Natural-frequency Analysis of Laminated Composite Shell

E. Kormanikova

Abstract—The paper deals with a numerical approach of mode-frequency analysis of a simply-supported laminated doubly curved shell. For laminated shell the first-order shear deformation theory is capable of accurately predicting the shell behaviour. Transverse displacement field leads to using shear correction factor. Theory is also based on the assumption that the thickness to radius ratio of shell is small compared to unity and hence negligible. The governing equations are derived in orthogonal curvilinear coordinates and then these equations are reduced to those of doubly curved shell. Linear layered structural shell elements are used in FEM analysis. The natural frequencies of laminated orthotropic doubly curved shell with simply supported ends are solved. The numerical analysis is conducted to determine the effect of symmetry with respect to mid-plane, fibre orientation and width-to-thickness ratio to change of resonant frequencies.

Keywords—Vibration Analysis, Laminated Composite Shell, FEM analysis.

I. INTRODUCTION

Composite materials like fibre reinforced plastics are often used in fields like automotive, aerospace, and civil engineering [1,2,3]. Composites are most often used in lightweight structures where the laminated shells tend to be thin with respect to their in-plane extensions. Layered shell models are used more and more in structural analysis with new material systems. The large amount of literature in this field indicates how many different problems and mechanical situations are addressed by shell analysis. Wung [4] presented a continuum-based shell element with transverse deformation. The element is based on first-order shear deformation theory (FSDT) and fourth-order transverse deformation. Whitney and Pagano [5] developed a Mindlin-type FSDT for multi-layered anisotropic plates. Similar classical laminate theory (CLT) and FSDTs are developed for multi-layered shells [6,7]. The free vibration investigation of simply supported sandwich plate is presented in [8]. The static and dynamic analyzes of single- and multi-layered plates and shells are investigated in [9-12]. Amabili and Reddy [13] worked on the use of higher order shear deformation nonlinear theory for shells of generic shape, taking geometric imperfections into account. They found that results were obtained by keeping non-linear terms of the Von Karman type for amplitudes of about two times the shell thickness.

II. ANALYTICAL ANALYSIS

Laminate shells can be also modelled as two-dimensional structural elements but with single or double curved reference surfaces (Fig. 1). Figure 1 shows a laminated doubly curved panel of rectangular platform, of total thickness $h$. The coordinates $x_1$ and $x_2$ represents the directions of the lines of curvature of the middle surface, while the $x_3$ - axis is a straight line perpendicular to the middle surface (Fig. 2). $R_i$ ($i=1, 2$) denotes the principal radii of curvature of the middle surface.

The displacement field, based on first-order shear deformation theory, is given by

$$u_1 = (1 + x_1 / R_1) \tilde{u}_1 + x_3 \frac{\partial u_1}{\partial x_1}$$

$$u_2 = (1 + x_1 / R_2) \tilde{u}_2 + x_3 \frac{\partial u_2}{\partial x_2}$$

$$u_3 = \tilde{u}_3$$

in which $u_i$ $(i = 1, 2, 3)$ represents the components of displacement at a point $x_i$ $(i = 1, 2, 3)$, while $\tilde{u}_i$ denotes the same for the corresponding point at the mid-surface.

Assumptions of shallowness, vanishing geodesic curvatures, transverse inextensibility and the strain displacement relations for a double curved shell, based on first-order deformation theory, are given by

$$\varepsilon_1 = \varepsilon_i + x_3 \kappa_1$$

$$\varepsilon_2 = \frac{\partial u_2}{\partial x_1} + x_3 \kappa_2$$

$$\varepsilon_3 = \frac{\partial u_3}{\partial x_1}$$

where

$$\varepsilon_i = \frac{\partial u_i}{\partial x_1} + \frac{u_i}{R_i}$$

$$\varepsilon_2 = \frac{\partial u_2}{\partial x_1} + \frac{u_2}{R_2}$$

$$\varepsilon_3 = \frac{\partial u_3}{\partial x_1} - \frac{u_3}{R_1}$$

$$\kappa_1 = \frac{\partial^2 u_1}{\partial x_1^2}$$

$$\kappa_2 = \frac{\partial^2 u_2}{\partial x_1 \partial x_2}$$

$$\kappa_3 = 2 \frac{\partial^2 u_3}{\partial x_1 \partial x_2} - \frac{1}{2} \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \left( \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_1} \right)$$

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It is assumed that the transverse normal stress is negligible in general, it is verified that $\sigma_t$ is small compared to $\tau_{11}$ and $\tau_{12}$, except near the shell edges, so that the hypothesis is a good approximation of the actual behavior of moderately thick shells. The stress–strain relations for the $n$-th orthotropic lamina of the shell, in the material principal coordinates are obtained under the hypothesis $\sigma_t = 0$.

$$
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_6
\end{bmatrix}^{(n)} =
\begin{bmatrix}
\sigma_{11}^{(n)} & \sigma_{12}^{(n)} & 0 & 0 & 0 \\
\sigma_{21}^{(n)} & \sigma_{22}^{(n)} & 0 & 0 & 0 \\
0 & 0 & G_{12}^{(n)} & 0 & 0 \\
0 & 0 & 0 & G_{11}^{(n)} & 0
\end{bmatrix}
\begin{bmatrix}
e_1^{(n)} \\
e_2^{(n)} \\
e_4^{(n)} \\
e_6^{(n)}
\end{bmatrix}
$$

where $G_{23}$, $G_{13}$, $G_{12}$ are the shear moduli, the superscript ($n$) refers to the $n$-th layer within a laminate. Eq. (5) is obtained under the transverse isotropy assumption with respect to planes orthogonal to axis 1, so that $E_1 = E_3$, $\nu_{12} = \nu_{13}$ and $G_{12} = G_{13}$.

The internal forces can be written in following form

$$
\begin{bmatrix}
N_1 \\
N_2 \\
N_6
\end{bmatrix} =
\begin{bmatrix}
M_1 \\
M_2 \\
M_6
\end{bmatrix},
V =
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
$$

The individual components are written in following general form

$$
N_1 = A_{11} \frac{\partial u_1}{\partial x_1} + A_{13} \frac{\partial u_3}{\partial x_1} + A_{16} \frac{\partial u_6}{\partial x_1} + \frac{1}{2} \left( \frac{1}{R_1} \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} \right) + \frac{1}{2} \left( \frac{1}{R_2} \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} \right),
$$

$$
N_2 = A_{21} \frac{\partial u_1}{\partial x_1} + A_{23} \frac{\partial u_3}{\partial x_1} + A_{26} \frac{\partial u_6}{\partial x_1} + \frac{1}{2} \left( \frac{1}{R_1} \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_1} \right) + \frac{1}{2} \left( \frac{1}{R_2} \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_1} \right),
$$

$$
N_6 = A_{61} \frac{\partial u_1}{\partial x_1} + A_{63} \frac{\partial u_3}{\partial x_1} + A_{66} \frac{\partial u_6}{\partial x_1} + \frac{1}{2} \left( \frac{1}{R_1} \frac{\partial u_6}{\partial x_1} - \frac{\partial u_5}{\partial x_1} \right) + \frac{1}{2} \left( \frac{1}{R_2} \frac{\partial u_6}{\partial x_1} - \frac{\partial u_5}{\partial x_1} \right),
$$

$$
V = (k') \int E(z) dz
$$
\[ M_i = B_{ii} \left( \frac{\partial u_i}{\partial x_i} \frac{u_i}{R_1} \right) + B_{ii} \left( \frac{\partial u_i}{\partial x_i} \frac{u_i}{R_2} \right) + \]
\[ + B_{6i} \left( \frac{\partial v_i}{\partial x_i} \right) + D_{ii} \frac{\partial^2 u_i}{\partial x_i^2} \frac{u_i}{R_1} + D_{ii} \frac{\partial^2 u_i}{\partial x_i^2} \frac{u_i}{R_2} + \]
\[ + D_{6i} \left( \frac{\partial^2 v_i}{\partial x_i^2} \right) \]
\[ \sum \sum \int \int \]
\[ \sum \sum \int \int \]
\[ \sum \sum \int \int \]
\[ M_2 = B_{ii} \left( \frac{\partial u_i}{\partial x_i} \frac{u_i}{R_1} \right) + B_{ii} \left( \frac{\partial u_i}{\partial x_i} \frac{u_i}{R_2} \right) + \]
\[ + B_{6i} \left( \frac{\partial v_i}{\partial x_i} \right) + D_{ii} \frac{\partial^2 u_i}{\partial x_i^2} \frac{u_i}{R_1} + D_{ii} \frac{\partial^2 u_i}{\partial x_i^2} \frac{u_i}{R_2} + \]
\[ + D_{6i} \left( \frac{\partial^2 v_i}{\partial x_i^2} \right) \]
\[ M_3 = B_{ii} \left( \frac{\partial u_i}{\partial x_i} \frac{u_i}{R_1} \right) + B_{ii} \left( \frac{\partial u_i}{\partial x_i} \frac{u_i}{R_2} \right) + \]
\[ + B_{6i} \left( \frac{\partial v_i}{\partial x_i} \right) + D_{ii} \frac{\partial^2 u_i}{\partial x_i^2} \frac{u_i}{R_1} + D_{ii} \frac{\partial^2 u_i}{\partial x_i^2} \frac{u_i}{R_2} + \]
\[ + D_{6i} \left( \frac{\partial^2 v_i}{\partial x_i^2} \right) \]
\[ V_i = k_i A_i \left( \frac{\partial u_i}{\partial x_i} \frac{u_i}{R_1} \right) \quad V_i = k_i A_i \left( \frac{\partial u_i}{\partial x_i} \frac{u_i}{R_2} \right) \]

The internal forces can be written in hypermatrix form

\[ \begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \phi \\ \nu \end{bmatrix} \]

\[ V = k A \gamma \]

where \( N \) is the membrane force resultant vector, \( M \) is the moment resultant vector and \( V \) is the transverse shear force resultant vector. In addition, \( A, B, D \) denote the classical extensional stiffness matrix, bending stiffness matrix and bending-extensional coupling stiffness matrix, respectively, whereas \( \bar{A} \) is the shear stiffness matrix [15,16].

The components of \( A, B, D, \bar{A} \) matrix are written as

\[ A = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)dz = \sum_{n=1}^{N} \int_{z_n}^{z_{n+1}} E(z)dz = \sum_{n=1}^{N} E z_n^3 \]

\[ B = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)dz = \sum_{n=1}^{N} \int_{z_n}^{z_{n+1}} E(z)dz = \sum_{n=1}^{N} E z_n^3 \]

\[ D = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)dz = \sum_{n=1}^{N} \int_{z_n}^{z_{n+1}} E(z)dz = \sum_{n=1}^{N} E z_n^3 \]

\[ \bar{A} = \int_{-\frac{h}{2}}^{\frac{h}{2}} E'(z)dz = \sum_{n=1}^{N} E' z_n^3 \]

III. Finite Element Analysis

The basic idea of the FEM is a discretisation of the continuous structure. The discretisation is defined by finite element mesh make up of elements nodes. The starting point for elastostatic problems is the total potential energy. In accordance with the Ritz method the approximation is used for displacement field vector by notation

\[ \ddot{u}(x) = [\phi] x \nu \]

where \( \phi(x) \) is the matrix of the shape functions, that are functions of the position vector \( x \) and \( \nu \) is the element displacement vector.

For the stresses and strains we obtain

\[ \sigma(x) = E \varepsilon(x) = ED \phi(x) \nu \]

\[ \varepsilon(x) = Du(x) = D[\phi](x) \nu = B(x) \nu \]

The total potential energy is a function of all the nodal displacement components arranged in the element displacement vector \( \nu \). The variation of the total potential energy

\[ \delta \tilde{U} = \delta \nu^T \left[ B^T EB \nu dV - \int \left[ \phi^T q dO \right] \right] \]

leads to

\[ \delta \nu^T (K \nu - f_p - f_q) = 0 \]

where \( p, q \) are volume and surface loadings, respectively and \( K \) is the symmetric stiffness matrix given by

\[ K = \int_V B^T EB dV \]

The vectors of the volume forces and the surface forces are written by

\[ f_p = \int_V \left[ \phi^T q dV \right] \]

\[ f_q = \int_{\partial V} \left[ \phi^T q dO \right] \]

If the components of \( \delta \nu \) are independent of each other, we obtain from Eq. (22) the system of linear equations

\[ K \nu = f \]

\[ f = f_p + f_q \]

All equations considered above are valid for a single finite element and they should have an additional index \( E \). We have the inner element energy

\[ U_E = \frac{1}{2} \nu^T \int_V B^T EB \nu dV = \frac{1}{2} \nu^T K \nu \]

with the element stiffness matrix
\[ K_v = \left[ B^T E B \right] / N \]
\[ E = \sum_{i=1}^{N} E \]

\[ ^* E = T^T (\alpha^*)^T E T(\beta^*) \]  \hspace{1cm} (27)

where \( E \) is the elasticity matrix obtained with suitable transformations in two stages, firstly from the principal material directions to the element local directions and secondly to the global directions. \( B \) is the strain matrix, \( T \) is the transformation matrix with

\[ T(\alpha) = (T^T (\alpha))^T \]  \hspace{1cm} (28)

The system stiffness matrix is also symmetric, but it is a singular matrix. After consideration of the boundary conditions of the whole system, \( K \) becomes a positive definite matrix and the system equations can be solved.

FE analysis is sensitive regarding to the strains and stresses (post-processing results) because the secondary solution converges slower than the primary solution. Quadratic elements have two basic disadvantages: the numerical effort increases and the meshing of a free-form surface is more complex because the thickness-to-curvature ratio has to be considered. Recently improvements in computing power, memory, and meshing algorithms make these elements more useful. The gain of higher shape-function approaches are better displacement, strain, and stress results. In addition, curved surfaces are mapped better because the shape functions are also used to describe the element geometry (isoparametric elements). The element descriptions are presented by Cook [17]. The quadratic shape functions of the 6-node and 8-node element (Fig. 3) are given in Equation 29 and 31, respectively [18].

The quadratic shape functions of the 6-node element have following form

\[ \Phi_1 = L_1(2L_1 - 1) \]
\[ \Phi_2 = L_2(2L_2 - 1) \]
\[ \Phi_3 = L_3(2L_3 - 1) \]
\[ \Phi_4 = 4L_2L_2 \]
\[ \Phi_5 = 4L_1L_1 \]
\[ \Phi_6 = 4L_3L_3 \]  \hspace{1cm} (29)

where

\[ L_1 = 1 - \xi - \eta \]
\[ L_2 = \xi \]
\[ L_3 = \eta \]

The quadratic shape functions of the 8-node element (Fig. 3) written in dimensionless coordinates are used in this study.

\[ \Phi_1 = 0.25(1 - \xi)(1 - \eta)(\xi - \eta - 1) \]
\[ \Phi_2 = 0.25(1 + \xi)(1 - \eta)(\xi + \eta - 1) \]
\[ \Phi_3 = 0.25(1 + \xi)(1 + \eta)(\xi + \eta - 1) \]
\[ \Phi_4 = 0.25(1 - \xi)(1 + \eta)(\xi - \eta - 1) \]
\[ \Phi_5 = 0.5(1 - \xi)(1 - \eta) \]
\[ \Phi_6 = 0.5(1 + \xi)(1 - \eta) \]
\[ \Phi_7 = 0.5(1 + \xi)(1 + \eta) \]
\[ \Phi_8 = 0.5(1 - \xi)(1 + \eta) \]  \hspace{1cm} (30)

The modal analysis is sensitive to the strains and stresses (post-processing results) because the secondary solution converges slower than the primary solution. Quadratic elements have two basic disadvantages: the numerical effort increases and the meshing of a free-form surface is more complex because the thickness-to-curvature ratio has to be considered. Recently improvements in computing power, memory, and meshing algorithms make these elements more useful. The gain of higher shape-function approaches are better displacement, strain, and stress results. In addition, curved surfaces are mapped better because the shape functions are also used to describe the element geometry (isoparametric elements). The element descriptions are presented by Cook [17]. The quadratic shape functions of the 6-node and 8-node element (Fig. 3) are given in Equation 29 and 31, respectively [18].

The particular solutions are

\[ \nu(t) = \nu_0 \sin(\omega_0 t) \quad \text{and} \quad \nu(t) = \nu_0 \cos(\omega_0 t) \]  \hspace{1cm} (33)

Than we get

\[ (K - \omega_0^2 \mu) \nu_0 = 0 \]  \hspace{1cm} (34)

where \( K \) is stiffness matrix, \( \nu_0 \) is mode shape vector of mode \( j \), \( \omega_0 \) is the natural circular frequency, \( \omega_0^2 \) is the eigenvalue and \( \mu \) is the mass matrix.

After modification of (34) we get

\[ M_0^{-1} \left[ K - \omega_0^2 \mu \right] \nu_0 = M_0^{-1} \omega_0^2 E \nu_0 = 0 \]  \hspace{1cm} (35)

where

\[ D^{-1} = M_0^{-1} K = K_0 \]  \hspace{1cm} (36)

The equation (35) means the eigenvalue problem, where \( \omega_0^2 \) are eigenvalues of the matrix \( D^{-1} \) and \( \nu_0 \) are natural modes of vibration.
V. EXAMPLE AND RESULTS

For this study, a double curved laminated shell with the following dimensions is selected: 
\[ a = b = 0.8 \text{ m}, R_1 = R_2 = 2.4 \text{ m}, h = 8 \text{ m}. \]
An unidirectional fiber reinforced composite laminate layer consists of isotropic fibers: 
\[ E_f = 270 \text{ GPa}, \nu_f = 0.3, \rho_f = 1800 \text{ kg m}^{-3} \]
and isotropic matrix: 
\[ E_m = 5 \text{ GPa}, \nu_m = 0.3, \rho_m = 1500 \text{ kg m}^{-3}. \]
The fiber volume fraction \( \xi = 0.4 \) and fiber diameter \( d = 9 \mu m \).
Most fiber reinforced composites have a random arrangement of the fibers at the micro-scale. A random microstructure results in transversely isotropic properties at the meso-scale. A simpler alternative is to assume that the random microstructure is well approximated by the hexagonal microstructure (Fig. 4).

![Hexagonal microstructure model](image)

An analysis of microstructure yields a transversely isotropic stiffness tensor

\[
\begin{pmatrix}
\sigma_x & \sigma_y & \sigma_z \\
\tau_{xy} & \tau_{xz} & \tau_{yz}
\end{pmatrix}
= \begin{pmatrix}
C_{11} & C_{12} & C_{12} \\
C_{12} & C_{22} & C_{23} \\
C_{12} & C_{23} & C_{33}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{xy} \\
\gamma_{xz} \\
\gamma_{yz}
\end{pmatrix}
\]

where the 1-axis aligned with the fiber direction and an overbar indicates the average computed over the volume of the RVE.

### Table I

<table>
<thead>
<tr>
<th>Volume ( V [\mu m^3] )</th>
<th>( C_{11} [\text{MPa}] )</th>
<th>( C_{12} [\text{MPa}] )</th>
<th>( C_{13} [\text{MPa}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>134.705</td>
<td>114277.5</td>
<td>5463.704</td>
<td>5463.3129</td>
</tr>
</tbody>
</table>

### Table II

<table>
<thead>
<tr>
<th>Volume ( V [\mu m^3] )</th>
<th>( C_{12} [\text{MPa}] )</th>
<th>( C_{22} [\text{MPa}] )</th>
<th>( C_{23} [\text{MPa}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>134.705</td>
<td>5464.704</td>
<td>13006.51</td>
<td>5209.964</td>
</tr>
</tbody>
</table>

### Table III

<table>
<thead>
<tr>
<th>Volume ( V [\mu m^3] )</th>
<th>( C_{13} [\text{MPa}] )</th>
<th>( C_{23} [\text{MPa}] )</th>
<th>( C_{33} [\text{MPa}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>134.705</td>
<td>5463.073</td>
<td>5209.964</td>
<td>13006.08</td>
</tr>
</tbody>
</table>

### Table IV

<table>
<thead>
<tr>
<th>Volume ( V [\mu m^3] )</th>
<th>( C_{66} [\text{MPa}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1074.64</td>
<td>2166.161</td>
</tr>
</tbody>
</table>

### Table V

**SUMMARY OF RESULTS OF HEXAGONAL NUMERICAL MODEL**


<table>
<thead>
<tr>
<th>( E_1 = E_3 [\text{GPa}] )</th>
<th>110.999</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_2 [\text{GPa}] )</td>
<td>10.824</td>
</tr>
<tr>
<td>( v_{12} = v_{13} )</td>
<td>0.3</td>
</tr>
<tr>
<td>( v_{23} )</td>
<td>0.388</td>
</tr>
<tr>
<td>( G_{12} = G_{13} [\text{GPa}] )</td>
<td>4.79</td>
</tr>
<tr>
<td>( G_{23} [\text{GPa}] )</td>
<td>3.899</td>
</tr>
</tbody>
</table>

Four-layer cross-ply [0/90/90/0], [0/90/0/90] and angle-ply [45/-45/-45/45], [45/45/45/-45] laminates are analysed to study the effect of symmetry to change the resonant frequencies (Tabs. 6).

From the Tables 6 can be seen, the frequencies in the case of angle-ply laminate are higher than in the case of cross-ply laminate. The frequencies in the case of symmetric layup are the same than in the case of anti-symmetric layup for both kinds of laminates.

### Table VI

**INFLUENCE OF FIBER ORIENTATION ON FREQUENCIES**

<table>
<thead>
<tr>
<th>Frequencies</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
<th>5th</th>
<th>6th</th>
<th>7th</th>
<th>8th</th>
<th>9th</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Frequencies</th>
<th>5th</th>
<th>6th</th>
<th>7th</th>
<th>8th</th>
<th>9th</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0/90/90/0]</td>
<td>7.676</td>
<td>7.949</td>
<td>8.064</td>
<td>7.676</td>
<td>7.949</td>
</tr>
<tr>
<td>[0/90/0/90]</td>
<td>7.676</td>
<td>7.949</td>
<td>8.064</td>
<td>7.676</td>
<td>7.949</td>
</tr>
</tbody>
</table>
Fig. 5 The first five natural modes of double curved laminated shell [0/90/90/0]

Fig. 6 The first five natural modes of double curved laminated shell [45/-45/-45/45]

Fig. 7 Displacements of the first five natural modes of double curved [0/90/90/0] laminated shell, respectively
From the Figs. 5-8 can be seen, the natural modes in the case of angle-ply laminate are more complicated than in the case of cross-ply laminate.

### Table VII

<table>
<thead>
<tr>
<th>b/h ratio</th>
<th>[0/90/90/0]</th>
<th>[45/-45/-45/45]</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>6.47196</td>
<td>8.14533</td>
</tr>
<tr>
<td>200</td>
<td>6.45032</td>
<td>7.54193</td>
</tr>
<tr>
<td>400</td>
<td>6.44501</td>
<td>7.14498</td>
</tr>
<tr>
<td>800</td>
<td>6.44344</td>
<td>6.88707</td>
</tr>
<tr>
<td>1000</td>
<td>6.44301</td>
<td>6.83161</td>
</tr>
</tbody>
</table>

In the Table 7 the width-to-thickness ratio is analysed. As can be seen from Table 7, as b/h increases, the fundamental frequency decreased. Decrease of fundamental frequency is more evident for [45/-45/-45/45] laminate than for [0/90/90/0] laminate.

### Table VIII

<table>
<thead>
<tr>
<th>α</th>
<th>[α/−α/−α/α]</th>
<th>[α/−α/α/−α]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5.65538</td>
<td>5.65538</td>
</tr>
<tr>
<td>15</td>
<td>5.91056</td>
<td>5.91056</td>
</tr>
<tr>
<td>30</td>
<td>6.63442</td>
<td>6.63442</td>
</tr>
<tr>
<td>45</td>
<td>8.14533</td>
<td>8.14533</td>
</tr>
</tbody>
</table>

Four-layer symmetric [α/−α/−α/α] and anti-symmetric [α/−α/α/−α] laminates with the angle of fibre orientation varying from 0° – 45° with b/h = 100 are analysed. As can be seen from Table 8, an increase of fibre orientation angle leads to an increase in the frequency of vibration. Also as it was written, the frequency in the case of symmetric layup is the same as in the case of anti-symmetric layup of laminates.

VI. DISCUSSION AND CONCLUSION

The material properties of unidirectional fiber reinforced composite are done by the numerical homogenization [20] of
unidirectional lamina (Table 5). Within the numerical homogenization, the hexagonal microstructure model was assumed in RVE. The numerical homogenization was done by using the Finite Element Method [21] in the program ANSYS.

In the paper, mode-frequency analysis of laminated double curved shell using a finite element model, based on first-order shear deformation theory is presented.

The frequencies in the case of angle-ply laminate are higher than in the case of cross-ply laminate. The frequencies in the case of symmetric layup are the same than in the case of anti-symmetric layup for both kinds of laminates. The natural modes in the case of angle-ply laminate are more complicated than in the case of cross-ply laminate. As width-to-thickness ratio \( b/h \) increases, the fundamental frequency decreased. Decrease of fundamental frequency is more evident for \([45/-45/45]\) laminate than for \([0/90/90/0]\) laminate. An increase of fibre orientation angle leads to an increase in the fundamental frequency of vibration. 

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