Influence of Coupling Effect in Laminated Composite Plate

E. Kormanikova, Kamila Kotrasova

Abstract**—**The paper deals with a numerical approach of modeling of laminated composite plate. In the frame of the numerical approach of modeling, the shear deformation theory for laminates is used. A very simple approach to introduce the shear correction factor is by considering the weight function for the distribution of the transverse shear stresses trough the thickness. The influence of coupling effect for complex response of laminated composite plate is investigated. The response of unsymmetric and symmetric laminated composite plate is solved by using FEM and results are shown in graphical form.

Keywords**—**Laminated Composite Plate, Coupling Effect, Shear Deformation Theory.

I. INTRODUCTION

The rapid growth in the use of composite materials in structures has required the development of the theory of mechanics of composite materials and the analysis of structural elements made of composite material [1-5]. For modelling of thick laminated composite plate, we have to take into account the shear deformation effects. The theory of laminate plates corresponds with the Reissner or Mindlin plate theory. Plate theories based upon Reissner-Mindlin assumption are called first order shear deformation theories [6].

Composite materials are heterogeneous, but in simplifying the analysis of composite structural elements in engineering applications, the heterogeneity of the material is neglected and approximately overlayed to a homogeneous material. Each single layer of laminates or sandwich faces is in general a fibre reinforced lamina. For investigation structural members or structures made of composite material, micro-macro modelling techniques are useful [7-14].

This work was supported by the Scientific Grant Agency of the Ministry of Education of Slovak Republic and the Slovak Academy of Sciences under Projects VEGA 1/0477/15 and 1/0078/16.

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The material response of a composite is determined by the material moduli of all constituents, the volume or mass fractions of the single constituents in the composite material, by the quality of their bonding, i.e. of the behaviour of the interfaces, and by the arrangement and distribution of the fibre reinforcement, i.e. the fibre architecture [15-21].

II. SHEAR DEFORMATION THEORY

Based upon that kinematical assumption of the first order shear deformation theory the displacements of the plate have the form [1]

$$u(x,y,z)=\bar{u}(x,y)-z\psi_x(x,y),$$
$$v(x,y,z)=\bar{v}(x,y)-z\psi_y(x,y),$$
$$w(x,y,z)=\bar{w}(x,y). \tag{1}$$

The strains relations are

$$\varepsilon_x=\frac{\partial u}{\partial x}=\frac{\partial \bar{u}}{\partial x}-z\frac{\partial \psi_x}{\partial x},$$
$$\varepsilon_y=\frac{\partial v}{\partial y}=\frac{\partial \bar{v}}{\partial y}-z\frac{\partial \psi_y}{\partial y},$$
$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=\frac{\partial \bar{u}}{\partial y}+\frac{\partial \bar{v}}{\partial x}-z\left(\frac{\partial \psi_x}{\partial y}+\frac{\partial \psi_y}{\partial x}\right),$$
$$\gamma_{xz}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}=\frac{\partial \bar{w}}{\partial x}-\psi_x,$$
$$\gamma_{yz}=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}=\frac{\partial \bar{w}}{\partial y}-\psi_y. \tag{2}$$

The vector notation is following

$$\boldsymbol{\varepsilon}(x,y,z)=\bar{\boldsymbol{\varepsilon}}(x,y)+z\boldsymbol{\kappa}(x,y), \tag{3}$$

with

$$\bar{\boldsymbol{\varepsilon}}=\left(\frac{\partial \bar{u}}{\partial x},\frac{\partial \bar{v}}{\partial y},\frac{\partial \bar{u}}{\partial y}+\frac{\partial \bar{v}}{\partial x}\right)^T,$$

$$\boldsymbol{\kappa}=-\left(\frac{\partial \psi_x}{\partial x},\frac{\partial \psi_y}{\partial y},\frac{\partial \psi_x}{\partial y}+\frac{\partial \psi_y}{\partial x}\right)^T \tag{4}$$

The last two kinematic relations in (2) we note in vector notation

$$\boldsymbol{\gamma}=\left(\frac{\partial \bar{w}}{\partial x}-\psi_x,\frac{\partial \bar{w}}{\partial y}-\psi_y\right)^T. \tag{5}$$

For the modelling of laminated 2-D structures we assume that each individual layer is considered to behave linear-elastic material, that all layers are assumed to be bonded together with a perfect bond and each lamina of composite material behaves macroscopically as if it were a homogeneous orthotropic material.

The stress resultant force vectors are [2]

$$\mathbf{N} = \int_{-h/2}^{+h/2} \mathbf{E}(z)dz\bar{\boldsymbol{\varepsilon}} + \int_{-h/2}^{+h/2} \mathbf{E}(z)zdz\boldsymbol{\kappa},$$

$$\mathbf{M} = \int_{-h/2}^{+h/2} \mathbf{E}(z)zdz\bar{\boldsymbol{\varepsilon}} + \int_{-h/2}^{+h/2} \mathbf{E}(z)z^2dz\boldsymbol{\kappa},$$

$$\mathbf{V}=\mathbf{k}^* \int_{-h/2}^{+h/2} \mathbf{E}^t(z)dz\boldsymbol{\gamma}, \tag{6}$$

with

$$\mathbf{A} = \int_{-h/2}^{+h/2} \mathbf{E}(z)dz = \sum_{n=1}^{N} \int_{{}^{n-1}z}^{{}^{n}z} {}^n\mathbf{E}dz = \sum_{n=1}^{N} {}^n\mathbf{E}^{\mathbf{n}}h,$$

$$\mathbf{B} = \int_{-h/2}^{+h/2} \mathbf{E}(z)zdz = \sum_{n=1}^{N} \int_{{}^{n-1}z}^{{}^{n}z} {}^n\mathbf{E}zdz = \sum_{n=1}^{N} {}^n\mathbf{E}\frac{{}^nz^2 - {}^{n-1}z^2}{2},$$

$$\mathbf{D} = \int_{-h/2}^{+h/2} \mathbf{E}(z)z^2dz = \sum_{n=1}^{N} \int_{{}^{n-1}z}^{{}^{n}z} {}^n\mathbf{E}z^2dz = \sum_{n=1}^{N} {}^n\mathbf{E}\frac{{}^nz^3 - {}^{n-1}z^3}{3}. \tag{7}$$

$$\overline{\mathbf{A}} = \mathbf{k}^* \int_{-h/2}^{+h/2} \mathbf{E}^t(z)dz = \mathbf{k}^* \sum_{n=1}^{N} {}^n\mathbf{E}^{t\,n}h. \tag{8}$$

The constitutive equations can be written in the condensed hypermatrix form [3]

$$\begin{pmatrix} \mathbf{N} \\ \mathbf{M} \\ \mathbf{V} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} \\ \mathbf{B} & \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \overline{\mathbf{A}} \end{pmatrix} \begin{pmatrix} \bar{\boldsymbol{\varepsilon}} \\ \boldsymbol{\kappa} \\ \boldsymbol{\gamma} \end{pmatrix}. \tag{9}$$

The stretching, coupling and bending stiffnesses A_{ij}, B_{ij}, D_{ij} stay unchanged in comparison to the classical laminate theory. The shear stiffness values can be improved with help of shear correction factors. The parameters $\mathbf{k}^*$ (Eq. 6) are the shear correction factors.

A very simple approach for calculation of shear stresses is to introduce a weighting function *f*(*z*) for the distribution of the transverse shear stresses trough the thickness *h*.

Assume a function *f*(*z*)

$$f(z) = \frac{5}{4}\left(1 - \left(\frac{z}{h/2}\right)^2\right), \tag{10}$$

it follows that transverse resultants are

$$V_{xz} = \sum_{n=1}^{N} \int_{{}^nh} {}^n\tau_{xz} f(z)dz,$$

$$V_{xz} = \frac{5}{4}\left(\begin{array}{l} \displaystyle\sum_{n=1}^{N} {}^nE_{44}\gamma_{xz} \int_{{}^nh}\left(1 - \left(\frac{z}{h/2}\right)^2\right)dz + \\ \displaystyle + \sum_{n=1}^{N} {}^nE_{45}\gamma_{yz} \int_{{}^nh}\left(1 - \left(\frac{z}{h/2}\right)^2\right)dz \end{array}\right), \tag{11}$$

$$V_{yz} = \sum_{n=1}^{N} \int_{{}^nh} {}^n\tau_{yz} f(z)dz,$$

$$V_{yz} = \frac{5}{4}\left(\begin{array}{l} \displaystyle\sum_{n=1}^{N} {}^nE_{45}\gamma_{xz} \int_{{}^nh}\left(1 - \left(\frac{z}{h/2}\right)^2\right)dz + \\ \displaystyle + \sum_{n=1}^{N} {}^nE_{55}\gamma_{yz} \int_{{}^nh}\left(1 - \left(\frac{z}{h/2}\right)^2\right)dz \end{array}\right). \tag{12}$$

$$V_{xz} = \overline{A}_{44}\gamma_{xz} + \overline{A}_{45}\gamma_{yz},$$

$$V_{yz} = \overline{A}_{45}\gamma_{xz} + \overline{A}_{55}\gamma_{yz}. \tag{13}$$

The shear stiffness coefficients $\overline{\mathbf{A}}$ are calculated by

$$\overline{A}_{ij} = \frac{5}{4}\sum_{n=1}^{N} {}^nE_{ij}\left(\left({}^nz - {}^{n-1}z\right) - \frac{4}{3h^2}\left({}^nz^3 - {}^{n-1}z^3\right)\right), \tag{14}$$

where $i, j = 4, 5$.

This approach yields, for the case of single layer with $E_{44} = E_{55} = G$ and $E_{45} = 0$, the shear correction factor $k^* = 5/6$.

An improved shear stiffness matrix which include the transverse shear stress distribution follows with the help of the complementary strain energy W^*

$$W^* = \frac{1}{2}\int_{(h)} \tau^T \mathbf{E}^{t^{-1}} \tau dz = \frac{1}{2}\mathbf{V}^T\left(\int_{(h)} \mathbf{F}^T \mathbf{E}^{t^{-1}} \mathbf{F}dz\right)\mathbf{V} = \frac{1}{2}\mathbf{V}^T\overline{\mathbf{A}}^{-1}\mathbf{V}, \tag{15}$$

with:

$$\mathbf{F}(z) = \left(\tilde{\mathbf{A}}(z)\mathbf{A}^{-1}\mathbf{B} - \tilde{\mathbf{B}}(z)\right)\mathbf{D}^{*-1}, \tag{16}$$

$$\tilde{\mathbf{A}} = \int_{-h/2}^{z} \mathbf{E}(z)dz = \sum_{n=1}^{m-1} {}^n\mathbf{E}\,{}^nh + {}^m\mathbf{E}\left(z - {}^{m-1}z\right), \tag{17}$$

$$\tilde{\mathbf{B}} = \int_{-h/2}^{z} \mathbf{E}(z)zdz = \sum_{n=1}^{m-1} {}^n\mathbf{E}\frac{{}^nz^2 - {}^{n-1}z^2}{2} + \frac{1}{2}{}^m\mathbf{E}\left(z^2 - {}^{m-1}z^2\right). \tag{18}$$

The equilibrium equations will be formulated for a plate element (Fig. 1) and yield three force and two moments equations

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} + p_1 = 0,$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + p_2 = 0,$$

$$\frac{\partial V_{xz}}{\partial x} + \frac{\partial V_{yz}}{\partial y} + p_3 = 0,$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} = V_{xz},$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = V_{yz}. \tag{19}$$

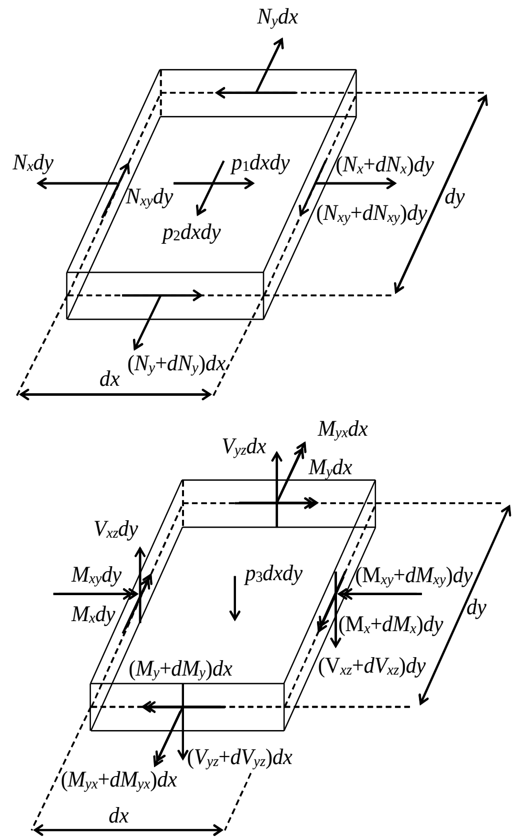

Fig. 1 Stress resultants applied to a plate element

Substituting the kinematic relations into the constitutive equations and then these equations into (19) we obtain the governing equations, written in matrix form

$$\begin{pmatrix} L_{11} & L_{12} & L_{13} & L_{14} & 0 \\ L_{21} & L_{22} & L_{23} & L_{24} & 0 \\ L_{31} & L_{32} & L_{33} & L_{34} & L_{35} \\ L_{41} & L_{42} & L_{43} & L_{44} & L_{45} \\ 0 & 0 & L_{53} & L_{54} & L_{55} \end{pmatrix} \begin{pmatrix} u \\ v \\ \psi \\ \varphi \\ w \end{pmatrix} = \begin{pmatrix} -p_1 \\ -p_2 \\ 0 \\ 0 \\ p_3 \end{pmatrix}. \tag{20}$$

The linear differential operators L_{ij} are defined in [2]. The necessary and sufficient number of boundary conditions for plates is three at each of the boundaries.

In the following we restrict to plates that are midplane symmetric (**B**=**0**), and the constitutive equations are then simplified to

$$\begin{pmatrix} \mathbf{N} \\ \mathbf{M} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \bar{\boldsymbol{\varepsilon}} \\ \boldsymbol{\kappa} \end{pmatrix},$$

$$\mathbf{V} = \overline{\mathbf{A}}\boldsymbol{\gamma}, \tag{21}$$

with

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{pmatrix},$$

$$\mathbf{D} = \begin{pmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{pmatrix},$$

$$\overline{\mathbf{A}} = \begin{pmatrix} k_{44}A_{44} & 0 \\ 0 & k_{55}A_{55} \end{pmatrix}. \tag{22}$$

Substituting the constitutive equations for M_x, M_y, M_{xy}, V_{xz}, V_{yz} into the three equilibrium equations (19) of the moments and transverse force resultants results in the set of governing differential equations for a symmetric laminated composite plate subjected to a lateral load p_3 and including transverse shear deformation

$$D_{11}\frac{\partial^2\psi_x}{\partial x^2} + (D_{12}+D_{66})\frac{\partial^2\psi_y}{\partial x\partial y} + D_{66}\frac{\partial^2\psi_x}{\partial y^2} - k_{44}A_{44}\left(\psi_x + \frac{\partial w}{\partial x}\right) = 0,$$

$$D_{66}\frac{\partial^2\psi_y}{\partial x^2} + (D_{12}+D_{66})\frac{\partial^2\psi_x}{\partial x\partial y} + D_{22}\frac{\partial^2\psi_y}{\partial y^2} - k_{55}A_{55}\left(\psi_y + \frac{\partial w}{\partial y}\right) = 0,$$

$$k_{44}A_{44}\left(\frac{\partial\psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + k_{55}A_{55}\left(\frac{\partial\psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right) + p_3 = 0. \tag{23}$$

In the next section, the governing equations are solved by using FEM for given boundary conditions.

III. Numerical Experiments, Results and Discussion

Firstly, the proposed model is verified for the square unsymmetrical laminated composite plate [45/0/45/0/45/0] (Fig. 2).

The material properties of each layer are $E_1 = E_2 = 76$ GPa, $G_{12} = 2.3$ GPa, $\nu = 0.34$, $G_{23} = 2.3$ GPa, $G_{13} = 2.3$ GPa.

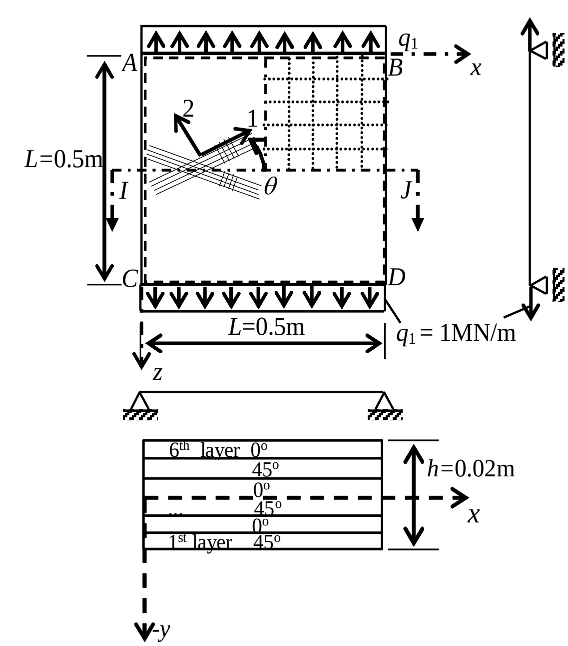

Fig. 2 Problem sketch and finite element mesh

Results:

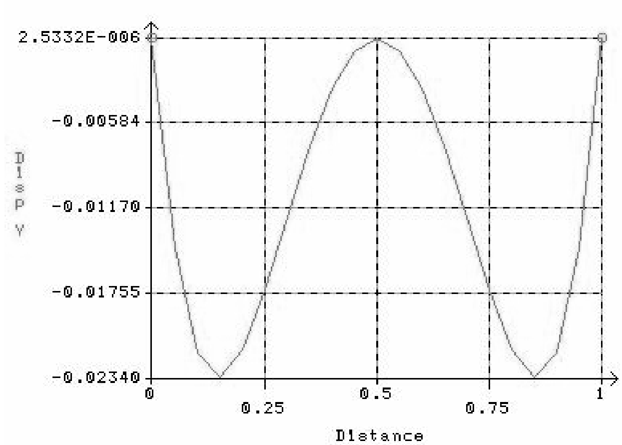

Fig. 3 Displacements *w* across the section I-J [mm]

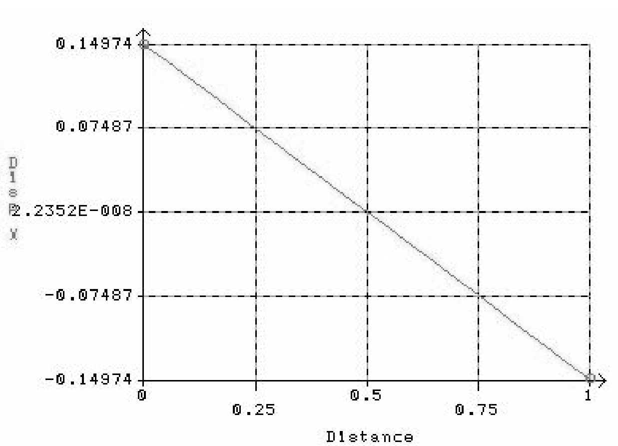

Fig. 4 Displacements *u* across the section I-J [mm]

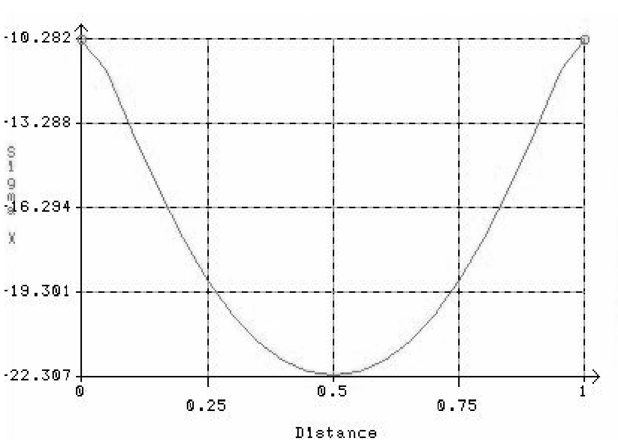

Fig. 5 Stresses σ_x for the bottom of the first layer across the section I-J [MPa]

The used coordinate system (x,y,z) is according the displacements (u,w,v), respectively (Figs. 3, 4 and 10).

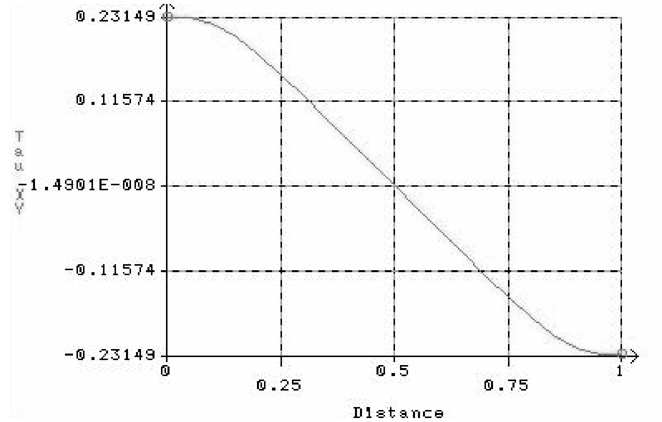

Fig. 6 Stresses τ_{xy} for the bottom of the first layer across the section I-J [MPa]

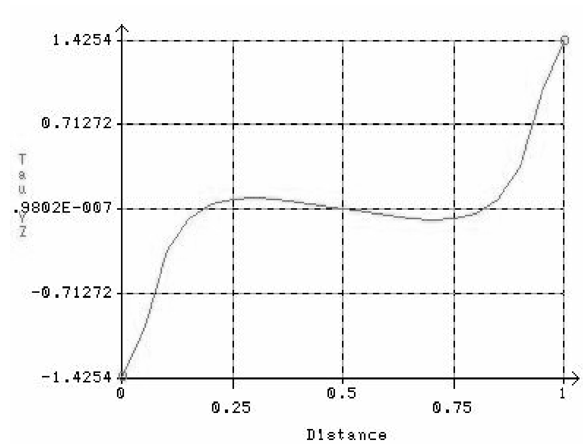

Fig. 7 Stresses τ_{yz} for the bottom of the first layer across the section A-C [MPa]

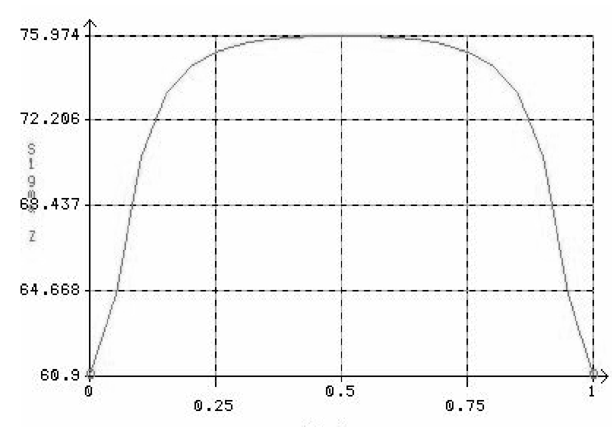

Fig. 8 Stresses σ_z for the bottom of the first layer across the section A-C [MPa]

Secondly, the proposed model is verified for the square symmetrical laminated composite plate [45/0/45] (Fig. 9).

The material properties of each layer are E_1 = 128 GPa, E_2 = 11GPa, $G_{12} = G_{23} = G_{13}$ = 45 GPa, ν = 0.25.

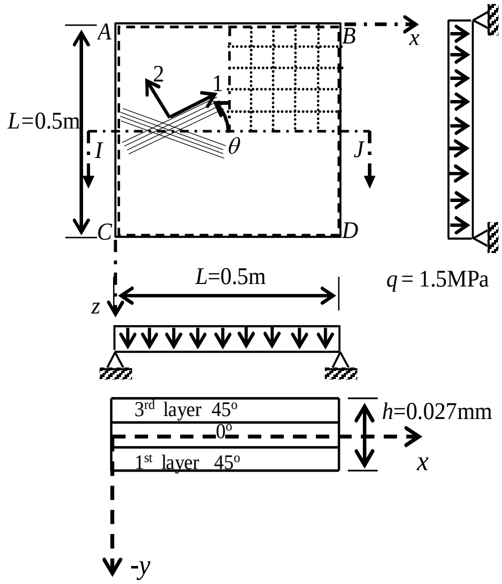

Fig. 9 Problem sketch and finite element mesh

Results:

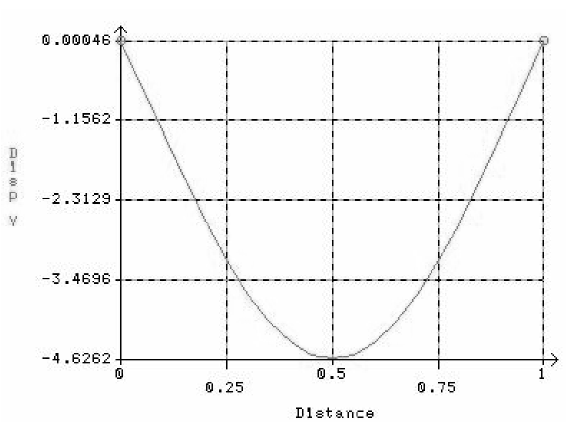

Fig. 10 Displacements *w* across the section I-J [mm]

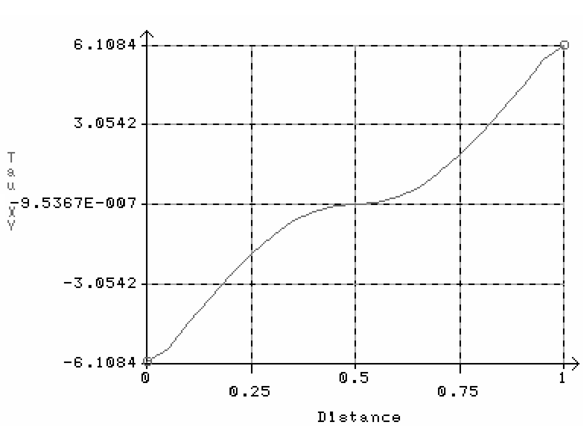

Fig. 11 Stresses τ_{xy} at the top of the third layer across the section I-J [MPa]

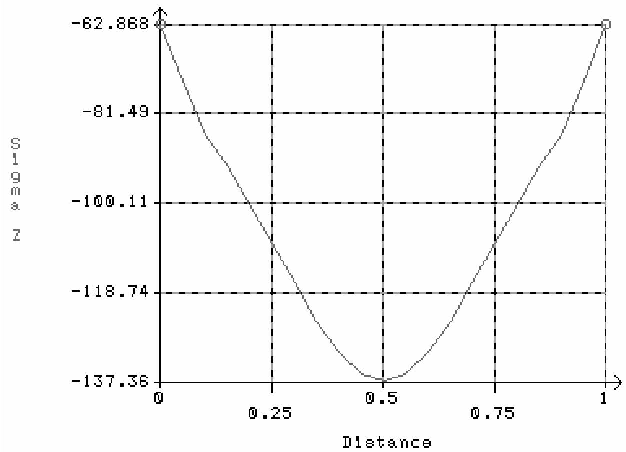

Fig. 12 Stresses σ_z at the top of the third layer across the section I-J [MPa]

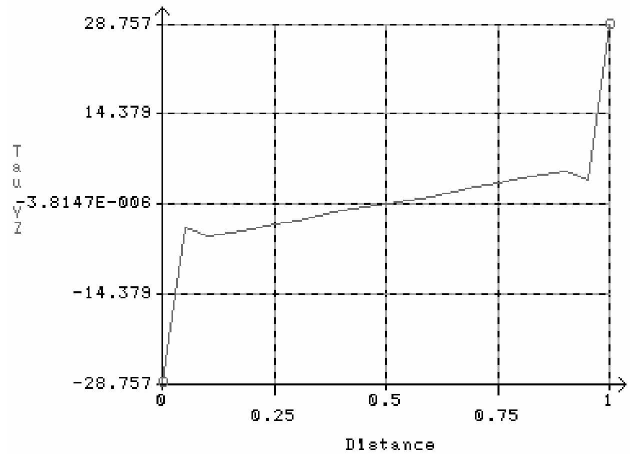

Fig. 13 Stresses τ_{yz} at the top of the third layer across the section I-J [MPa]

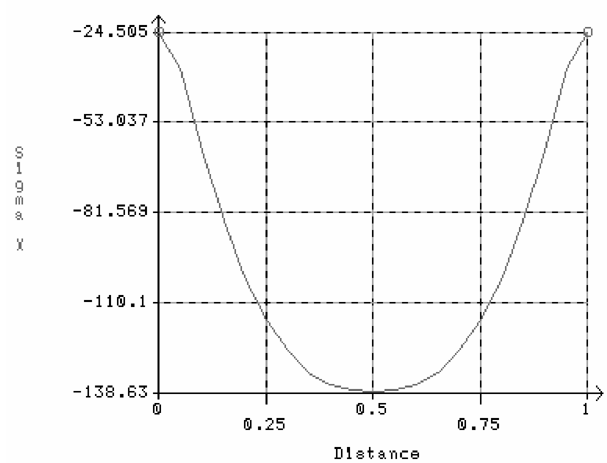

Fig. 14 Stresses σ_x at the top of the third layer across the section I-J [MPa]

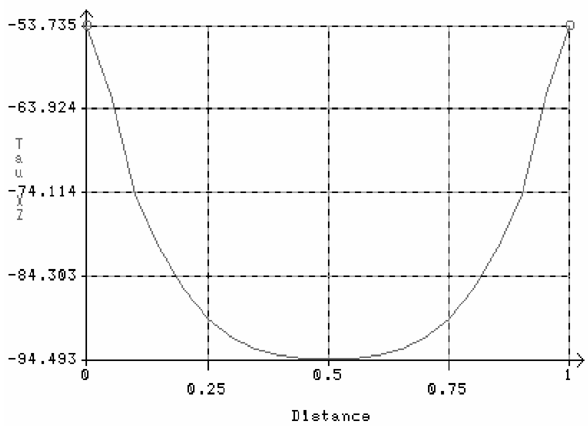

Fig. 15 Stresses τ_{xz} at the top of the third layer across the section I-J [MPa]

In the first example, there is solved unsymmetric laminated composite plate under in-plane loads. We can see, that there is coupling ($\mathbf{B} \neq \mathbf{0}$) between in-plane and bending deformations of the laminate structure (Figs. 3-8).

In the second example, there is solved symmetric laminated composite plate under bending loads. We can see, that there is not coupling ($\mathbf{B}=\mathbf{0}$) between in-plane and bending deformations of the laminate structure (Figs. 10-15). Transversal shear stresses τ_{yz} in both, symmetric and unsymmetric laminates are negligibly instead of free edge regions (Figs. 7, 13), that are critical in design process.

IV. CONCLUSION

The numerical approach of modelling of laminated composite plates was investigated in this paper. Within the numerical approach of modelling, there was described the shear deformation theory of first order for laminates. For this approach of modeling, there was solved the unsymmetric and symmetric laminated composite plate. In the first numerical example, there is solved unsymmetric laminated composite plate under in-plane loads. We saw that there is coupling effect between in-plane and bending deformations of the laminate structure. In the second example, there is solved symmetric laminated composite plate under bending loads. We saw that there is not coupling effect between in-plane and bending deformations of the laminate structure. The symmetric laminates are designed for flat structural elements and unsymmetric laminates for curved structural elements.

ACKNOWLEDGMENT

This work was supported by the Scientific Grant Agency of the Ministry of Education of Slovak Republic and the Slovak Academy of Sciences under Projects VEGA 1/0477/15 and 1/0078/16.

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