^2_{L^2(\Omega)^2} + \int_0^{t_m} ||\mathbf{u}(s) - \mathbf{u}_h(s)||^2_X ds$$
$$\leq 2||\mathbf{u}(t_m) - \mathbf{u}_\tau(t_m)||^2_{L^2(\Omega)^2}$$
$$+2\int_0^{t_m} ||\mathbf{u}(s) - \mathbf{u}_\tau(s)||^2_X ds + 2||\mathbf{u}_\tau(t_m) - \mathbf{u}_h(t_m)||^2_{L^2(\Omega)^2}$$
$$+2\int_0^{t_m} ||\mathbf{u}_\tau(s) - \mathbf{u}_h(s)||^2_X ds.$$

For the two first terms of second member, we use Theorem (V.2) to obtain:

$$||\mathbf{u}(t_m) - \mathbf{u}_h^m||^2_{L^2(\Omega)^2} + \int_0^{t_m} ||\mathbf{u}(s) - \mathbf{u}_h(s)||^2_X ds$$
$$\leq 2c\Big(\sum_{n=1}^m \sum_{\kappa\in\mathcal{T}_{nh}} (\eta^\tau_{n,\kappa})^2 + \sum_{n=1}^m \tau_n||\mathbf{u}^n - \mathbf{u}_h^n||^2_X$$
$$+||\mathbf{f} - \pi_\tau\mathbf{f}||^2_{L^2(0,t_m,X')}\Big) + 2||\mathbf{u}_\tau(t_m) - \mathbf{u}_h(t_m)||^2_{L^2(\Omega)^2}$$
$$+2\int_0^{t_m} ||\mathbf{u}_\tau(s) - \mathbf{u}_h(s)||^2_X ds.$$

Second, the fact that $\mathbf{u}_\tau - \mathbf{u}_h$ is piecewise affine equal to $\mathbf{u}^n - \mathbf{u}_h^n$ on t_n, gives by using Lemma V.6:

$$||\mathbf{u}(t_m) - \mathbf{u}_h^m||^2_{L^2(\Omega)^2} + \int_0^{t_m} ||\mathbf{u}(s) - \mathbf{u}_h(s)||^2_X ds$$
$$\leq 2c\Big(\sum_{n=1}^m \sum_{\kappa\in\mathcal{T}_{nh}} (\eta^\tau_{n,\kappa})^2 + \sum_{n=1}^m \tau_n||\mathbf{u}^n - \mathbf{u}_h^n||^2_X$$
$$+||\mathbf{f} - \pi_\tau\mathbf{f}||^2_{L^2(0,t_m,X')}\Big) + 2||\mathbf{u}^m - \mathbf{u}_h^m||^2_{L^2(\Omega)^2}$$
$$+2\frac{1+\sigma_\tau}{2}\sum_{n=1}^m \tau_n||\mathbf{u}^n - \mathbf{u}_h^n||^2_X.$$

We use Theorem (V.5) for the last two terms of this inequality to obtain the result. □

Next, we bound the function

$$\frac{\partial}{\partial t}(\mathbf{u} - \mathbf{u}_h) + \big(\mathbf{u}\nabla\mathbf{u} - \pi_{l,\tau}\mathbf{u}_h\nabla\pi_\tau\mathbf{u}_h, \mathbf{v}\big)$$
$$-\frac{1}{2}(\mathrm{div}\,\pi_{l,\tau}\mathbf{u}_h\pi_\tau\mathbf{u}_h, v) + \nabla(p - p_h).$$

Theorem V.8. *The following a posteriori error estimate holds between the solution ($\mathbf{u}$,p) of Problem (FV) and $\big(\mathbf{u}_h, \pi_\tau p_\tau\big)$ associated with the solutions of Problem $(FV_{n,h})$: For $1 \leq n \leq N$,*

$$||\frac{\partial}{\partial t}(\mathbf{u} - \mathbf{u}_h) + \big(\mathbf{u}\nabla\mathbf{u} - \pi_{l,\tau}\mathbf{u}_h\nabla\pi_\tau\mathbf{u}_h, \mathbf{v}\big) - \frac{1}{2}(div\,\pi_{l,\tau}\mathbf{u}_h\pi_\tau\mathbf{u}_h, \mathbf{v})$$
$$+\nabla(p - p_h)||_{L^2(0,t_m,X')}$$
$$\leq C\Big(\sum_{n=1}^m \sum_{\kappa\in\mathcal{T}_{nh}} (\tau_n(\eta^h_{n,\kappa})^2 + (\eta^\tau_{n,\kappa})^2)$$
$$+\sum_{n=1}^m \sum_{\kappa\in\mathcal{T}_{nh}} \tau_n h_\kappa^2||\mathbf{f}^n - \mathbf{f}_h^n||^2_{0,\kappa} + ||\mathbf{f} - \pi_\tau\mathbf{f}||^2_{L^2(0,t_m,X')}\Big). \tag{46}$$

Proof. The proof of this theorem follows exactly the same steps of Theorem 4.10 in [5]. □

To conclude the upper bound, we bound the quantity $\sum_{n=1}^m \int_{t_{m-1}}^{t_m} ||\mathbf{u}(t) - \pi_\tau\mathbf{u}_h(t)||^2_X dt.$

Theorem V.9. *The following a posteriori error estimate holds between the velocity $\mathbf{u}$ solution of Problem (FV) and the velocity $\mathbf{u}_h$ corresponding to the solutions $\mathbf{u}_h^n$ of Problem $(FV_{n,h})$:*

$$\sum_{n=1}^m \int_{t_{n-1}}^{t_n} ||\mathbf{u}(s) - \pi_\tau\mathbf{u}_h(s)||^2_X ds$$
$$\leq c\left(\int_0^{t_m} ||\mathbf{u}(s) - \mathbf{u}_h(s)||^2_X ds + \sum_{n=1}^m \sum_{\kappa\in\tau_{nh}} (\eta^\tau_{n,\kappa})^2\right). \tag{47}$$

Proof. For the proof of this lemma, we refer to Theorem 4.10 in [4]. □

Corollary V.10. *The pression and the velocity verify the following a posteriori error:*

$$[[\mathbf{u} - \mathbf{u}_h]]^2(t_m) + ||\frac{\partial}{\partial t}(\mathbf{u} - \mathbf{u}_h) + \mathbf{u}\nabla\mathbf{u} - \pi_{l,\tau}\mathbf{u}_h\nabla\pi_\tau\mathbf{u}_h$$
$$-\frac{1}{2}div\,\pi_{l,\tau}\mathbf{u}_h\pi_\tau\mathbf{u}_h + \nabla(p - p_h)||_{L^2(0,t_m,X')}$$
$$\leq C\big(\sum_{n=1}^m \sum_{\kappa\in\mathcal{T}_{nh}} (\tau_n(\eta^h_{n,\kappa})^2 + (\eta^\tau_{n,\kappa})^2)$$
$$+\sum_{n=1}^m \sum_{\kappa\in\mathcal{T}_{nh}} \tau_n h_\kappa^2||\mathbf{f}^n - \mathbf{f}_h^n||^2_{0,\kappa} + ||\mathbf{f} - \pi_\tau\mathbf{f}||^2_{L^2(0,t_m,X')}\big). \tag{48}$$

Proof. We start form the definition of $[[\mathbf{u} - \mathbf{u}_h]]^2(t_n)$:

$$[[\mathbf{u} - \mathbf{u}_h]]^2(t_m) \leq ||\mathbf{u}(t_m) - \mathbf{u}_h(t_m)||^2_{L^2(\Omega)^2}$$
$$+\nu\max\Big(\int_0^{t_m} ||\mathbf{u}(t) - \mathbf{u}_h(t)||^2_X dt,$$
$$\textstyle\sum_{n=1}^m \int_{t_{n-1}}^{t_n} ||\mathbf{u}(t) - \pi_\tau\mathbf{u}_h(t)||^2_X dt\Big).$$

By using (47) and the definition of $\mathbf{u}_h$, we obtain:

$$[[\mathbf{u} - \mathbf{u}_h]]^2(t_m) \leq ||\mathbf{u}(t_m) - \mathbf{u}_h(t_m)||^2_{L^2(\Omega)^2}$$
$$+\nu\max\Big(\int_0^{t_m} ||\mathbf{u}(t) - \mathbf{u}_h(t)||^2_X dt, \sum_{n=1}^m \sum_{\kappa\in\tau_{nh}} (\eta^\tau_{n,\kappa})^2\Big).$$

By using the corollary (V.7) and the equation (46), we get the result. □

VI. Upper bounds of the indicators

In this section, we prove the upper bounds of the indicators. We follow exactly the same steps of Theorems 4.11 and 4.12 in [5] by simply changing the form of the non-linear terms.

Theorem VI.1. *The following estimate holds*

$$\tau_n(\eta^h_{n,\kappa})^2 \leq c\Big(\nu||\mathbf{u} - \mathbf{u}_h^n||_{L^2(t_{n-1},t_n,X(w_\kappa))}$$
$$+||\mathbf{f} - \mathbf{f}^n||^2_{L^2(t_{n-1},t_n,X(w_\kappa)')} + \tau_n h_\kappa^2||\mathbf{f}^n - \mathbf{f}_h^n||^2_{0,w_\kappa}$$
$$+||\frac{\partial}{\partial t}(\mathbf{u} - \mathbf{u}_h) + \mathbf{u}\nabla\mathbf{u} - \pi_{l,\tau}\mathbf{u}_h\nabla\pi_\tau\mathbf{u}_h$$
$$-\frac{1}{2}div\,\pi_{l,\tau}\mathbf{u}_h\pi_\tau\mathbf{u}_h + \nabla(p - p_h)||_{L^2(t_{n-1},t_n,X(w_\kappa)')}\Big), \tag{49}$$

where w_κ denotes the union of the elements of τ_{nh} that share at least a face with κ.

Theorem VI.2. *We have the following estimate:*

$$(\eta^{\tau}_{n,\kappa})^2 \le c\Big(||\mathbf{u}-\mathbf{u}_h||^2_{L^2(t_{n-1},t_n,X(\kappa))} + ||\mathbf{u}-\pi_\tau\mathbf{u}_h||^2_{L^2(t_{n-1},t_n,X(\kappa))}\Big). \quad (50)$$

We have proved that the pressure and the velocity verify the upper bound:

$$\begin{aligned} &||\mathbf{u}-\mathbf{u}_h||^2_{L^\infty(0,t_m,L^2(\Omega)^2)} + \int_0^{t_m} ||\mathbf{u}(s)-\mathbf{u}_h(s)||^2_X ds \\ &+ \int_0^{t_m} ||\mathbf{u}-\pi_\tau\mathbf{u}_h||^2_X ds + ||\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h) + \mathbf{u}\nabla\mathbf{u} \\ &-\pi_{l,\tau}\mathbf{u}_h\nabla\pi_\tau\mathbf{u}_h - \frac{1}{2}\mathrm{div}\,\pi_{l,\tau}\mathbf{u}_h\pi_\tau\mathbf{u}_h + \nabla(p-p_h)||_{L^2(0,t_m,X')} \\ &\le C\Big(\sum_{n=1}^{m}\sum_{\kappa\in\tau_{n,\kappa}} \big(\tau_n(\eta^h_{n,\kappa})^2 + (\eta^\tau_{n,\kappa})^2\big) \\ &\quad + \sum_{n=1}^{m}\sum_{\kappa\in\tau_{n,\kappa}} \tau_n h_\kappa^2 ||\mathbf{f}^n - \mathbf{f}^n_h||^2_{0,\kappa} + ||\mathbf{f}-\pi_\tau\mathbf{f}||^2_{L^2(0,t_m,X')}\Big), \end{aligned} \quad (51)$$

where C is a positive constant. On the other hand, the lower bounds follow from (49) and (50).

We observe the estimate (51) is optimal: Up to the terms involving the data, the full error is bounded from above and from below by a constant times the sum indicators. Estimates (49) and (50) are local in space and local in time. The indicator $\eta^\tau_{n,\kappa}$ can be interpreted as a measure for the error of the time discretization. Correspondingly, they can be used for controlling the step-size in times. On the other hand, the other indicator $\eta^h_{n,\kappa}$ can be viewed as a measure for the error of the space discretization and can be used to adapt the mesh size in the space. We refer to[[6] section 6] for the detailed description for a simple adaptivity strategy relying on similar estimates.

REFERENCES

[1] M. AINSWORTH & J. T. ODEN, A posteriori error estimation in finite element analysis, *Pure and Applied Mathematics (New York)*, Wiley-Interscience [John Wiley & Sons], New York, 2000.

[2] A. BERGAM, C. BERNARDI & Z. MGHAZLI, A posteriori analysis of the finite element discretization of some parabolic equations, *Math. Comp.*, vol. 74, no. 251, pp. 1117-1138, 2005.

[3] C. BERNARDI, Y. MADAY & F. RAPETTI, Discrétisations variationnelles de problèmes aux limites elliptiques, *Collection "Mathématiques et Applications"*, vol. 45, Springer-Verlag, 2004.

[4] C. BERNARDI & T. SAYAH, A posteriori error analysis of the time dependent Stokes equations with mixed boundary conditions, *IMA J Numer Anal*, vol. 35, no. 1, pp 179-198, 2015.

[5] C. BERNARDI & T. SAYAH, A posteriori error analysis of the time dependent Navier-Stokes equations with mixed boundary conditions, *SeMA Journal*, vol. 69, no. 1, pp 1-23, 2015.

[6] C. BERNARDI & E. SÜLI, Time and space adaptivity for the second–order wave equation, *Math. Models and Methods in Applied Sciencesn*, vol. 15, pp. 199–225, 2005.

[7] C. BERNARDI & R. VERFÜRTH, A posteriori error analysis of the fully discretized time-dependent Stokes equations, *Math. Model. and Numer. Anal.*, vol. 38, pp. 437-455, 2004.

[8] P. CLÉMENT, Approximation by finite element functions using local regularisation, *R.A.I.R.O. Anal. Numer.*, vol. 9, pp.77-84, 1975.

[9] A. ERN & M. VORALIÏK, A posteriori error estimation based on potential and flux reconstruction for the heat equation, *SIAM J. Numer. Anal.*, vol. 48, no. 1, pp. 198-223, 201).

[10] V. GIRAULT AND P.A RAVIART, Finite Element Methods for the Navier-Stockes Equations Theory and Algorithms., *In Springer Series in Computational Mathematics 5*, Springer-Verlag, Berlin, Heideberg, New York, 1979.

[11] P. LADEVZE, CONSTITUTIVE RELATION ERROR ESTIMATORS FOR TIME-DEPENDENT NONLINEAR FE ANALYSIS, *Comput. Methods Appl. Mech. Engrg.*, vol. 188, no. 4, 2000, (775-788. IV WCCM, Part II (Buenos Aires, 1998)).

[12] J.-L. LIONS, Quelques méthodes de résolution des problèmes aux limites non linéaires., *Dunod, Paris*, 1969.

[13] J. POUSIN & J. RAPPAZ, Consistency, stability, a priori and a posteriori errors for Petrov-Galerkin methods applied to nonlinear problems, *Numer. Math.*, vol. 69, pp. 213–231, 1994.

[14] L.R. SCOTT & S. ZHANG, Finite element interpolation of nonsmooth functions satisfying boundary conditions. Math Comp., vol. 5, pp. 483-493, 1990.

[15] R. TEMAM, Theory and Numerical Analysis of the Naiver-Stockes Equations, *North-Holland*, 1977.

[16] R. VERFÜRTH, A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques, *Wiley and Teubner Mathematics*, 1996.

[17] R. VERFÜRTH, A posteriori error estimates for finite element discretizations of the heat equation, *Calcolo*, vol. 40, no. 3, pp. 195-212, 2003.