Rheology and decay rate for Frenkel-Biot P1 waves in porous media with gas bubbles

Dmitry V. Strunin and Adham A. Ali

Abstract—We study the effect of using different rheological schemes in models of Frenkel-Biot elastic waves of P-type in porous media. Two rheological schemes are considered -one with the bubbles and the other without. The bubble-including scheme consists of segments representing the solid continuum and bubbles inside the fluid, while the bubble-free scheme is represented by the standard solid-fluid rheological model. We derived the dispersion relations for the wave equations in their linear forms and analyzed the decay rate, λ, versus the wave number, k. We compared the $\lambda(k)$-dependence for the two rheologies under consideration using typical values of the mechanical parameters of the model. We observed, in particular, that an increase of the radius and the number of the bubbles leads to an increase in the decay rate.

Keywords—Porous, rheology, fluid, bubbles.

I. Introduction

THE problem of elastic wave propagation through the liquid-saturated granular medium has been studied in many theoretical works. This phenomenon is usually described by Biot's equations of poroelasticity [1]–[4]. The Biot's dynamic equations are commonly used to describe the wave propagation in a porous solid fully saturated with a single-phase fluid. Many researchers re-derived the Biot's equations using different mathematical approaches, for example, homogenization for periodic structures [5]–[7] and volume averaging processes [8]. The Biot's equations involve four basic assumptions [9]: first, the porous rock is isotropic and homogeneous; second, the porous rock is fully saturated with only one fluid; third, the motion between the solid and fluid is governed by the Darcy's law and fourth, the wavelength of the wave is larger than the size of the biggest grains or pores. From the Biot's theory, there are two types of longitudinal waves propagating in a saturated porous medium. The first type is the fast wave with week attenuation, called P1-wave, whereas the second type is the slow wave with strong attenuation, called P2-wave.

It is clear that the presence of bubbles affects the properties of the gas-liquid mixture [10]–[15] such as compressibility, pressure, velocity, and attenuation. For more details of the effects of the gas bubbles on the attenuation and wave velocities in liquid-saturated porous rocks, the reader is referred to [16]–[20].

This paper studies the influence of different rheologies including the bubbles on the wave attenuation in the liquid-saturated porous media. We will use an extended stress-strain relation relative to the standard linear solid model to take into account the bubbles in modelling the P-type waves. In [16], Dunin et al. used the simple stress-strain relation, $\sigma = Ee$, in such modelling. However, Nikolaevskiy [21], [22] used a considerably complicated stress-strain relation that involves higher-order time derivatives of the stress σ and strain e. This relation is the result of the rheological scheme shown in Fig. 1. Eventually it leads to a higher-order partial differential equation with respect to the velocity of the solid matrix. However, the original rheological scheme [21] does not include gas bubbles. Nikolaevskiy and Strunin [23] pointed out the place in this scheme that the bubbles should take, see Fig. 2. In the present work we aim to include the bubble into the rheological scheme and, based on this, derive the P-wave equations, where the coefficients will depend on the bubble-related parameters.

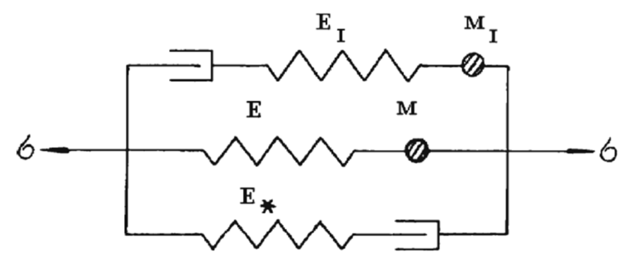

Fig. 1. The model of the viscoelastic medium with internal oscillators [21].

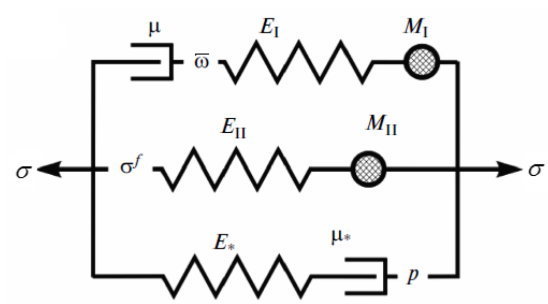

Fig. 2. The modified model of Fig. 1 to include the bubble represented by ϖ [23].

II. Basic Equations of Motion

A. Conservation of mass and momentum

For a one-dimensional case the momentum and mass balance equations [24] are

$$\frac{\partial}{\partial t}(1-m)\rho^{(s)}v+\frac{\partial}{\partial x}(1-m)\rho^{(s)}vv = \frac{\partial}{\partial x}\sigma-(1-m)\frac{\partial p}{\partial x}-I\,,$$

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$$\frac{\partial}{\partial t} m\rho^{(f)} u + \frac{\partial}{\partial x} m\rho^{(f)} uu = -m\frac{\partial p}{\partial x} + I\,,$$
$$\frac{\partial}{\partial t}(1-m)\rho^{(s)} + \frac{\partial}{\partial x}(1-m)\rho^{(s)} v = 0\,, \qquad (1)$$
$$\frac{\partial}{\partial t} m\rho^{(f)} + \frac{\partial}{\partial x} m\rho^{(f)} u = 0\,,$$

where, the subscripts s and f label the solid and gas-liquid mixture respectively, ρ, v, and u are the corresponding densities and mass velocities, m is the porosity, σ is the stress, p is the pore pressure, and I is the interfacial viscous force approximated by

$$I = \delta m(v-u), \quad \delta = \frac{\mu^{(f)} m}{\ell}\,,$$

where $\mu^{(f)}$ is the gas-liquid mixture viscosity and ℓ is the intrinsic permeability.

Now we add to the system (1) the equation of the dynamics of a bubble [16]

$$R\frac{\partial^2}{\partial t^2}R + \frac{3}{2}\left(\frac{\partial}{\partial t}R\right)^2 + \frac{4\mu}{\rho^{(L)}}\left(\frac{1}{R} + \frac{m}{4\ell}R\right)\frac{\partial}{\partial t}R$$
$$= (p_g - p)/\rho^{(L)}\,, \qquad (2)$$

where R is the bubble radius, p is the pressure in the liquid, $p_g = p_0(R_0/R)^{\chi}$ is the gas pressure inside the bubble (here $\chi = 3\varsigma$, ς is the adiabatic exponent), $\rho^{(L)}$ is the density of the liquid without the bubbles, and μ is the viscosity of the liquid without the bubbles. The density equations for the solid and liquid without gas are

$$\rho^{(s)} = \rho_0^{(s)}(1 - \beta^{(s)}\sigma)\,, \qquad (3)$$

$$\rho^{(L)} = \rho_0^{(L)}(1 + \beta^{(L)}p)\,. \qquad (4)$$

The mean density of the gas-liquid mixture is

$$\rho^{(f)} = (1-\phi)\rho^{(L)} + \phi\rho^{(g)}, \qquad (5)$$

where

$$\phi = (4\pi/3)R^3 n_0\,.$$

Here σ is the stress, ϕ is the volume gas content and n_0 is the number density of the bubbles per unit volume. In Eq. (5) we can neglect the density of the gas $\rho^{(g)}$ due to the low gas content. The change in ϕ is due to the change in the bubble radius R. Then Eq. (5) becomes

$$\rho^{(f)} = \rho_0^{(L)}(1 + \beta^{(L)}p)\left(1 - \frac{4\pi}{3}R_0^3 n_0\right)\,. \qquad (6)$$

Similarly to [16] we also assume that the pore pressure p is equal to the pressure in the liquid far from the bubble.

B. Rheological model to derive stress-strain relation

In this section we consider a simplified rheological model compared to Fig. 1 and Fig. 2. It includes three elastic springs with the elastic moduli E_1, E_2, and E_3, and one dashpot with viscosity μ as shown in Fig. 3. Applying the Newton's laws , this scheme generates the following equations

$$\begin{aligned} &e = e_2 = e_1 + e_3 + e_4\,,\\ &E_3 e_3 - \mu\frac{de_4}{dt} = 0\,,\\ &E_1 e_1 - E_3 e_3 = 0\,,\\ &E_1 e_1 + E_2 e_2 = \sigma\,. \end{aligned} \qquad (7)$$

Using system (7) we arrive at the following matrix system

$$\begin{bmatrix} E_2 & E_1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & E_1 & 0 & 0\\ 0 & 0 & E_3 & 0 & 0 & 0 & -\mu\\ -1 & 1 & 1 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 1 & 1\\ 0 & E_1 & -E_3 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & E_1 & -E_3 & 0 \end{bmatrix} \begin{bmatrix} e\\ e_1\\ e_3\\ e_4\\ \frac{de_1}{dt}\\ \frac{de_3}{dt}\\ \frac{de_4}{dt} \end{bmatrix} = \begin{bmatrix} \sigma\\ \frac{d\sigma}{dt} - E_2\frac{de}{dt}\\ 0\\ 0\\ \frac{de}{dt}\\ 0\\ 0 \end{bmatrix}. \qquad (8)$$

Solving system (8) for e we get

$$e = \frac{-((E_1+E_3)E_2 + E_1E_3)\mu\frac{de}{dt} + E_1E_3\sigma + (E_1+E_3)\mu\frac{d\sigma}{dt}}{E_1E_2E_3}\,. \qquad (9)$$

Equation (9) leads to the following stress-strain relation

$$\sigma + b_1\frac{d\sigma}{dt} = E_2 e + a_1\frac{de}{dt}\,, \qquad (10)$$

where $a_1 = ((E_1+E_3)E_2 + E_1E_3)\theta$, $b_1 = (E_1+E_3)\theta$, and $\theta = \mu/E_1E_3$.

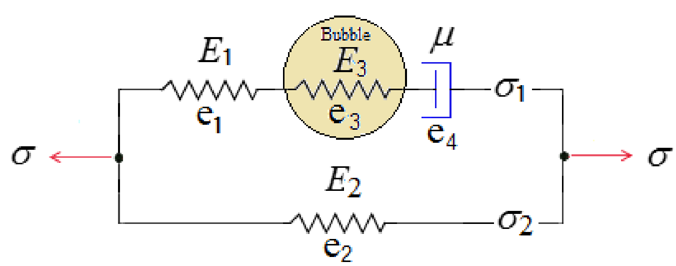

Fig. 3. A simplified rheological model including the bubble.

The system of equations (1)-(10) is closed by the relation between the deformation e and the velocity of the solid v,

$$\frac{De}{Dt} \equiv \frac{\partial e}{\partial t} + v\frac{\partial e}{\partial x} = \frac{\partial v}{\partial x}\,. \qquad (11)$$

III. Propagation of P-waves including gas bubbles

Following Nikolaevskiy [22] we consider the slowly varying wave in space and time. Accordingly we use the running coordinate system with simultaneous scale change,

$$\begin{aligned} &\xi = \varepsilon(x - ct)\,, \qquad \tau = \frac{1}{2}\varepsilon^2 t\,,\\ &\frac{\partial}{\partial x} = \varepsilon\frac{\partial}{\partial \xi}\,, \qquad \frac{\partial}{\partial t} = \varepsilon\left(\frac{1}{2}\varepsilon\frac{\partial}{\partial \tau} - c\frac{\partial}{\partial \xi}\right)\,, \end{aligned} \qquad (12)$$

where ε is the small parameter. Thus, the constitutive law (10) transforms into the following form

$$\sigma + b_1 \varepsilon \left(\frac{1}{2}\varepsilon \frac{\partial}{\partial \tau} + (v-c)\frac{\partial}{\partial \xi} \right) \sigma = E_2 e + a_1 \varepsilon \left(\frac{1}{2}\varepsilon \frac{\partial}{\partial \tau} + (v-c)\frac{\partial}{\partial \xi} \right) e . \qquad (13)$$

Now, we seek the unknown functions as power series

$$v = \varepsilon v_1 + \varepsilon^2 v_2 + \ldots , \; m = m_0 + \varepsilon m_1 + \varepsilon^2 m_2 \ldots ,$$
$$\sigma = \sigma_0 + \varepsilon \sigma_1 + \varepsilon^2 \sigma_2 \ldots , \; p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 \ldots ,$$
$$\phi = \phi_0 + \varepsilon \phi_1 + \varepsilon^2 \phi_2 \ldots , \; R = R_0(1 + \varepsilon R_1 + \varepsilon^2 R_2 \ldots) , \qquad (14)$$
$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \ldots , \; e = e_0 + \varepsilon e_1 + \varepsilon^2 e_2 \ldots .$$

A. First linear approximation

Using Eqs. (14), we collect the linear terms $\sim \varepsilon$ in Eqs. (1), (2), (11) and (13) to get

$$\begin{aligned}
&\rho_0^{(s)} c \frac{\partial m_1}{\partial \xi} - (1-m_0) c \frac{\partial \rho_1^{(s)}}{\partial \xi} + (1-m_0)\rho_0^{(s)} \frac{\partial v_1}{\partial \xi} \\
&= -\tfrac{1}{2}(1-m_0)\tfrac{\partial \rho_0^{(s)}}{\partial \tau} , \\
&-m_0 c \frac{\partial \rho_1^{(f)}}{\partial \xi} - \rho_0^{(f)} c \frac{\partial m_1}{\partial \xi} + m_0 \rho_0^{(f)} \frac{\partial u_1}{\partial \xi} \\
&= -\tfrac{1}{2} m_0 \tfrac{\partial \rho_0^{(f)}}{\partial \tau} , \\
&-(1-m_0)\rho_0^{(s)} c \frac{\partial v_1}{\partial \xi} = \frac{\partial \sigma_1}{\partial \xi} - (1-m_0)\frac{\partial p_1}{\partial \xi} , \\
&-m_0 \rho_0^{(f)} c \frac{\partial u_1}{\partial \xi} = -m_0 \frac{\partial p_1}{\partial \xi} , \\
&\mu c \left[\frac{4}{R_0} + \frac{m_0 R_0}{\ell} \right] \frac{\partial R_0}{\partial \xi} = (p_0 \chi R_1 + p_1) , \\
&\frac{1}{2}\frac{\partial e_0}{\partial \tau} - c \frac{\partial e_1}{\partial \xi} + v_1 \frac{\partial e_0}{\partial \xi} = \frac{\partial v_1}{\partial \xi} , \; \sigma_1 - E_2 e_1 \\
&= -a_1 c \tfrac{\partial e_0}{\partial \xi} + b_1 c \tfrac{\partial \sigma_0}{\partial \xi} .
\end{aligned} \qquad (15)$$

Further,

$$\rho_1^{(s)} = -\rho_0^{(s)} \beta^{(s)} \sigma_1 ,$$
$$\rho_1^{(f)} = \rho_0^{(L)} \left(\beta^{(L)} \kappa_1 p_1 - 4\pi n_0 \kappa_2 R_0^3 R_1 \right) , \qquad (16)$$
$$\rho_0^{(f)} = \kappa_1 \kappa_2 \rho_0^{(L)} ,$$

where

$$\kappa_1 = 1 - \frac{4\pi}{3} R_0^3 n_0 , \; \kappa_2 = 1 + \beta^{(L)} p .$$

Inserting Eqs. (16) into the system (15) gives the following integrals,

$$(1-m_0)\rho_0^{(s)} v_1 + c(1-m_0)\rho_0^{(s)} \beta^{(s)} \sigma_1 + c\rho_0^{(s)} m_1 = 0 ,$$
$$m_0 \rho_0^{(L)} \kappa_1 \kappa_2 u_1 - c\rho_0^{(L)} \kappa_1 \kappa_2 m_1 - c m_0 \kappa_1 \rho_0^{(L)} \beta^{(L)} p_1$$
$$+ 4 c \kappa_2 \rho_0^{(L)} \pi n_0 m_0 R_0^3 R_1 = 0 ,$$
$$c(1-m_0)\rho_0^{(s)} v_1 + \sigma_1 - (1-m_0) p_1 = 0 , \qquad (17)$$
$$c \kappa_1 \kappa_2 m_0 \rho_0^{(L)} u_1 - m_0 p_1 = 0 ,$$
$$c e_1 + v_1 = 0 , \quad \sigma_1 - E_2 e_1 = 0 , \quad p_1 + p_0 \chi R_1 = 0 .$$

Now we have seven equations with seven unknowns: v_1, u_1, p_1, σ_1, m_1, R_1, and e_1. In order to find the velocity of the wave from the system (17) we require that

$$\det(a_{nm}) = 0 , \qquad (18)$$

where
$a_{11} = c(1-m_0)\rho_o^{(s)}$, $a_{12} = 0$, $a_{13} = (1-m_0)$, $a_{14} = 1$, $a_{15} = 0$, $a_{16} = 0$, $a_{17} = 0$, $a_{21} = 0$, $a_{22} = c\kappa_1\kappa_2 m_0 \rho_0^{(L)}$, $a_{23} = -m_0$, $a_{24} = 0$, $a_{25} = 0$, $a_{26} = 0$, $a_{27} = 0$, $a_{31} = (1-m_0)$, $a_{32} = 0$, $a_{33} = 0$, $a_{34} = c\beta^s(1-m_0)$, $a_{35} = c$, $a_{36} = 0$, $a_{37} = 0$, $a_{41} = 0$, $a_{42} = \kappa_1\kappa_2 m_0$, $a_{43} = -c\kappa_1\beta^{(L)} m_0$, $a_{44} = 0$, $a_{45} = -c\kappa_1\kappa_2$, $a_{46} = 4c\kappa_2\pi m_0 n_o R_0$, $a_{47} = 0$, $a_{51} = 1$, $a_{52} = 0$, $a_{53} = 0$, $a_{54} = 0$, $a_{55} = 0$, $a_{56} = 0$, $a_{57} = c$, $a_{61} = 0$, $a_{62} = 0$, $a_{63} = 0$, $a_{64} = 1$, $a_{65} = 0$, $a_{66} = 0$, $a_{67} = -E_2$, $a_{71} = 0$, $a_{72} = 0$, $a_{73} = 1$, $a_{74} = 0$, $a_{75} = 0$, $a_{76} = \chi p_0$, $a_{77} = 0$.

This gives

$$c^4 \left[m_0(1-m_0)\rho_0^{(s)}\rho_0^{(L)} \left(\kappa_1 \chi \beta^{(L)} p_0 + 4\kappa_2 \pi n_0 R_0^3 \right) \right]$$
$$-c^2 \left[\rho_0^{(L)} \left(\kappa_1 \chi p_0 (\kappa_2 (1 - E_2 \beta^{(s)})(-1+m_0)^2 + E_2 \beta^{(L)} m_0) \right.\right.$$
$$\left.\left. + 4E_2 \kappa_2 \pi m_0 n_0 R_0^3 \right) + \chi(1-m_0) m_0 p_0 \rho_0^{(s)} \right] + E_2 \chi m_0 p_0 = 0 . \qquad (19)$$

From equation (19) we find the velocity of the wave with the bubbles,

$$c^2 = \frac{-\beta_1 \pm \sqrt{\beta_1^2 - 4\alpha_1\gamma_1}}{2\alpha_1} , \qquad (20)$$

where

$$\alpha_1 = m_0(1-m_0)\rho_0^{(s)}\rho_0^{(L)} \left(\kappa_1 \chi \beta^{(L)} p_0 + 4\kappa_2 \pi n_0 R_0^3 \right) ,$$
$$\beta_1 = -\left[\rho_0^{(L)} \left(\kappa_1 \chi p_0 (\kappa_2 (1 - E_2 \beta^{(s)})(-1+m_0)^2 \right.\right.$$
$$\left.\left. + E_2 \beta^{(L)} m_0) + 4E_2 \kappa_2 \pi m_0 n_0 R_0^3 \right) + \chi(1-m_0) m_0 p_0 \rho_0^{(s)}) \right] ,$$
$$\gamma_1 = E_2 m_0 \chi p_0 .$$

Thus, all the variables are expressed through any one selected variable, for example, the velocity v_1. From the last three equations of system (17), we have

$$e_1 = -\frac{v_1}{c}, \quad \sigma_1 = -E_2 \frac{v_1}{c}, \quad p_1 = -p_0 \chi R_1 . \qquad (21)$$

Substituting of σ_1, and p_1 into the remaining equations of the system (17), we obtain

$$(1-m_0)\rho_0^{(s)} v_1 - E_2(1-m_0)\rho_0^{(s)} \beta^{(s)} v_1 + c\rho_0^{(s)} m_1 = 0 . \qquad (22)$$

$$m_0 u_1 - c m_1 + \left(\frac{p_0 \chi \beta^{(L)}}{\kappa_2} + \frac{4\pi n_0 R_0^3}{\kappa_1} \right) c m_0 R_1 = 0 . \qquad (23)$$

$$c(1-m_0)\rho_0^{(s)} v_1 - \frac{E_2}{c} v_1 + (1-m_0) p_0 \chi R_1 = 0 . \qquad (24)$$

$$c \kappa_1 \kappa_2 m_0 \rho_0^{(L)} u_1 + m_0 p_0 \chi R_1 = 0 . \qquad (25)$$

Equation (22) gives

$$m_1 = -(1-m_0)(1 - E_2 \beta^{(s)}) \frac{v_1}{c} . \qquad (26)$$

Then from (24) we get

$$R_1 = -\left(\frac{c^2(1-m_0)\rho_0^{(s)} - E_2}{(1-m_0)p_0\,\chi}\right)\frac{v_1}{c}\,. \tag{27}$$

Substituting (27) into the value of p_1 from (21), leads to

$$p_1 = \left(\frac{c^2(1-m_0)\rho_0^{(s)} - E_2}{(1-m_0)}\right)\frac{v_1}{c}\,. \tag{28}$$

Moreover, we derive the proportionality between v_1 and u_1 as

$$u_1 = \left(\frac{c(1-m_0)\rho_0^{(s)} - \frac{E_2}{c}}{\kappa_1\kappa_2\rho_0^{(L)}(1-m_0)}\right)\frac{v_1}{c}\,. \tag{29}$$

In the second approximation for the full system we have

B. Second linear approximation

$$\begin{aligned}
&\frac{\partial}{\partial\xi}\Big((1-m_0)v_2 + c(1-m_0)\beta^{(s)}\sigma_2 + cm_2\Big) = \Lambda^{(s)}\,,\\
&\frac{\partial}{\partial\xi}\Big(m_0u_2 - \Big[m_2 + \frac{m_0\beta^{(L)}p_2}{\kappa_2} - \frac{4\pi\, m_0\, n_0 R_0^3(R_2+R_1^2)}{\kappa_1}\\
&+\frac{4\pi\, m_0\, n_0 R_0^3\, p_0\,\chi\,\beta^{(L)}R_1^2}{\kappa_1\kappa_2}\Big]c\Big) = \Lambda^{(L)}\,,\\
&\frac{\partial}{\partial\xi}\Big(c(1-m_0)\rho_0^{(s)}v_2 + \sigma_2 - (1-m_0)p_2\Big) = \Sigma_1,\\
&\frac{\partial}{\partial\xi}\Big(c\kappa_1\kappa_2 m_0\rho_0^{(L)}u_2 - m_0p_2\Big) = \Sigma_2\,,\\
&\frac{\partial}{\partial\xi}\left(p_1 + p_0\,\chi\,R_1\right) = \frac{\partial\Gamma}{\partial\xi},\ \frac{\partial}{\partial\xi}\left(ce_2 + v_2\right) = F,\\
&\frac{\partial}{\partial\xi}\left(\sigma_2 - E_2\,e_2\right) = \frac{\partial T}{\partial\xi}\,,
\end{aligned} \tag{30}$$

where

$$\Lambda^{(s)} = \frac{1}{2}\frac{\partial}{\partial\tau}\left[(m_1 +\ (1-m_0)\beta^{(s)}\sigma_1)\right]\,,$$

$$\Lambda^{(L)} = -\frac{1}{2}\frac{\partial}{\partial\tau}\left[\kappa_1\left(m_1\kappa_2 + m_0\beta^{(L)}p_1\right) - 4\pi n_0\kappa_2 R_0^3 R_1\right],$$

$$\Sigma_1 = (1-m_0)\rho_0^{(s)}\frac{1}{2}\frac{\partial v_1}{\partial\tau}\,,\ \Sigma_2 = m_0\rho_0^{(f)}\frac{1}{2}\frac{\partial u_1}{\partial\tau}\,,$$

$$\Gamma = \mu c\left(4 + \frac{m_0R_0^2}{\ell}\right)\frac{\partial R_1}{\partial\xi}\,,\ F = -\frac{1}{2c}\frac{\partial v_1}{\partial\tau}\,,$$

$$T = -a_1c\frac{\partial e_1}{\partial\xi} + b_1c\frac{\partial\sigma_1}{\partial\xi}\,.$$

The determinant of the left-hand side of the system (30) coincides with the determinant of (18), which equals zero. Therefore, a non-zero solution for v_2 exists only if the following compatibility condition takes place,

$$\det(b_{nm}) = 0\,, \tag{31}$$

where
$b_{11} = \frac{\partial T}{\partial\xi}\,, b_{12} = 0\,, a_{13} = 0\,, b_{14} = 1\,, b_{15} = 0\,, b_{16} = 0\,, b_{17} = -E_2\,, b_{21} = \Sigma_1\,, b_{22} = 0\,, b_{23} = (1-m_0)\,, b_{24} = 1\,, b_{25} = 0\,, b_{26} = 0\,, b_{27} = 0\,, b_{31} = \Sigma_2\,, b_{32} = c\kappa_1\kappa_2 m_0\rho_0^{(L)}\,, a_{33} = -m_o\,, b_{34} = 0\,, b_{35} = 0\,, b_{36} = 0\,, b_{37} = 0\,, b_{41} = \frac{\partial\Gamma}{\partial\xi}\,, b_{42} = 0\,, b_{43} = 1\,, b_{44} = 0\,, b_{45} = 0\,, b_{46} = \chi p_0\,, b_{47} = 0\,, b_{51} = F\,, b_{52} = 0\,, b_{53} = 0\,, b_{54} = 0\,, b_{55} = 0\,, b_{56} = 0\,, b_{57} = c\,, b_{61} = \Lambda^{(s)}\,, b_{62} = 0\,, b_{63} = 0\,, b_{64} = c(1-m_0)\beta^{(s)}\,, b_{65} = c\,, b_{66} = 0\,, b_{67} = 0\,, b_{71} = \Lambda^{(L)}\,, b_{72} = m_0\,, b_{73} = -\frac{c\beta^{(L)}m_0}{\kappa_2}\,, b_{74} = 0\,, b_{75} = -c\,, b_{76} = \frac{4c\pi m_0 n_0 R_0^3}{\kappa_1}\,, b_{77} = 0\,.$

This gives the evolution equation for $v \cong v_1$

$$\begin{aligned}
&\chi\Big(c\Sigma_2 + \Big(E_2F - c\Big(\Sigma_1 + \Sigma_2 - \frac{\partial T}{\partial\xi}\Big)\Big)m_0\Big)p_0\\
&+c^2\Big(-4\kappa_2\pi m_0\Big(E_2F - c\Big(\Sigma_1 - \frac{\partial T}{\partial\xi} + \frac{\partial\Gamma}{\partial\xi} + c\frac{\partial\Gamma}{\partial\xi}m_0\Big)n_oR_0^3\Big)\\
&+\kappa_1\chi p_0\Big(\Big(-E_2F - c\Big(\frac{\partial T}{\partial\xi} - \Sigma_1\Big)\Big)\beta^{(L)}m_0 - \kappa_2(1-m_0)\Big(\Big(E_2F\\
&+c\frac{\partial T}{\partial\xi}\Big)\beta^{(s)}(-1+m_0) + \Lambda^{(L)} + \Lambda^{(s)}\Big)\Big)\Big)\rho_0^{(L)} = 0\,.
\end{aligned} \tag{32}$$

We re-write equation (32) in terms of v and re-arrange with the help of Mathematica software,

$$\begin{aligned}
&\frac{1}{2}\big[c(1-m_0)\rho_0^{(s)}\ \Upsilon_1 + c\ \Upsilon_2 + c^2\kappa_1\kappa_2\chi p_0\rho_0^{(L)}((1-m_0)^2\\
&-\Upsilon_3) - E_2\ \Upsilon_4\,\big]\frac{\partial v}{\partial\tau} + c^2\left[\Upsilon_4(a_1 - b_1E_2) - 4\Upsilon_5\right]\frac{\partial^2 v}{\partial\xi^2}\\
&+c\ \Upsilon_6\ \frac{\partial vv}{\partial\xi} = 0\,,
\end{aligned} \tag{33}$$

where

$\Upsilon_1 = cm_0\big(-\chi p_0 + c^2\rho_0^{(L)}(\kappa_1\chi p_0\beta^{(L)} + 4\kappa_2\pi n_0R_0^3)\big)\,,$

$\Upsilon_2 = \chi p_0 m_0\Big(c(1-m_0)\rho_0^{(s)} - \dfrac{E_2}{c}\Big)\,,$

$\Upsilon_3 = \kappa_1\kappa_2(1-m_0)^2(1-E_2\beta^{(s)}) - (c^2(1-m_0)\rho_0^{(s)} - E_2)(\kappa_1 m_0\rho_0^{(L)} + 4\kappa_2\pi n_0R_0^3)\,,$

$\Upsilon_4 = p_0\,\chi m_0 + c^2\rho_0^{(L)}\Big(\kappa_1 p_0\,\chi\big(\kappa_2\beta^{(s)}(-1+m_0)^2 - \beta^{(L)}m_0\big) - 4\pi n_0 m_0\kappa_2R_0^3\Big)\,,$

$\Upsilon_5 = c^2\pi n_0 m_0\kappa_2R_0^3\rho_0^{(L)}\mu\big(4 + \dfrac{m_0R_0^2}{\ell}\big)\Big(\dfrac{c^2(1-m_0)\rho_0^{(s)} - E_2}{p_0\,\chi}\Big)$

and Υ_6 is the nonlinearity coefficient, which we do not present here because our further analysis focuses on the linear part of equation (33).

Finally, we re-write the wave equation (33) as

$$A_1\frac{\partial v}{\partial\tau} + A_2\frac{\partial^2 v}{\partial\xi^2} + A_N\frac{\partial vv}{\partial\xi} = 0\,, \tag{34}$$

where

$A_1 = \dfrac{1}{2}\big[c(1-m_0)\rho_0^{(s)}\ \Upsilon_1 + c\ \Upsilon_2 + c^2\kappa_1\kappa_2\chi p_0\rho_0^{(L)}((1-m_0)^2 - \Upsilon_3) - E_2\ \Upsilon_4\,\big]\,,$

$A_2 = c^2\left[\Upsilon_4(a_1 - b_1E_2) - 4\Upsilon_5\right],\ A_N = c\ \Upsilon_6\ .$

IV. P-WAVES WITHOUT GAS BUBBLES

A. Rheological model to derive stress-strain relation

By removing one elastic spring segment, which represents the gas bubble, from the rheological model in Fig. (3) (see Fig. 4) we get the following constitutive law [25]

$$\sigma + b_1\frac{d\sigma}{dt} = E_2e + a_1\frac{de}{dt}\,, \tag{35}$$

where $a_1 = (E_1+E_2)\theta,\ b_1 = \theta,\ \ \theta = \mu/E_1.$

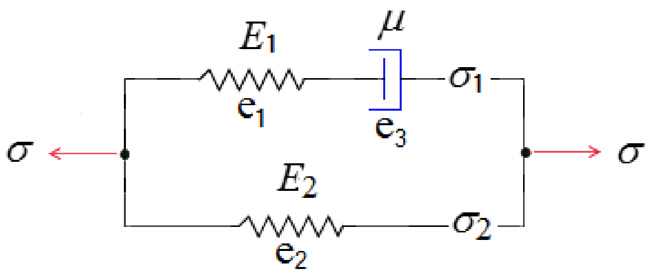

Fig. 4. Rheological model for standard linear model.

B. The equations of dynamics

In the case without the bubbles, the equations of motion reduce to the six equations

$$\begin{aligned}
&\tfrac{\partial}{\partial t}(1-m)\rho^{(s)} + \tfrac{\partial}{\partial x}(1-m)\rho^{(s)}v = 0\,,\\
&\frac{\partial}{\partial t}m\rho^{(L)} + \frac{\partial}{\partial x}m\rho^{(L)}u = 0\,,\\
&\frac{\partial}{\partial t}(1-m)\rho^{(s)}v + \frac{\partial}{\partial x}(1-m)\rho^{(s)}vv\\
&= \tfrac{\partial \sigma}{\partial x} - (1-m)\tfrac{\partial p}{\partial x} - I\,,\\
&\frac{\partial}{\partial t}m\rho^{(L)}u + \frac{\partial}{\partial x}m\rho^{(L)}uu = -m\frac{\partial p}{\partial x} + I\,,\\
&\sigma + b_1\frac{d\sigma}{dt} = E_2 e + a_1\frac{de}{dt}\,,\\
&\frac{De}{Dt} \equiv \frac{\partial e}{\partial t} + v\frac{\partial e}{\partial x} = \frac{\partial v}{\partial x}\,.
\end{aligned} \tag{36}$$

The density equation (3) for the solid remains unchanged, but for the gas-liquid mixture we neglect the volume gas content ϕ in equation (5),

$$\rho^{(f)} = \rho^{(L)} = \rho_0^{(L)}(1+\beta^{(L)}p)\,. \tag{37}$$

C. First linear approximation

The linear terms in the order $\sim \varepsilon$ in system (36)

$$\begin{aligned}
&c(1-m_0)\rho_0^{(s)}v_1 + \sigma_1 - (1-m_0)p_1 = 0\,,\\
&m_0u_1 - cm_1 - cm_0\beta^{(L)}p_1 = 0\,,\\
&(1-m_0)v_1 + c(1-m_0)\beta^{(s)}\sigma_1 + cm_1 = 0\,,\\
&cm_0\rho_0^{(L)}u_1 - m_0p_1 = 0\,,\\
&\sigma_1 - E_2\,e_1 = 0\,, \quad ce_1 + v_1 = 0\,.
\end{aligned} \tag{38}$$

In system (38), to find the velocity c we require

$$\det(a_{nm}) = 0\,, \tag{39}$$

where
$a_{11} = c(1-m_0)\rho_0^{(s)}$, $a_{12} = 0$, $a_{13} = (1-m_0)$, $a_{14} = 1$, $a_{15} = 0$, $a_{16} = 0$, $a_{21} = 0$, $a_{22} = m_0$, $a_{23} = -c\beta^{(L)}m_0$, $a_{24} = 0$, $a_{25} = -c$, $a_{26} = 0$, $a_{31} = (1-m_0)$, $a_{32} = 0$, $a_{33} = 0$, $a_{34} = c\beta^{(s)}(1-m_0)$, $a_{35} = c$, $a_{36} = 0$, $a_{41} = 0$, $a_{42} = cm_o\rho_0^{(L)}$, $a_{43} = -m_0$, $a_{44} = 0$, $a_{45} = 0$, $a_{46} = 0$, $a_{51} = 0$, $a_{52} = 0$, $a_{53} = 0$, $a_{54} = 1$, $a_{55} = 0$, $a_{56} = -E_2$, $a_{61} = 1$, $a_{62} = 0$, $a_{63} = 0$, $a_{64} = 0$, $a_{65} = 0$, $a_{66} = c$.

This gives

$$\begin{aligned}
&c^4\rho_0^{(s)}\rho_0^{(L)}\beta^{(s)}(1-m_0)m_0 - c^2\Big(\rho_0^{(L)}\big((1-E_2\beta^{(s)})(-1+m_0)^2\big)\\
&+E_2\beta^{(L)}m_0 + \rho_0^{(s)}m_0(1-m_0)\Big) + E_2m_0 = 0\,.
\end{aligned} \tag{40}$$

Equation (40) gives the velocity of the wave without the bubbles,

$$c^2 = \frac{-\beta_2 \pm \sqrt{\beta_2^2 - 4\alpha_2\gamma_2}}{2\alpha_2}\,, \tag{41}$$

where

$$\begin{aligned}
&\alpha_2 = \rho_0^{(s)}\rho_0^{(L)}\beta^{(s)}(1-m_0)m_0\,,\\
&\beta_2 = -\Big(\rho_0^{(L)}\big((1-E_2\beta^{(s)})(-1+m_0)^2 + E_2\beta^{(L)}m_0\big)\\
&+\rho_0^{(s)}m_0(1-m_0)\Big)\,,\\
&\gamma_2 = E_2m_0\,.
\end{aligned}$$

The terms for e_1, σ_1, p_1, and m_1 remain the same, while the proportion between v_1 and u_1 becomes

$$u_1 = \left(\frac{c(1-m_0)\rho_0^{(s)} - \frac{E_2}{c}}{\rho_0^{(L)}(1-m_0)}\right)\frac{v_1}{c}\,. \tag{42}$$

D. Second linear approximation

Collecting the quadratic terms$\sim \varepsilon^2$ in system (36) gives

$$\begin{aligned}
&\tfrac{\partial}{\partial \xi}(\sigma_2 - E_2\,e_2) = \tfrac{\partial T}{\partial \xi}\,, \quad \tfrac{\partial}{\partial \xi}(ce_2 + v_2) = F\,,\\
&c(1-m_0)\rho_0^{(s)}\frac{\partial v_2}{\partial \xi} + \frac{\partial \sigma_2}{\partial \xi} - (1-m_0)\frac{\partial p_2}{\partial \xi} = \Sigma_1\,,\\
&(cm_0\rho_0^{(L)}\frac{\partial u_2}{\partial \xi} - m_0\frac{\partial p_2}{\partial \xi}) = \Sigma_2\,,\\
&(1-m_0)\frac{\partial v_2}{\partial \xi} + c(1-m_0)\beta^{(s)}\frac{\partial \sigma_2}{\partial \xi} + c\frac{\partial m_2}{\partial \xi} = \Lambda^{(s)}\,,\\
&m_0\frac{\partial u_2}{\partial \xi} - c\frac{\partial m_2}{\partial \xi} - cm_0\beta^{(L)}\frac{\partial p_2}{\partial \xi} = \Lambda^{(L)}\,,
\end{aligned} \tag{43}$$

where the formulas of F, Σ_1, and $\Lambda^{(s)}$ are the same, while the formulas for Σ_2, $\Lambda^{(L)}$, and T are changed to

$$\Sigma_2 = m_0\rho_0^{(L)}\frac{1}{2}\frac{\partial u_1}{\partial \tau}\,,$$

$$\Lambda^{(L)} = -\frac{1}{2}\frac{\partial}{\partial \tau}\left(m_1 + m_0\beta^{(L)}p_1\right)\,,$$

$$T = -a_1c\frac{\partial e_1}{\partial \xi} + b_1c\frac{\partial \sigma_1}{\partial \xi}\,.$$

In analogy to (31), the compatibility condition for the system (43) has the form

$$\det(b_{nm}) = 0\,, \tag{44}$$

where
$b_{11} = \frac{\partial T}{\partial \xi}$, $b_{12} = 0$, $b_{13} = 0$, $b_{14} = 1$, $b_{15} = 0$, $b_{16} = -E_2$, $b_{21} = F$, $b_{22} = 0$, $b_{23} = 0$, $b_{24} = 0$, $b_{25} = 0$, $b_{26} = c$, $b_{31} = \Sigma_1$, $b_{32} = 0$, $b_{33} = (1-m_0)$, $b_{34} = 1$, $b_{35} = 0$, $b_{36} = 0$, $b_{41} = \Sigma_2$, $b_{42} = cm_o\rho_0^{(L)}$, $b_{43} = -m_0$, $b_{44} = 0$, $b_{45} = 0$, $b_{46} = 0$, $b_{51} = \Lambda^{(s)}$, $b_{52} = 0$, $b_{53} = 0$, $b_{54} = c\beta^{(s)}(1-m_0)$, $b_{55} = c$, $b_{56} = 0$, $b_{61} = \Lambda^{(L)}$, $b_{62} = m_0$, $b_{63} = -c\beta^{(L)}m_0$, $b_{64} = 0$, $b_{65} = -c$, $b_{66} = 0$.
Then the evolution equation for $v \cong v_1$ is

$$\begin{aligned}
&c\Sigma_2 + \left(E_2F - c\left(\Sigma_1 + \Sigma_2 - \frac{\partial T}{\partial \xi}\right)\right)m_0\beta^{(L)}m_0\\
&+c^2\left(\left(-E_2F - c\left(\frac{\partial T}{\partial \xi} - \Sigma_1\right)\right) - (1-m_0)\right.\\
&\left.\left(\left(E_2F + c\frac{\partial T}{\partial \xi}\right)\beta^s(-1+m_0) + \Lambda^{(L)} + \Lambda^{(s)}\right)\right)\rho_0^{(L)} = 0\,.
\end{aligned} \tag{45}$$

Now, we re-write equation (45) in terms of v and re-arrange,

$$\begin{aligned}
&\frac{1}{2}\left(\lambda_1 + \lambda_2 - (\lambda_3 + \lambda_4 + E_2\lambda_5)\right)\frac{\partial v}{\partial \tau}\\
&+c^2\,\lambda_5\,(a_1 - b_1E_2)\frac{\partial^2 v}{\partial \xi^2} + c\,\lambda_6\,\frac{\partial vv}{\partial \xi} = 0\,,
\end{aligned} \tag{46}$$

where

$$\lambda_1 = cm_0^2\Big(c(1-m_0)\rho_0^{(s)} - \frac{E_2}{c}\Big)\,,$$

$$\lambda_2 = c^2 m_0(1-m_0)^2\rho_0^{(L)}\,,$$

$$\lambda_3 = c^2 m_0\rho_0^{(L)}\Big((1-m_0)^2(1-E_2\beta^{(s)}) - m_0\beta^{(L)}(c^2(1-m_0)\rho_0^{(s)} - E_2))\Big)\,,$$

$$\lambda_4 = c^2 m_0^2\rho_0^{(s)}(1-m_0)(1-c^2\rho_0^{(L)}\beta^{(L)})\,,$$

$$\lambda_5 = m_0\Big(m_0 + c^2\rho_0^{(L)}(\beta^{(s)}(-1+m_0)^2 - m_0\beta^{(L)})\Big)\,,$$

and λ_6 is the nonlinearity coefficient. The above equation then becomes

$$B_1\frac{\partial v}{\partial \tau} + B_2\frac{\partial^2 v}{\partial \xi^2} + B_N\frac{\partial vv}{\partial \xi} = 0\,, \qquad (47)$$

where

$$B_1 = \frac{1}{2}\big(\lambda_1 + \lambda_2 - (\lambda_3 + \lambda_4 + E_2\lambda_5)\big)\,,$$

$$B_2 = c^2\,\lambda_5\,(a_1 - b_1E_2)\,,\ B_N = c\,\lambda_6\ .$$

V. Linearized model

In this section we consider the linearized version of the models (34) and (47). Our main interest is its dissipative part responsible for decay (attenuation) of the wave.

A. Evaluation of the parameters and the wave velocity

From [15], [16], [26], the values of the parameters are: densities, $\rho_0^{(L)} = 1000\,\mathrm{kg/m^3}$ for water, $\rho^{(g)} = 2\,\mathrm{kg/m^3}$ for gas, $\rho_0^{(s)} = 2500\,\mathrm{kg/m^3}$ for solid; porosity $m_0 = 0.25$; compressibility $\beta^{(L)} = 2\times 10^{-9}\,\mathrm{Pa}^{-1}$ for water, $\beta^{(L)} = 2.4\times 10^{-6}\,\mathrm{Pa}^{-1}$ for gas, $\beta^{(s)} = 2\times 10^{-10}\,\mathrm{Pa}^{-1}$ for solid; steady pressure $p_0 = 10^3\,\mathrm{Pa}$; bubble radius $R_0 = 10^{-4}\,\mathrm{m}$; volume gas content $\phi_0 = 10^{-3}$; viscosity $\mu = 10^{-3}\,\mathrm{Pa{\cdot}s}$; adiabatic exponent $\zeta = 1.4$, and permeability $\ell = 1.8\times 10^{-11}\,\mathrm{m^2}$. Using the data from [23], [27], [28], [29], the values of the parameters of the rheological scheme in Fig. 3 are

$$(a)\quad E_1 = 1/\beta^{(L)} = 4\times 10^5\,\mathrm{Pa}\,,\ E_2 = c^2\rho_0 = 2\times 10^7\,\mathrm{Pa}\,,$$

$$E_3 = 3\chi\,p_0 = 4\times 10^7\,\mathrm{Pa}\,,$$

where we used, just for the purpose of evaluating of E_i and M_i, the typical velocity $c \sim 100\,\mathrm{m/s}$ and the linear size of the oscillator $L_s = 0.3\,\mathrm{cm}$ from [27], [30].

We will also explore the values of E_i obtained by a different method, namely by using the formula $c^2\rho$ for all three phases, with ρ being the density of the liquid, solid and gas, respectively,

$$(b)\quad E_1 = c^2\rho^{(L)} = 1000\times 10^4\,\mathrm{Pa}\,,$$

$$E_2 = c^2\rho^{(s)} = 25\times 10^6\,\mathrm{Pa}\,,\ E_3 = c^2\rho^{(g)} = 2\times 10^4\,\mathrm{Pa}\,.$$

According to [1], [2], the equation (20) gives the velocity of P-waves with the bubbles: for the P1-wave $c \approx 103\,\mathrm{m/s}$ and $c \approx 116\,\mathrm{m/s}$ for the both variants (a) and (b) respectively; for the P2-wave $c \approx 3\,\mathrm{m/s}$ for the both variants (a) and (b). We see that the velocity of P2-wave is indeed smaller than the velocity of P1-wave.

The results of equation (41) gives the velocity of P-waves without the bubbles. For the P1-wave using the variants (a) and (b) gives $c \approx 1050\,\mathrm{m/s}$, while for the P2-wave $c \approx 70\,\mathrm{m/s}$ and $c \approx 78\,\mathrm{m/s}$ for the both variants (a) and (b) respectively. These results confirm that the presence of gas bubbles significantly decreases the P-waves velocities [20].

Furthermore, we observed that the velocity (41) of the P-waves without the bubbles is almost the same as the velocity of P-waves with the bubbles (20) when we set ($n_0 = 0$ and $R_0 = 0$) for the both variants (a) and (b).

B. Dissipation rate

In this section we are again interested in the effect of the bubbles on the wave dissipation. Therefore, we consider the linearized wave equations (34) and (47). The linearized form of Eq. (34) can be written as

$$\frac{\partial v}{\partial \tau} = -\frac{A_2}{A_1}\frac{\partial^2 v}{\partial \xi^2}\,. \qquad (48)$$

Now using the Fourier modes $v \sim \exp(\lambda t + ikx)$, we get the dissipation relation

$$\lambda(k) = \frac{A_2}{A_1}k^2\,, \qquad (49)$$

where λ is the decay rate and k is the wave number. For the case without the bubbles the linearized form of equation (47) is

$$\frac{\partial v}{\partial \tau} = -\frac{B_2}{B_1}\frac{\partial^2 v}{\partial \xi^2}\,. \qquad (50)$$

Then the dissipation relation is

$$\lambda(k) = \frac{B_2}{B_1}k^2\,. \qquad (51)$$

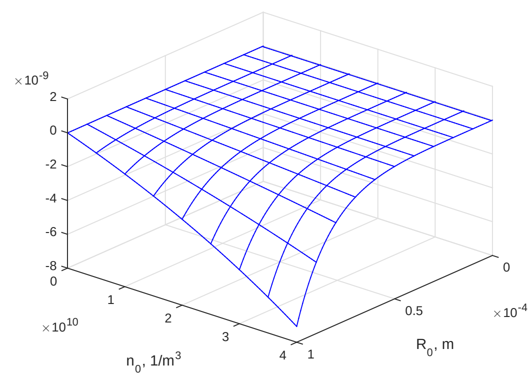

Fig. 5. The decay rate by formula (49) for variant (a), $k_* = 0.25$ 1/m.

The plot in Fig. 5 shows the decay rate at fixed $k_* = 0.25$ 1/m [21] against R_0 and n_0. As mentioned earlier, the decay rate is significantly affected by the increase in R_0 and becomes large in absolute value; this is because the bubbles affect the system through the pressure $p_1 = -p_0\,\chi\,R_1$. As for n_0, one should disregard the region of small n_0 in Fig. 5 where the equations of continuum mechanics cease to be valid. This is because the used assumption that each bubble is embedded in its own fluid particle (see Eq. (2)) is no longer inapplicable due to the large size of the particle.

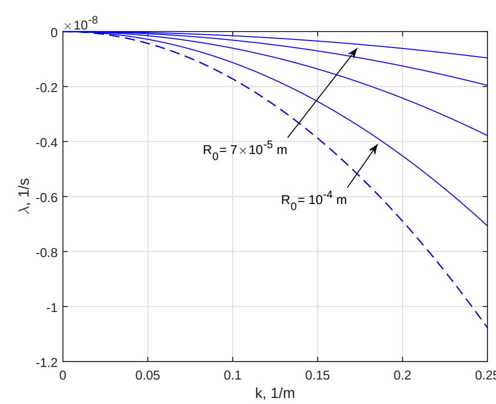

Fig. 7. The decay rate by formulas (49) and (51) for variant (a): R_0 varies, $n_0 = 4\times 10^{10}$.

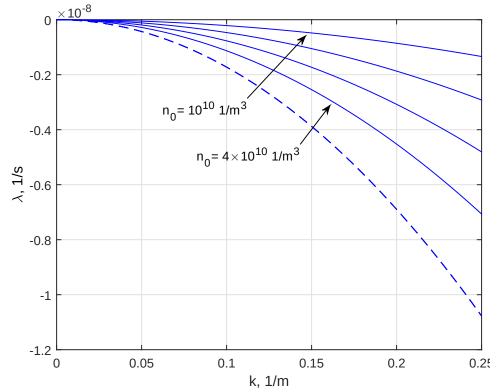

Fig. 6. The decay rate by formulas (49) and (51) for variant (a): n_0 varies, $R_0 = 10^{-4}$.

Figs. 6 and 7 compare the decay curves of the wave with the bubbles and the wave without the bubbles. The dashed line describes the case without the bubbles and the solid lines correspond to the wave with the bubbles. Fig. 6 is for varying n_0 and fixed R_0. Fig. 7 is for varying R_0 and fixed n_0. We clearly see that the curves lie entirely below zero, which means that the wave decays and the decay rate depends on the number and radius of the bubbles. This result agrees with the conception emphasized in [32], [33] about the essentially dissipative nature of the freely propagating elastic wave. Similar results are obtained for variants (b) as shown in Figs. 8–10.

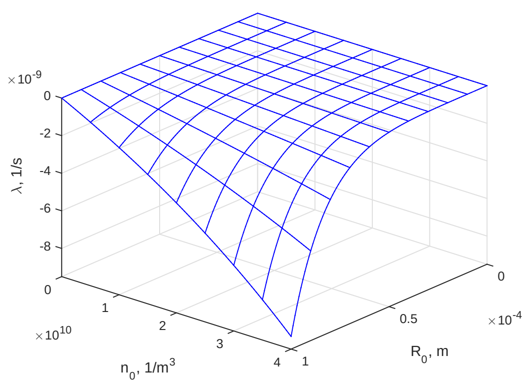

Fig. 8. The decay rate by formula (49) for variant (b), $k_* = 0.25$ 1/m.

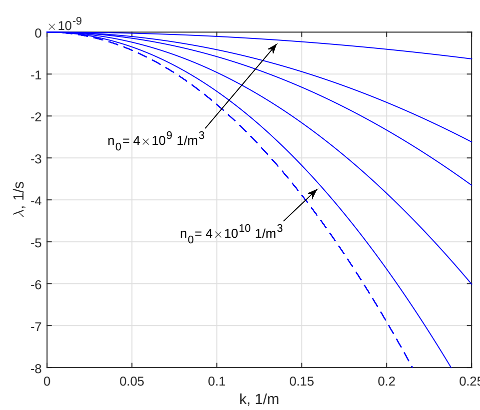

Fig. 9. The decay rate by formulas (49) and (51) for variant (b): n_0 varies, $R_0 = 10^{-4}$.

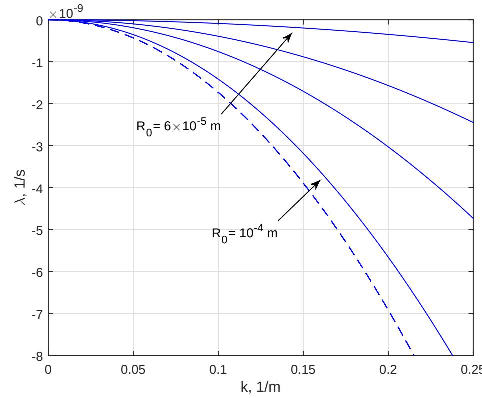

Fig. 10. The decay rate by formulas (49) and (51) for variant (b): R_0 varies, $n_0 = 4 \times 10^{10}$.

VI. Conclusions

We studied the effect of the rheology including bubbles on the Frenkel-Biot P-waves in porous rocks. Using two-segment rheology, we derived the P-type wave equations with and without the bubbles describing the velocity of the solid matrix in the medium. The linearized versions of the equations are compared in terms of the decay rate $\lambda(k)$ of the Fourier modes. For the both cases with and without the bubbles, the $\lambda(k)$-curve lies entirely below zero. We discovered that —$\lambda(k)$— increases with the increase of the radius and the number of the bubbles.

Nomenclature

$\beta^{(s)}$	Compressibility for solid, Pa^{-1}
$\beta^{(L)}$	Compressibility for water and gas, Pa^{-1}
$\rho_0^{(s)}$	Density for solid, kg/m^3
$\rho_0^{(L)}$	Density for water, kg/m^3
$\rho_0^{(g)}$	Density for gas, kg/m^3
R_0	Bubble radius, m
μ	Viscosity, Pa·s
ℓ	Permeability, m^2
p	Pressure, Pa
σ	Stress, Pa
ϕ	Volume gas content
n_0	Number of bubbles , 1/m^3
m_0	Porosity
ζ	Adiabatic exponent
k	Wave number, 1/m
λ	Decay rate, 1/s

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