Then from (24) we get

$$R_1 = -\left(\frac{c^2(1-m_0)\rho_0^{(s)} - E_2}{(1-m_0)p_0\,\chi}\right)\frac{v_1}{c}\,. \tag{27}$$

Substituting (27) into the value of p_1 from (21), leads to

$$p_1 = \left(\frac{c^2(1-m_0)\rho_0^{(s)} - E_2}{(1-m_0)}\right)\frac{v_1}{c}\,. \tag{28}$$

Moreover, we derive the proportionality between v_1 and u_1 as

$$u_1 = \left(\frac{c(1-m_0)\rho_0^{(s)} - \frac{E_2}{c}}{\kappa_1\kappa_2\rho_0^{(L)}(1-m_0)}\right)\frac{v_1}{c}\,. \tag{29}$$

In the second approximation for the full system we have

B. Second linear approximation

$$\begin{aligned}
&\frac{\partial}{\partial\xi}\Big((1-m_0)v_2 + c(1-m_0)\beta^{(s)}\sigma_2 + cm_2\Big) = \Lambda^{(s)}\,,\\
&\frac{\partial}{\partial\xi}\Big(m_0u_2 - \Big[m_2 + \frac{m_0\beta^{(L)}p_2}{\kappa_2} - \frac{4\pi\, m_0\, n_0 R_0^3(R_2+R_1^2)}{\kappa_1}\\
&+\tfrac{4\pi\, m_0\, n_0 R_0^3\, p_0\,\chi\,\beta^{(L)} R_1^2}{\kappa_1\kappa_2}\Big]c\Big) = \Lambda^{(L)}\,,\\
&\frac{\partial}{\partial\xi}\Big(c(1-m_0)\rho_0^{(s)}v_2 + \sigma_2 - (1-m_0)p_2\Big) = \Sigma_1,\\
&\frac{\partial}{\partial\xi}\Big(c\kappa_1\kappa_2 m_0\rho_0^{(L)}u_2 - m_0p_2\Big) = \Sigma_2\,,\\
&\frac{\partial}{\partial\xi}\left(p_1 + p_0\,\chi\,R_1\right) = \frac{\partial\Gamma}{\partial\xi},\ \frac{\partial}{\partial\xi}\left(ce_2 + v_2\right) = F,\\
&\frac{\partial}{\partial\xi}\left(\sigma_2 - E_2\,e_2\right) = \frac{\partial T}{\partial\xi}\,,
\end{aligned} \tag{30}$$

where

$$\Lambda^{(s)} = \frac{1}{2}\frac{\partial}{\partial\tau}\left[(m_1 + \,(1-m_0)\beta^{(s)}\sigma_1)\right]\,,$$

$$\Lambda^{(L)} = -\frac{1}{2}\frac{\partial}{\partial\tau}\left[\kappa_1\left(m_1\kappa_2 + m_0\beta^{(L)}p_1\right) - 4\pi n_0\kappa_2R_0^3R_1\right],$$

$$\Sigma_1 = (1-m_0)\rho_0^{(s)}\frac{1}{2}\frac{\partial v_1}{\partial\tau}\,,\ \Sigma_2 = m_0\rho_0^{(f)}\frac{1}{2}\frac{\partial u_1}{\partial\tau}\,,$$

$$\Gamma = \mu c\left(4 + \frac{m_0R_0^2}{\ell}\right)\frac{\partial R_1}{\partial\xi}\,,\ F = -\frac{1}{2c}\frac{\partial v_1}{\partial\tau}\,,$$

$$T = -a_1c\frac{\partial e_1}{\partial\xi} + b_1c\frac{\partial\sigma_1}{\partial\xi}\,.$$

The determinant of the left-hand side of the system (30) coincides with the determinant of (18), which equals zero. Therefore, a non-zero solution for v_2 exists only if the following compatibility condition takes place,

$$\det(b_{nm}) = 0\,, \tag{31}$$

where
$b_{11} = \frac{\partial T}{\partial\xi}$, $b_{12} = 0$, $a_{13} = 0$, $b_{14} = 1$, $b_{15} = 0$, $b_{16} = 0$, $b_{17} = -E_2$, $b_{21} = \Sigma_1$, $b_{22} = 0$, $b_{23} = (1-m_0)$, $b_{24} = 1$, $b_{25} = 0$, $b_{26} = 0$, $b_{27} = 0$, $b_{31} = \Sigma_2$, $b_{32} = c\kappa_1\kappa_2m_0\rho_0^{(L)}$, $a_{33} = -m_o$, $b_{34} = 0$, $b_{35} = 0$, $b_{36} = 0$, $b_{37} = 0$, $b_{41} = \frac{\partial\Gamma}{\partial\xi}$, $b_{42} = 0$, $b_{43} = 1$, $b_{44} = 0$, $b_{45} = 0$, $b_{46} = \chi p_0$, $b_{47} = 0$, $b_{51} = F$, $b_{52} = 0$, $b_{53} = 0$, $b_{54} = 0$, $b_{55} = 0$, $b_{56} = 0$, $b_{57} = c$, $b_{61} = \Lambda^{(s)}$, $b_{62} = 0$, $b_{63} = 0$, $b_{64} = c(1-m_0)\beta^{(s)}$, $b_{65} = c$, $b_{66} = 0$, $b_{67} = 0$, $b_{71} = \Lambda^{(L)}$, $b_{72} = m_0$, $b_{73} = -\frac{c\beta^{(L)}m_0}{\kappa_2}$, $b_{74} = 0$, $b_{75} = -c$, $b_{76} = \frac{4c\pi m_0 n_0 R_0^3}{\kappa_1}$, $b_{77} = 0$.

This gives the evolution equation for $v \cong v_1$

$$\begin{aligned}
&\chi\Big(c\Sigma_2 + \Big(E_2F - c\Big(\Sigma_1 + \Sigma_2 - \frac{\partial T}{\partial\xi}\Big)\Big)m_0\Big)p_0\\
&+c^2\Big(-4\kappa_2\pi m_0\Big(E_2F - c\Big(\Sigma_1 - \frac{\partial T}{\partial\xi} + \frac{\partial\Gamma}{\partial\xi} + c\frac{\partial\Gamma}{\partial\xi}m_0\Big)n_oR_0^3\Big)\\
&+\kappa_1\chi p_0\Big(\Big(-E_2F - c\Big(\frac{\partial T}{\partial\xi} - \Sigma_1\Big)\Big)\beta^{(L)}m_0 - \kappa_2(1-m_0)\Big(\Big(E_2F\\
&+c\frac{\partial T}{\partial\xi}\Big)\beta^{(s)}(-1+m_0) + \Lambda^{(L)} + \Lambda^{(s)}\Big)\Big)\Big)\rho_0^{(L)} = 0\,.
\end{aligned} \tag{32}$$

We re-write equation (32) in terms of v and re-arrange with the help of Mathematica software,

$$\begin{aligned}
&\frac{1}{2}\big[c(1-m_0)\rho_0^{(s)}\ \Upsilon_1 + c\ \Upsilon_2 + c^2\kappa_1\kappa_2\chi p_0\rho_0^{(L)}((1-m_0)^2\\
&-\Upsilon_3) - E_2\ \Upsilon_4\,\big]\frac{\partial v}{\partial\tau} + c^2\left[\Upsilon_4(a_1 - b_1E_2) - 4\Upsilon_5\right]\frac{\partial^2 v}{\partial\xi^2}\\
&+c\ \Upsilon_6\ \frac{\partial vv}{\partial\xi} = 0\,,
\end{aligned} \tag{33}$$

where

$\Upsilon_1 = cm_0\big(-\chi p_0 + c^2\rho_0^{(L)}(\kappa_1\chi p_0\beta^{(L)} + 4\kappa_2\pi n_0R_0^3)\big)$,

$\Upsilon_2 = \chi p_0 m_0\Big(c(1-m_0)\rho_0^{(s)} - \dfrac{E_2}{c}\Big)$,

$\Upsilon_3 = \kappa_1\kappa_2(1-m_0)^2(1 - E_2\beta^{(s)}) - (c^2(1-m_0)\rho_0^{(s)} - E_2)(\kappa_1m_0\rho_0^{(L)} + 4\kappa_2\pi n_0R_0^3)$,

$\Upsilon_4 = p_0\,\chi m_0 + c^2\rho_0^{(L)}\Big(\kappa_1p_0\,\chi\big(\kappa_2\beta^{(s)}(-1+m_0)^2 - \beta^{(L)}m_0\big) - 4\pi n_0m_0\kappa_2R_0^3\Big)$,

$\Upsilon_5 = c^2\pi n_0m_0\kappa_2R_0^3\rho_0^{(L)}\mu\big(4 + \dfrac{m_0R_0^2}{\ell}\big)\Big(\dfrac{c^2(1-m_0)\rho_0^{(s)} - E_2}{p_0\,\chi}\Big)$

and Υ_6 is the nonlinearity coefficient, which we do not present here because our further analysis focuses on the linear part of equation (33).

Finally, we re-write the wave equation (33) as

$$A_1\frac{\partial v}{\partial\tau} + A_2\frac{\partial^2 v}{\partial\xi^2} + A_N\frac{\partial vv}{\partial\xi} = 0\,, \tag{34}$$

where

$A_1 = \dfrac{1}{2}\big[c(1-m_0)\rho_0^{(s)}\ \Upsilon_1 + c\ \Upsilon_2 + c^2\kappa_1\kappa_2\chi p_0\rho_0^{(L)}((1-m_0)^2 - \Upsilon_3) - E_2\ \Upsilon_4\,\big]$,

$A_2 = c^2\left[\Upsilon_4(a_1 - b_1E_2) - 4\Upsilon_5\right]$, $A_N = c\ \Upsilon_6$.

IV. P-WAVES WITHOUT GAS BUBBLES

A. Rheological model to derive stress-strain relation

By removing one elastic spring segment, which represents the gas bubble, from the rheological model in Fig. (3) (see Fig. 4) we get the following constitutive law [25]

$$\sigma + b_1\frac{d\sigma}{dt} = E_2e + a_1\frac{de}{dt}\,, \tag{35}$$

where $a_1 = (E_1 + E_2)\theta,\ b_1 = \theta,\ \ \theta = \mu/E_1$.

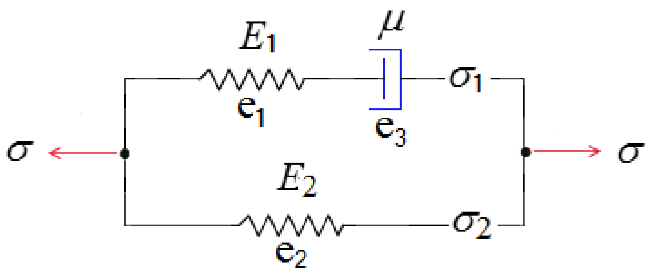

Fig. 4. Rheological model for standard linear model.

B. The equations of dynamics

In the case without the bubbles, the equations of motion reduce to the six equations

$$\begin{aligned}
&\tfrac{\partial}{\partial t}(1-m)\rho^{(s)} + \tfrac{\partial}{\partial x}(1-m)\rho^{(s)}v = 0\,,\\
&\frac{\partial}{\partial t}m\rho^{(L)} + \frac{\partial}{\partial x}m\rho^{(L)}u = 0\,,\\
&\frac{\partial}{\partial t}(1-m)\rho^{(s)}v + \frac{\partial}{\partial x}(1-m)\rho^{(s)}vv\\
&= \tfrac{\partial\sigma}{\partial x} - (1-m)\tfrac{\partial p}{\partial x} - I\,,\\
&\frac{\partial}{\partial t}m\rho^{(L)}u + \frac{\partial}{\partial x}m\rho^{(L)}uu = -m\frac{\partial p}{\partial x} + I\,,\\
&\sigma + b_1\frac{d\sigma}{dt} = E_2 e + a_1\frac{de}{dt}\,,\\
&\frac{De}{Dt} \equiv \frac{\partial e}{\partial t} + v\frac{\partial e}{\partial x} = \frac{\partial v}{\partial x}\,.
\end{aligned} \tag{36}$$

The density equation (3) for the solid remains unchanged, but for the gas-liquid mixture we neglect the volume gas content ϕ in equation (5),

$$\rho^{(f)} = \rho^{(L)} = \rho_0^{(L)}(1+\beta^{(L)}p)\,. \tag{37}$$

C. First linear approximation

The linear terms in the order $\sim\varepsilon$ in system (36)

$$\begin{aligned}
&c(1-m_0)\rho_0^{(s)}v_1 + \sigma_1 - (1-m_0)p_1 = 0\,,\\
&m_0u_1 - cm_1 - cm_0\beta^{(L)}p_1 = 0\,,\\
&(1-m_0)v_1 + c(1-m_0)\beta^{(s)}\sigma_1 + cm_1 = 0\,,\\
&cm_0\rho_0^{(L)}u_1 - m_0p_1 = 0\,,\\
&\sigma_1 - E_2\,e_1 = 0\,,\quad ce_1 + v_1 = 0\,.
\end{aligned} \tag{38}$$

In system (38), to find the velocity c we require

$$\det(a_{nm}) = 0\,, \tag{39}$$

where
$a_{11} = c(1-m_0)\rho_0^{(s)}$, $a_{12} = 0$, $a_{13} = (1-m_0)$, $a_{14} = 1$, $a_{15} = 0$, $a_{16} = 0$, $a_{21} = 0$, $a_{22} = m_0$, $a_{23} = -c\beta^{(L)}m_0$, $a_{24} = 0$, $a_{25} = -c$, $a_{26} = 0$, $a_{31} = (1-m_0)$, $a_{32} = 0$, $a_{33} = 0$, $a_{34} = c\beta^{(s)}(1-m_0)$, $a_{35} = c$, $a_{36} = 0$, $a_{41} = 0$, $a_{42} = cm_o\rho_0^{(L)}$, $a_{43} = -m_0$, $a_{44} = 0$, $a_{45} = 0$, $a_{46} = 0$, $a_{51} = 0$, $a_{52} = 0$, $a_{53} = 0$, $a_{54} = 1$, $a_{55} = 0$, $a_{56} = -E_2$, $a_{61} = 1$, $a_{62} = 0$, $a_{63} = 0$, $a_{64} = 0$, $a_{65} = 0$, $a_{66} = c$.

This gives

$$\begin{aligned}
&c^4\rho_0^{(s)}\rho_0^{(L)}\beta^{(s)}(1-m_0)m_0 - c^2\Big(\rho_0^{(L)}\big((1-E_2\beta^{(s)})(-1+m_0)^2\big)\\
&+E_2\beta^{(L)}m_0 + \rho_0^{(s)}m_0(1-m_0)\Big) + E_2m_0 = 0\,.
\end{aligned} \tag{40}$$

Equation (40) gives the velocity of the wave without the bubbles,

$$c^2 = \frac{-\beta_2 \pm \sqrt{\beta_2^2 - 4\alpha_2\gamma_2}}{2\alpha_2}\,, \tag{41}$$

where

$$\begin{aligned}
&\alpha_2 = \rho_0^{(s)}\rho_0^{(L)}\beta^{(s)}(1-m_0)m_0\,,\\
&\beta_2 = -\Big(\rho_0^{(L)}\big((1-E_2\beta^{(s)})(-1+m_0)^2 + E_2\beta^{(L)}m_0\big)\\
&+\rho_0^{(s)}m_0(1-m_0)\Big)\,,\\
&\gamma_2 = E_2m_0\,.
\end{aligned}$$

The terms for e_1, σ_1, p_1, and m_1 remain the same, while the proportion between v_1 and u_1 becomes

$$u_1 = \left(\frac{c(1-m_0)\rho_0^{(s)} - \frac{E_2}{c}}{\rho_0^{(L)}(1-m_0)}\right)\frac{v_1}{c}\,. \tag{42}$$

D. Second linear approximation

Collecting the quadratic terms$\sim\varepsilon^2$ in system (36) gives

$$\begin{aligned}
&\tfrac{\partial}{\partial\xi}(\sigma_2 - E_2\,e_2) = \tfrac{\partial T}{\partial\xi}\,,\quad \tfrac{\partial}{\partial\xi}(ce_2 + v_2) = F\,,\\
&c(1-m_0)\rho_0^{(s)}\frac{\partial v_2}{\partial\xi} + \frac{\partial\sigma_2}{\partial\xi} - (1-m_0)\frac{\partial p_2}{\partial\xi} = \Sigma_1\,,\\
&(cm_0\rho_0^{(L)}\frac{\partial u_2}{\partial\xi} - m_0\frac{\partial p_2}{\partial\xi}) = \Sigma_2\,,\\
&(1-m_0)\frac{\partial v_2}{\partial\xi} + c(1-m_0)\beta^{(s)}\frac{\partial\sigma_2}{\partial\xi} + c\frac{\partial m_2}{\partial\xi} = \Lambda^{(s)}\,,\\
&m_0\frac{\partial u_2}{\partial\xi} - c\frac{\partial m_2}{\partial\xi} - cm_0\beta^{(L)}\frac{\partial p_2}{\partial\xi} = \Lambda^{(L)}\,,
\end{aligned} \tag{43}$$

where the formulas of F, Σ_1, and $\Lambda^{(s)}$ are the same, while the formulas for Σ_2, $\Lambda^{(L)}$, and T are changed to

$$\Sigma_2 = m_0\rho_0^{(L)}\frac{1}{2}\frac{\partial u_1}{\partial\tau}\,,$$

$$\Lambda^{(L)} = -\frac{1}{2}\frac{\partial}{\partial\tau}\left(m_1 + m_0\beta^{(L)}p_1\right)\,,$$

$$T = -a_1c\frac{\partial e_1}{\partial\xi} + b_1c\frac{\partial\sigma_1}{\partial\xi}\,.$$

In analogy to (31), the compatibility condition for the system (43) has the form

$$\det(b_{nm}) = 0\,, \tag{44}$$

where
$b_{11} = \frac{\partial T}{\partial\xi}$, $b_{12} = 0$, $b_{13} = 0$, $b_{14} = 1$, $b_{15} = 0$, $b_{16} = -E_2$, $b_{21} = F$, $b_{22} = 0$, $b_{23} = 0$, $b_{24} = 0$, $b_{25} = 0$, $b_{26} = c$, $b_{31} = \Sigma_1$, $b_{32} = 0$, $b_{33} = (1-m_0)$, $b_{34} = 1$, $b_{35} = 0$, $b_{36} = 0$, $b_{41} = \Sigma_2$, $b_{42} = cm_o\rho_0^{(L)}$, $b_{43} = -m_0$, $b_{44} = 0$, $b_{45} = 0$, $b_{46} = 0$, $b_{51} = \Lambda^{(s)}$, $b_{52} = 0$, $b_{53} = 0$, $b_{54} = c\beta^{(s)}(1-m_0)$, $b_{55} = c$, $b_{56} = 0$, $b_{61} = \Lambda^{(L)}$, $b_{62} = m_0$, $b_{63} = -c\beta^{(L)}m_0$, $b_{64} = 0$, $b_{65} = -c$, $b_{66} = 0$.
Then the evolution equation for $v \cong v_1$ is

$$\begin{aligned}
&c\Sigma_2 + \left(E_2F - c\left(\Sigma_1 + \Sigma_2 - \frac{\partial T}{\partial\xi}\right)\right)m_0\beta^{(L)}m_0\\
&+c^2\left(\left(-E_2F - c\left(\frac{\partial T}{\partial\xi} - \Sigma_1\right)\right) - (1-m_0)\right.\\
&\left.\left(\left(E_2F + c\frac{\partial T}{\partial\xi}\right)\beta^s(-1+m_0) + \Lambda^{(L)} + \Lambda^{(s)}\right)\right)\rho_0^{(L)} = 0\,.
\end{aligned} \tag{45}$$

Now, we re-write equation (45) in terms of v and re-arrange,

$$\begin{aligned}
&\frac{1}{2}\big(\lambda_1 + \lambda_2 - (\lambda_3 + \lambda_4 + E_2\lambda_5)\big)\frac{\partial v}{\partial\tau}\\
&+c^2\,\lambda_5\,(a_1 - b_1E_2)\frac{\partial^2 v}{\partial\xi^2} + c\,\lambda_6\,\frac{\partial vv}{\partial\xi} = 0\,,
\end{aligned} \tag{46}$$

where

$$\lambda_1 = cm_0^2\Big(c(1-m_0)\rho_0^{(s)} - \frac{E_2}{c}\Big)\,,$$

$$\lambda_2 = c^2 m_0 (1-m_0)^2 \rho_0^{(L)}\,,$$

$$\lambda_3 = c^2 m_0 \rho_0^{(L)}\Big((1-m_0)^2(1-E_2\beta^{(s)})$$
$$-m_0\beta^{(L)}(c^2(1-m_0)\rho_0^{(s)} - E_2))\Big)\,,$$

$$\lambda_4 = c^2 m_0^2 \rho_0^{(s)}(1-m_0)(1-c^2\rho_0^{(L)}\beta^{(L)})\,,$$

$$\lambda_5 = m_0\Big(m_0 + c^2\rho_0^{(L)}(\beta^{(s)}(-1+m_0)^2 - m_0\beta^{(L)})\Big)\,,$$

and λ_6 is the nonlinearity coefficient. The above equation then becomes

$$B_1\frac{\partial v}{\partial \tau} + B_2\frac{\partial^2 v}{\partial \xi^2} + B_N\frac{\partial vv}{\partial \xi} = 0\,, \qquad (47)$$

where

$$B_1 = \frac{1}{2}\big(\lambda_1 + \lambda_2 - (\lambda_3 + \lambda_4 + E_2\lambda_5)\big)\,,$$

$$B_2 = c^2\,\lambda_5\,(a_1 - b_1E_2)\,,\ B_N = c\,\lambda_6\ .$$

V. Linearized model

In this section we consider the linearized version of the models (34) and (47). Our main interest is its dissipative part responsible for decay (attenuation) of the wave.

A. Evaluation of the parameters and the wave velocity

From [15], [16], [26], the values of the parameters are: densities, $\rho_0^{(L)} = 1000\,\mathrm{kg/m^3}$ for water, $\rho^{(g)} = 2\,\mathrm{kg/m^3}$ for gas, $\rho_0^{(s)} = 2500\,\mathrm{kg/m^3}$ for solid; porosity $m_0 = 0.25$; compressibility $\beta^{(L)} = 2\times10^{-9}\,\mathrm{Pa^{-1}}$ for water, $\beta^{(L)} = 2.4\times10^{-6}\,\mathrm{Pa^{-1}}$ for gas, $\beta^{(s)} = 2\times10^{-10}\,\mathrm{Pa^{-1}}$ for solid; steady pressure $p_0 = 10^3\,\mathrm{Pa}$; bubble radius $R_0 = 10^{-4}\,\mathrm{m}$; volume gas content $\phi_0 = 10^{-3}$; viscosity $\mu = 10^{-3}\,\mathrm{Pa{\cdot}s}$; adiabatic exponent $\zeta = 1.4$, and permeability $\ell = 1.8\times10^{-11}\,\mathrm{m^2}$. Using the data from [23], [27], [28], [29], the values of the parameters of the rheological scheme in Fig. 3 are

$$(a)\quad E_1 = 1/\beta^{(L)} = 4\times10^5\,\mathrm{Pa}\,,\ E_2 = c^2\rho_0 = 2\times10^7\,\mathrm{Pa}\,,$$
$$E_3 = 3\chi\,p_0 = 4\times10^7\,\mathrm{Pa}\,,$$

where we used, just for the purpose of evaluating of E_i and M_i, the typical velocity $c\sim 100\,\mathrm{m/s}$ and the linear size of the oscillator $L_s = 0.3\,\mathrm{cm}$ from [27], [30].

We will also explore the values of E_i obtained by a different method, namely by using the formula $c^2\rho$ for all three phases, with ρ being the density of the liquid, solid and gas, respectively,

$$(b)\quad E_1 = c^2\rho^{(L)} = 1000\times10^4\,\mathrm{Pa}\,,$$
$$E_2 = c^2\rho^{(s)} = 25\times10^6\,\mathrm{Pa}\,,\ E_3 = c^2\rho^{(g)} = 2\times10^4\,\mathrm{Pa}\,.$$

According to [1], [2], the equation (20) gives the velocity of P-waves with the bubbles: for the P1-wave $c\approx 103\,\mathrm{m/s}$ and $c\approx 116\,\mathrm{m/s}$ for the both variants (a) and (b) respectively; for the P2-wave $c\approx 3\,\mathrm{m/s}$ for the both variants (a) and (b). We see that the velocity of P2-wave is indeed smaller than the velocity of P1-wave.

The results of equation (41) gives the velocity of P-waves without the bubbles. For the P1-wave using the variants (a) and (b) gives $c\approx 1050\,\mathrm{m/s}$, while for the P2-wave $c\approx 70\,\mathrm{m/s}$ and $c\approx 78\,\mathrm{m/s}$ for the both variants (a) and (b) respectively. These results confirm that the presence of gas bubbles significantly decreases the P-waves velocities [20].

Furthermore, we observed that the velocity (41) of the P-waves without the bubbles is almost the same as the velocity of P-waves with the bubbles (20) when we set ($n_0 = 0$ and $R_0 = 0$) for the both variants (a) and (b).

B. Dissipation rate

In this section we are again interested in the effect of the bubbles on the wave dissipation. Therefore, we consider the linearized wave equations (34) and (47). The linearized form of Eq. (34) can be written as

$$\frac{\partial v}{\partial \tau} = -\frac{A_2}{A_1}\frac{\partial^2 v}{\partial \xi^2}\,. \qquad (48)$$

Now using the Fourier modes $v\sim\exp(\lambda t + ikx)$, we get the dissipation relation

$$\lambda(k) = \frac{A_2}{A_1}k^2\,, \qquad (49)$$

where λ is the decay rate and k is the wave number. For the case without the bubbles the linearized form of equation (47) is

$$\frac{\partial v}{\partial \tau} = -\frac{B_2}{B_1}\frac{\partial^2 v}{\partial \xi^2}\,. \qquad (50)$$

Then the dissipation relation is

$$\lambda(k) = \frac{B_2}{B_1}k^2\,. \qquad (51)$$

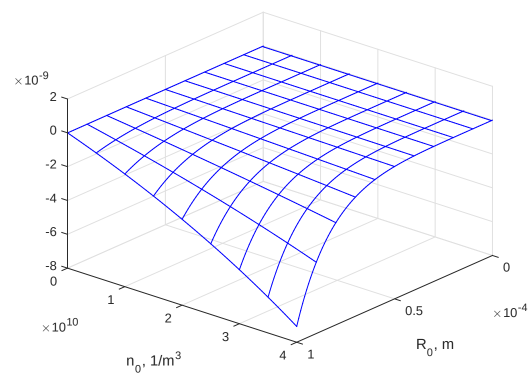

Fig. 5. The decay rate by formula (49) for variant (a), $k_* = 0.25$ 1/m.

The plot in Fig. 5 shows the decay rate at fixed $k_* = 0.25$ 1/m [21] against R_0 and n_0. As mentioned earlier, the decay rate is significantly affected by the increase in R_0 and becomes large in absolute value; this is because the bubbles affect the system through the pressure $p_1 = -p_0\,\chi\,R_1$. As for n_0, one should disregard the region of small n_0 in Fig. 5 where the equations of continuum mechanics cease to be valid. This is because the used assumption that each bubble is embedded in its own fluid particle (see Eq. (2)) is no longer inapplicable due to the large size of the particle.

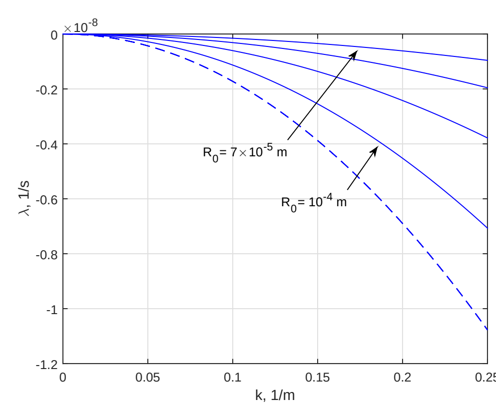

Fig. 7. The decay rate by formulas (49) and (51) for variant (a): R_0 varies, $n_0 = 4\times10^{10}$.

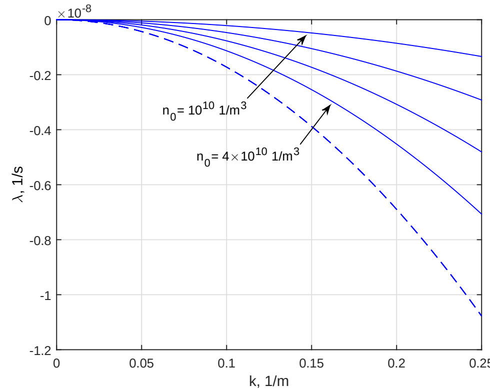

Fig. 6. The decay rate by formulas (49) and (51) for variant (a): n_0 varies, $R_0 = 10^{-4}$.

Figs. 6 and 7 compare the decay curves of the wave with the bubbles and the wave without the bubbles. The dashed line describes the case without the bubbles and the solid lines correspond to the wave with the bubbles. Fig. 6 is for varying n_0 and fixed R_0. Fig. 7 is for varying R_0 and fixed n_0. We clearly see that the curves lie entirely below zero, which means that the wave decays and the decay rate depends on the number and radius of the bubbles. This result agrees with the conception emphasized in [32], [33] about the essentially dissipative nature of the freely propagating elastic wave. Similar results are obtained for variants (b) as shown in Figs. 8–10.

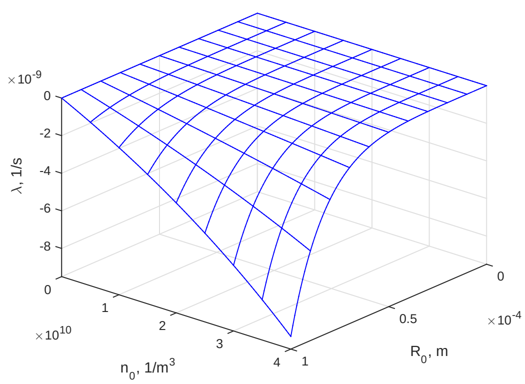

Fig. 8. The decay rate by formula (49) for variant (b), $k_* = 0.25\ 1/\mathrm{m}$.

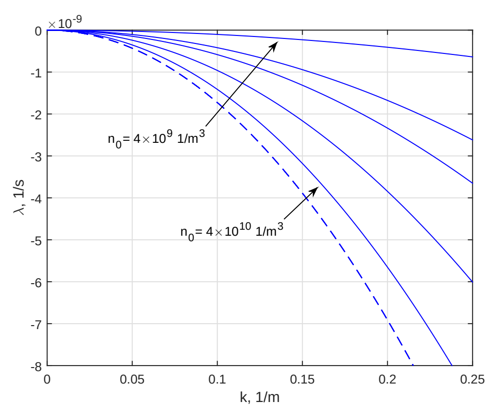

Fig. 9. The decay rate by formulas (49) and (51) for variant (b): n_0 varies, $R_0 = 10^{-4}$.

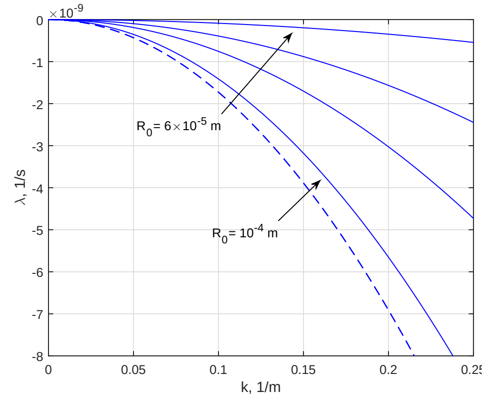

Fig. 10. The decay rate by formulas (49) and (51) for variant (b): R_0 varies, $n_0 = 4 \times 10^{10}$.

VI. Conclusions

We studied the effect of the rheology including bubbles on the Frenkel-Biot P-waves in porous rocks. Using two-segment rheology, we derived the P-type wave equations with and without the bubbles describing the velocity of the solid matrix in the medium. The linearized versions of the equations are compared in terms of the decay rate $\lambda(k)$ of the Fourier modes. For the both cases with and without the bubbles, the $\lambda(k)$-curve lies entirely below zero. We discovered that —$\lambda(k)$— increases with the increase of the radius and the number of the bubbles.

Nomenclature

$\beta^{(s)}$	Compressibility for solid, Pa^{-1}
$\beta^{(L)}$	Compressibility for water and gas, Pa^{-1}
$\rho_0^{(s)}$	Density for solid, $\mathrm{kg/m^3}$
$\rho_0^{(L)}$	Density for water, $\mathrm{kg/m^3}$
$\rho_0^{(g)}$	Density for gas, $\mathrm{kg/m^3}$
R_0	Bubble radius, m
μ	Viscosity, Pa·s
ℓ	Permeability, m^2
p	Pressure, Pa
σ	Stress, Pa
ϕ	Volume gas content
n_0	Number of bubbles , $1/\mathrm{m}^3$
m_0	Porosity
ζ	Adiabatic exponent
k	Wave number, 1/m
λ	Decay rate, 1/s

References

[1] Biot, M. A. "Theory of propagation of elastic waves in a fluid-saturated porous solid. I. Low-frequency range." The Journal of the Acoustical Society of America 28.2 (1956): 168-178.

[2] Biot, M. A. "Theory of propagation of elastic waves in a fluid-saturated porous solid. II. Higher frequency range." the Journal of the Acoustical Society of America 28.2 (1956): 179-191.

[3] Biot, M. A. "Mechanics of deformation and acoustic propagation in porous media." Journal of Applied Physics 33.4 (1962): 1482-1498.

[4] Biot, M. A. "Generalized theory of acoustic propagation in porous dissipative media." The Journal of the Acoustical Society of America 34.9A (1962): 1254-1264.

[5] Lvy, Thrse. "Propagation of waves in a fluid-saturated porous elastic solid." International Journal of Engineering Science 17.9 (1979): 1005-1014.

[6] Auriault, J. L. "Dynamic behaviour of a porous medium saturated by a Newtonian fluid." International Journal of Engineering Science 18.6 (1980): 775-785.

[7] Burridge, Robert, and Joseph B. Keller. "Poroelasticity equations derived from microstructure." The Journal of the Acoustical Society of America 70.4 (1981): 1140-1146.

[8] Pride, Steven R., Anthony F. Gangi, and F. Dale Morgan. ”Deriving the equations of motion for porous isotropic media.” The Journal of the Acoustical Society of America 92.6 (1992): 3278-3290.
[9] Mller, Tobias M., Boris Gurevich, and Maxim Lebedev. ”Seismic wave attenuation and dispersion resulting from wave-induced flow in porous rocksA review.” Geophysics 75.5 (2010): 75A147-75A164.
[10] Silberman, Edward. ”Sound velocity and attenuation in bubbly mixtures measured in standing wave tubes.” The journal of the Acoustical Society of America 29.8 (1957): 925-933.
[11] Wijngaarden, L. van. ”On the equations of motion for mixtures of liquid and gas bubbles.” Journal of Fluid Mechanics 33.3 (1968): 465-474.
[12] Wijngaarden, L. van. ”One-dimensional flow of liquids containing small gas bubbles.” Annual review of fluid Mechanics 4.1 (1972): 369-396.
[13] Anderson, A. L., and Hampton, L. D. ”Acoustics of gas-bearing sediments I. Background.” The Journal of the Acoustical Society of America 67.6 (1980): 1865-1889.
[14] Commander, Kerry W., and Andrea Prosperetti. ”Linear pressure waves in bubbly liquids: Comparison between theory and experiments.” The Journal of the Acoustical Society of America 85.2 (1989): 732-746.
[15] Dunin, S. Z., and Nikolaevskii, V. N. ”Nonlinear waves in porous media saturated with live oil.” Acoustical Physics 51.1 (2005): S61-S66.
[16] Dunin, S. Z., Mikhailov, D. N., and Nikolayevskii, V. N. ”Longitudinal waves in partially saturated porous media: the effect of gas bubbles.” Journal of Applied Mathematics and Mechanics 70.2 (2006): 251-263.
[17] Leighton, T. G. ”Theory for acoustic propagation in marine sediment containing gas bubbles which may pulsate in a nonstationary nonlinear manner.” Geophysical Research Letters 34.17 (2007).
[18] Mantouka, A., and T. G. Leighton. ”Theory for the dynamics of acoustically-excited gas bubbles in porous media (with specific application to marine sediment).” (2008).
[19] Toms, J., et al. ”Comparative review of theoretical models for elastic wave attenuation and dispersion in partially saturated rocks.” Soil Dynamics and Earthquake Engineering 26.6-7 (2006): 548-565.
[20] Gubaidullin, A. A., O. Yu Boldyreva, and D. N. Dudko. ”Waves in porous media saturated with bubbly liquid.” Journal of Physics: Conference Series. Vol. 899. No. 3. IOP Publishing, 2017.
[21] Nikolaevskii, V. N. Dynamics of viscoelastic media with internal oscillations, S.L. Koh et al. (eds.), Recent Advances in Engineering Science, Springer-Verlag, Berlin, 1989, 210–221.
[22] Nikolaevskiy, V. N. ”Non-linear Evolution of P-waves in ViscousElastic Granular Saturated Media.” Transport in Porous Media 73.2 (2008): 125-140.
[23] Nikolaevskiy, V. N., and Strunin, D. V. ”The role of natural gases in seismics of hydrocarbon reservoirs.” Elastic Wave Effect on Fluid in Porous Media, Proceedings 2012 (2012): 25-29.
[24] Nikolaevskiy, Viktor N. Mechanics of porous and fractured media. Vol. 8. World Scientific, 1990.
[25] Findley, William N., and Francis A. Davis. Creep and relaxation of nonlinear viscoelastic materials. Courier Corporation, 2013.
[26] Mikhailov, D. N. ”The influence of gas saturation and pore pressure on the characteristics of the Frenkel-Biot P waves in partially saturated porous media.” Izvestiya, Physics of the Solid Earth 46.10 (2010): 897-909.
[27] Nikolaevskiy, V. N. ”Viscoelasticity with internal oscillators as a possible model of seismoactive medium.” Doklady Akademii Nauk SSSR 283.6 (1985): 1321-1324.
[28] Nikolaevskiy, V. N. ”A real P-wave and its dependence on the presence of gas.” Izvestiya, Physics of the Solid Earth 52.1 (2016): 1-13.
[29] Nikolaevskiy, V. N., and Stepanova, G. S. ”Nonlinear seismics and the acoustic action on the oil recovery from an oil pool.” Acoustical Physics 51.1 (2005): S131-S139.
[30] Vilchinska, N., Nikolajevskiy, V. N , and Lisin, V. ”Slow waves and natural oscillations in sandy marine soils.” Izvestiya Acad. Nauk SSSR, Oceanology 4.
[31] Ali, A. A., and Strunin, D. V. ”The role of rheology in modelling elastic waves with gas bubbles in granular fluid-saturated media.” Journal of Mechanics of Materials and Structures 14.1 (2019): 1-24.
[32] Strunin, D. V. ”On dissipative nature of elastic waves.” Journal of Coupled Systems and Multiscale Dynamics 2.2 (2014): 70-73.
[33] Strunin, D. V., and Ali, A. A. ”On nonlinear dynamics of neutral modes in elastic waves in granular media.” Journal of Coupled Systems and Multiscale Dynamics 4.3 (2016): 163-169.