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On the long-time behavior of the solution of a non linear viscoelastic plate equation with infinite memory and general kernel

Adel M. Al-Mahdi,
The Preparatory Year Program, King Fahd University of Petroleum and Mineralsy,
Dhahran 31261,
Saudi Arabia.

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Abstract- In this paper, we investigate the asymptotic behavior of the solution of a nonlinear viscoelastic plate equation with infinite memory. The nonlinearity in this problem is of a logarithmic type. We use a minimal condition on a relaxation function $h \in L^1(0,\infty)$; that is

$$h'(t) \le -\xi(t)H(h(t)),$$

where ξ is a nonincreasing function and H is an increasing and convex function near the origin. We establish an explicit energy decay formula under this very general assumption on the behavior of the relaxation function at infinity. Our results substantially improve some earlier results in the literature.

Keywords- Asymptotic behavior, Convex functions, Infinite memory, Logarithmic Sobolev inequalities, Plate equation.

I. Introduction

Viscoelastic plate equations have been studied by many authors and several stability results have been established. For example, Dafermos [1] considered the following abstract problem with infinite memory

$$u_{tt} + Au - \int_0^{+\infty} h(s)Au(t-s)ds = 0,\ t>0, \tag{1}$$

where A is a strictly positive self-adjoint linear operator and he proved that the energy tends asymptotically to zero, but no decay rate was given. Appleby et al. [2] studied the linear integro-differential equation

$$u_{tt} + Au(t) + \int_{-\infty}^{t} h(t-s)Au(s)ds = 0,\ t>0, \tag{2}$$

and established an exponential decay result for strong solutions in a Hilbert space. Pata [3] discussed the decay properties of the semigroup generated by the following equation:

$$u_{tt} + \alpha Au(t) + \beta u_t(t) - \int_0^{+\infty} h(s)Au(t-s)ds = 0,\ t>0, \tag{3}$$

where $\alpha > 0, \beta \ge 0$ and the memory kernel h is a decreasing function satisfying specific conditions. Subsequently, they established necessary as well as the sufficient conditions for the exponential stability. In [4], Guesmia considered

$$u_{tt} + Au - \int_0^{+\infty} h(s)Au(t-s)ds = 0,\ t>0, \tag{4}$$

and introduced a new approach for proving a more general decay result based on the properties of convex functions and the use of the generalized Young inequality. He used a larger class of infinite history relaxation functions satisfies the following condition

$$\int_0^{+\infty} \frac{h(s)}{H^{-1}(-h'(s))}ds + \sup_{s\in\mathbb{R}_+} \frac{h(s)}{H^{-1}(-h'(s))} < +\infty, \tag{5}$$

such that

$$H(0) = H'(0) = 0 \text{ and } \lim_{t\to+\infty} H'(t) = +\infty, \tag{6}$$

where $H : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing strictly convex function. Al-Mahdi and Al-Gharabli [5] considered the following

$$u_{tt} - \Delta u + \int_0^{+\infty} h(s)\Delta u(t-s)ds + |u_t|^{m-2}u_t = 0, \text{ in } D, \tag{7}$$

where $D = \Omega \times (0,+\infty)$. They established decay results with using a relaxation function h, satisfying the following condiation

$$h'(t) \le -\xi(t)h^p(t),\ t\ge 0,\ 1\le p < \frac{3}{2}. \tag{8}$$

Mustafa [6] consider the following coupled quasilinear system

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g_1(s)\Delta u(t-s)ds + f_1(u,v) = 0 \\ |v_t|^\rho v_{tt} - \Delta v - \Delta v_{tt} + \int_0^\infty g_2(s)\Delta v(t-s)ds + f_2(u,v) = 0 \end{cases} \tag{9}$$

and established more general decay rate results where the relaxation functions satisfy $g_i'(t) \le -H(g_i(t)),\ i = 1,2$. He provided more general decay rates for which the usual exponential and polynomial rates are only special cases. Recently, Al-Mahdi [7] consider the following viscoelastic plate problem with a velocity-dependent material density and a logarithmic nonlinearity:

$$|u_t|^\rho u_{tt} + \Delta^2 u + \Delta^2 u_{tt} - \int_0^{+\infty} g(s)\Delta^2 u(t-s)ds = ku \ln|u|, , \tag{10}$$

in D, where $D = \Omega \times (0,\infty)$ and Ω is a bounded domain of $\mathbb{R}^2$, with a smooth boundary $\partial\Omega$. He established an explicit and general decay rate results with imposing a minimal condition on the relaxation function, that is,

$$g'(t) \le -\xi(t)H(g(t)), \tag{11}$$

where the two functions ξ and H satisfy some conditions. Very recently, Al-Mahdi [8] considered the following plate problem:

$$u_{tt} - \sigma\Delta u_{tt} + \Delta^2 u - \int_0^{+\infty} g(s)\Delta^2 u(t-s)ds = 0,$$

and proved that the stability of this problem holds for which the relaxation function g satisfies the same condition (11). For a numerical study of a viscoelastic flow between porous moving walls, we refer to see [9]. For more result in this direction, we refer the reader to see [10–20].

A. Problems with Logarithmic Nonlinearity and their Applications

The logarithmic nonlinearity has many applications in physics such as nuclear physics, optics and geophysics [21–26]. For the problems with logarithmic nonlinearity, we start with the works of Birula and Mycielski [21] and [27] where they proved that the wave equations with the logarithmic nonlinearity have stable and localized solutions. Cazenave and Haraux [28] looked into the following Cauchy problem

$$u_{tt} - \Delta u = u \ln|u|^\alpha, \tag{12}$$

in $\mathbb{R}^3$. They established the existence and uniqueness of the solution. The corresponding one-dimensional problem of (12) is studied by Gorka [22] where he establish the global existence of weak solutions provided that $(u_0, u_1) \in H_0^1 \times L^2$. In [23], Bartkowski and Gorka investigated the weak solutions and also proved existence results of the classical solutions. Hiramatsu et al. [24] considered the following problem

$$u_{tt} - \Delta u + u + u_t + |u|^2 u = u \ln|u|, \tag{13}$$

and they investigated the numerical solutions of this problem without theoretical analysis. Recently, Al-Gharabli et al. [25] considered the following

$$u_{tt} + \Delta^2 u + u - \int_0^t h(t-s)\Delta^2 u(s)ds = \alpha u \ln|u|, \tag{14}$$

in $\Omega \times (0,\infty)$ and proved the existence and decay results of the solutions with imposing the following condition on the relaxation function

$$h'(t) \le -\xi(t)h^p(t),\ 1 \le p < \frac{3}{2}, \tag{15}$$

where α in (14) is a positive constant satisfies $\alpha < \alpha_0$ and

$$\sqrt{\frac{2\pi\ell}{\alpha_0 c_p}} = e^{-\frac{3}{2}-\frac{1}{\alpha_0}}, \tag{16}$$

where c_p is a positive constant satisfies the poincare inequality. In [26], Al-Gharabli et al. considered the following problem

$$|u_t|^\rho u_{tt} + \Delta^2 u + \Delta^2 u_{tt} - \int_0^t h(t-s)\Delta^2 u(s)ds = \alpha u \ln|u|,$$

in $\Omega \times (0,\infty)$ and as in [25] the authors proved the existence and decay results of the solutions with imposing the same two conditions (15) and (16). Very recently, Al-Gharabli [29] considered the same problem (14)with finite memory and established a general decay result for which the relaxation function h satisfies $h'(t) \le -\xi(t)H(h(t))$. For more results of some problems with logarithmic nonlinearity, we refer to the recent works in [30–34].

B. Our Problem and Motivations

Motivated by these importance and applications of the problems with logarithmic nonlinearity and the the lake of decay results of such those problems with using a wider class of the relaxation functions, **we consider the following viscoelastic plate problem with logarithmic nonlinearity:**

$$\begin{cases} u_{tt} + \Delta^2 u + u - \int_0^{+\infty} h(s)\Delta^2 u(t-s)ds = \alpha u \ln|u|, & \text{in } D, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{in } \partial\Omega \times (0,\infty), \\ u(x,-t) = u_0(x,t), \quad u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{17}$$

where $D = \Omega \times (0,\infty)$ and $\Omega \subseteq \mathbb{R}^2$ is a bounded domain with a smooth boundary $\partial\Omega$. The vector ν is the unit outer normal to $\partial\Omega$ and the constant α is a small positive real number measures the force of the nonlinear interaction and the nonlinear effects in quantum mechanics. The function h is the kernel and satisfies some conditions to be specified later.

In the present paper, we investigate the stability of the solution of our problem. In fact, we extend some earlier works for wave equations to the plate equation with logarithmic nonlinearity. We also extended some general decay results, known for the case of finite history, to the case of infinite history where the relaxation function satisfies a wider class of relaxation functions. Moreover, we drop the boundedness assumptions on the history data considered in many earlier results in the literature.

Our Methodology: We obtained our results by using the multiplier method with some logarithmic inequalities and some properties of integro-differential equations and inequalities. Our decay result is based on ξ, H and α.

Existing and Alternative Approaches: The existing approaches in the literature to prove the stability of our problem (17) exist in [25] and [26], however, the relaxation function h is especial case of our relaxation function defined in (19). Moreover, the same relaxation function used in the present paper is used in [7] and [8], however, that approach is completely different. The decay rate obtained in [7] and [8] is better than the decay rate obtained in the present paper but the current approach is much better and easier than the one in [7] and [8].

This paper is organized as follows. In section (II.), we present some notations, assumptions and the local and global existence result of our problem. In section (III.), we establish some lemmas needed in the proof of our result. The stability results with an example are presented in section (IV.). Some conclusions and future works are mentioned in Section (V.).

II. Preliminaries

In this section, we present some notations and material needed in the proof of our results. We use the standard Lebesgue space $L^2(\Omega)$ and Sobolev space $H_0^2(\Omega)$ with their usual scalar products and norms. Throughout this paper, c is used to denote a generic positive constant and we consider the following hypotheses:

$(A1)$ $h:\mathbb{R}^+\to\mathbb{R}^+$ is a C^1- nonincreasing function satisfying, for some $\beta_0>0$,

$$-\beta_0 h(s)\le h'(s),\ h(t)>0,\ 1-\int_0^{+\infty}h(s)ds:=\ell>0, \tag{18}$$

$(A2)$ There exists a function $H:(0,\infty)\to(0,\infty)$ in $C^1(\mathbb{R}_+)\cap C^2(\mathbb{R}_+^*)$ which is increasing and strictly convex, with $H(0)=H'(0)=0$, $\lim_{s\to+\infty}H'(s)=+\infty$, $s\mapsto sH'(s)$ and $s\mapsto s\left(H'\right)^{-1}(s)$ are convex on $(0,r]$ and there exists a nonincreasing function $\xi:\mathbb{R}_+\to\mathbb{R}_+$, such that

$$h'(t)\le -\xi(t)H(h(t)),\quad \forall\ \ t\ge 0. \tag{19}$$

$(A3)$ The constant α in (17) is such that

$$0<\alpha<\alpha_0=\frac{2\pi\ell e^3}{c_p}, \tag{20}$$

where c_p is the smallest positive number satisfying

$$\|\nabla u\|_2^2\le c_p\|\Delta u\|_2^2,\quad \forall u\in H_0^2(\Omega),$$

where $\|.\|_2=\|.\|_{L^2(\Omega)}$.

Remark 1 *Assumption $(A3)$ is needed for establishing the local existence of the solutions of the problem (17). For more details we refer to see [25].*

Remark 2 *If H is a strictly increasing and strictly convex C^2 function on $(0,r]$, with $H(0)=H'(0)=0$, then it has an extension $\overline{H}$, which is strictly increasing and strictly convex C^2 function on $(0,+\infty)$. For instance, if $H(r)=a, H'(r)=b, H''(r)=C$, we can define $\overline{H}$, for $t>r$, by*

$$\overline{H}(t)=\frac{C}{2}t^2+(b-Cr)t+\left(a+\frac{C}{2}r^2-br\right). \tag{21}$$

For simplicity, in the rest of this paper, we use H instead of $\overline{H}$

Remark 3 *Since H is strictly convex on $(0,r]$ and $H(0)=0$, then*

$$H(\theta t)\le\theta H(t),\ 0\le\theta\le 1 \text{ and } t\in(0,r]. \tag{22}$$

Remark 4 *The function $g(s)=\sqrt{\frac{2\pi\ell}{c_p s}}-e^{-\frac{3}{2}-\frac{1}{s}}$ is a continuous and decreasing function on $(0,\infty)$, with*

$$\lim_{s\to0^+}g(s)=\infty \text{ and } \lim_{s\to\infty}g(s)=-e^{-\frac{3}{2}}.$$

Then, there exists a unique $\alpha_0>0$ such that $g(\alpha_0)=0$. Moreover,

$$e^{-\frac{3}{2}-\frac{1}{s}}<\sqrt{\frac{2\pi\ell}{c_p s}},\ \forall s\in(0,\alpha_0), \tag{23}$$

which implies that the selection of α in $(A3)$ is possible.

The modified energy functional associated with problem (17) is given by

$$E(t)=\frac{1}{2}\left(\|u_t\|_2^2+\ell\|\Delta u\|_2^2+\frac{\alpha+2}{2}\|u\|_2^2\right)-\frac{1}{2}\int_\Omega u^2\ln|u|^\alpha dx+\frac{1}{2}(ho\Delta u)(t), \tag{24}$$

where

$$(ho\Delta u)(t)=\int_0^{+\infty}h(s)\|\Delta u(t)-\Delta u(t-s)\|_2^2ds.$$

Direct differentiation of (24) with using (17), leads to

$$E'(t)=\frac{1}{2}(h'o\Delta u)(t)\le 0. \tag{25}$$

Lemma 1 *[35, 36] (Logarithmic Sobolev inequality) Let u be any function in $H_0^1(\Omega)$ and a be any positive real number. Then*

$$\int_\Omega u^2\ln|u|dx\le\frac{1}{2}\|u\|_2^2\ln\|u\|_2^2+\frac{a^2}{2\pi}\|\nabla u\|_2^2-(1+\ln a)\|u\|_2^2. \tag{26}$$

Corollary 1 *Let u be any function in $H_0^2(\Omega)$ and a be any positive real number. Then*

$$\int_\Omega u^2\ln|u|dx\le\frac{1}{2}\|u\|_2^2\ln\|u\|_2^2+\frac{c_pa^2}{2\pi}\|\Delta u\|_2^2-(1+\ln a)\|u\|_2^2. \tag{27}$$

Now, we state without proofs the following existence result of the solution of our problem (17).

Theorem 1 *Let* $(u_0, u_1) \in H_0^2(\Omega)\times L^2(\Omega)$. *Assume that* $(A1)-(A3)$ *and the following selection of a*

$$e^{-\frac{3}{2}-\frac{1}{\alpha}} < a < \sqrt{\frac{2\pi\ell}{\alpha c_p}} \tag{28}$$

hold. Then problem (17) *has a weak solution*

$$u \in C([0,T], H_0^2(\Omega))\cap C^1([0,T], L^2(\Omega))\cap C^2([0,T], H^{-2}(\Omega)). \tag{29}$$

For the global existence, we introduce the following functionals:

$$J(t) = \frac{1}{2}\left(\ell\|\Delta u\|_2^2 + \|u\|_2^2 + (ho\Delta u)(t) - \int_\Omega u^2 \ln|u|^\alpha dx\right) + \frac{\alpha}{4}\|u\|_2^2 \tag{30}$$

and

$$I(t) = \ell\|\Delta u\|_2^2 + \|u\|_2^2 + (ho\Delta u)(t) - 3\int_\Omega u^2 \ln|u|^\alpha dx. \tag{31}$$

From (30) and (31), one can easily see that

$$J(t) = \frac{1}{3}\left[\ell\|\Delta u\|_2^2 + \|u\|_2^2 + (ho\Delta u)(t)\right] + \frac{\alpha}{4}\|u\|_2^2 + \frac{1}{6}I(t). \tag{32}$$

Therefor, we have the following important lemma

Lemma 2 *Assume that* $(A1)-(A3)$. *Let* $(u_0,u_1) \in H_0^2(\Omega)\times L^2(\Omega)$ *such that*

$$I(0) > 0 \text{ and } \sqrt{54}kc_*^3\left(\frac{E(0)}{\ell}\right)^{\frac{1}{2}} < \ell. \tag{33}$$

Then

$$I(t) > 0, \ \forall t \in [0,T). \tag{34}$$

The proofs of the above existence results can be established by following the same arguments in [25] with adapting the finite memory to infinite memory.

III. Technical lemmas

In this section, we start by establishing several lemmas needed for the proof of our main result.

Lemma 3 *There exists a positive constant* M_1 *such that*

$$\int_t^\infty h(s)\left(\triangle u(t) - \triangle u(t-s)\right)^2 ds dx \le M_1 h_1(t), \tag{35}$$

where $h_1(t) := \int_0^{+\infty} h(t+s)\left(1+||\triangle u_0(s)||^2\right)ds$.

Proof *The proof is based on the same arguments in [8]. In fact, we have*

$$\begin{aligned}
&\int_t^{+\infty} h(s)||\triangle u(t) - \triangle u(t-s)||^2 ds \\
&\le 2||\triangle u(t)||^2\int_t^{+\infty} h(s)ds + 2\int_t^{+\infty} h(s)||\triangle u(t-s)||^2 ds \\
&\le 2\sup_{s\ge 0}||\triangle u(s)||^2\int_0^{+\infty} h(t+s)ds + 2\int_0^{+\infty} g(t+s)||\triangle u(-s)||^2 ds \\
&\le \left(\frac{4}{\ell}E(s)\right)\int_0^{\infty} h(t+s)ds + 2\int_0^{\infty} h(t+s)||\triangle u_0(s)||^2 ds \\
&\le \left(\frac{4}{\ell}E(0)\right)\int_0^{+\infty} h(t+s)ds + 2\int_0^{+\infty} h(t+s)||\triangle u_0(s)||^2 ds \\
&\le M_1\int_0^{+\infty} h(t+s)\left(1+||\triangle u_0(s)||^2\right)ds,
\end{aligned} \tag{36}$$

where $M_1 = \max\left\{2, \frac{4E(0)}{\ell}\right\}$.

Lemma 4 *Assume that h satisfies* $(A1)$. *Then, for* $u \in H_0^2(\Omega)$,

$$\int_\Omega\left(\int_0^{+\infty} h(s)(\Delta u(t) - \Delta u(t-s))ds\right)^2 dx \le c(h\circ\Delta u)(t),$$

$$\int_\Omega\left(\int_0^{+\infty} h'(s)(\Delta u(t) - \Delta u(t-s))ds\right)^2 dx \le -c(h'\circ\Delta u)(t).$$

Proof *Using Cauchy-Schwarz inequality, we get*

$$\begin{aligned}
&\int_\Omega\left(\int_0^{+\infty} h(s)(\Delta u(t) - \Delta u(t-s))ds\right)^2 dx \\
&\le \int_\Omega\left(\int_0^{+\infty}\sqrt{h(s)}\sqrt{h(s)}(\Delta u(t) - \Delta u(t-s))ds\right)^2 dx \\
&\le c\int_\Omega\int_0^{+\infty} h(s)\left(\Delta u(t) - \Delta u(t-s))ds\right)^2 dx \le c(h\circ\Delta u)(t).
\end{aligned}$$

Similarly, we can establish the second estimate in this lemma.

Lemma 5 *Assume that* $(A1)-(A3)$ *and* (33) *are hold. Then the functionals*

$$\psi(t) = \int_\Omega uu_t dx,$$

$$\chi(t) = -\int_\Omega u_t\int_0^t h(t-s)(u(t)-u(s))dsdx$$

satisfy, along the solutions of (17), *for any* $\epsilon_0 \in (0,1)$ *and* $\delta > 0$, *the following estimates*

$$\psi'(t) \le ||u_t||_2^2 - \frac{\ell}{2}||\Delta u||_2^2 - ||u||_2^2 + \int_\Omega u^2\ln|u|^\alpha dx + c(ho\Delta u)(t). \tag{37}$$

$$\chi'(t) \le \delta||\Delta u||_2^2 + \frac{c}{\delta}(ho\Delta u)(t) + \frac{c}{\delta}(-h'o\Delta u)(t) + (\delta-(1-\ell))\|u_t\|_2^2 + c_{\epsilon_0,\delta}(ho\Delta u)^{\frac{1}{1+\epsilon_0}}(t). \tag{38}$$

Proof *Direct differentiations, using* (17), *we get*

$$\psi_1' = ||u_t||_2^2 - ||\Delta u||_2^2 - ||u||_2^2 + \int_\Omega \Delta u \int_0^\infty h(s)\Delta u(t-s)dsdx + \int_\Omega u^2 \ln|u|^\alpha dx. \quad (39)$$

We now use Lemma 4 and Young's inequality, to obtain, for any $\mu > 0$,

$$\int_\Omega \Delta u(t)\left(\int_0^\infty h(s)\Delta u(t-s)ds\right)dx \leq \left(1-\ell+\frac{\mu}{2}\right)||\Delta u||_2^2 + \frac{1}{2\mu}(1-\ell)(ho\Delta u)(t). \quad (40)$$

By choosing $\mu = \ell$ *and combining* (39) *and* (40), *we obtain* (37). *To prove* (38), *direct differentiations, using* (17), *gives*

$$\begin{aligned}\psi_2'(t) &= \int_\Omega \Delta u \int_0^\infty h(s)(\Delta u(t)-\Delta u(t-s))dsdx \\ &+ \int_\Omega u \int_0^\infty h(s)(u(t)-u(t-s))dsdx \\ &+ \int_\Omega \int_0^\infty h(s)(\Delta u(t)-\Delta u(t-s))ds \int_0^\infty h(s)\Delta u(s)dsdx \\ &- \int_\Omega u\ln|u|^\alpha \int_0^\infty h(s)(u(t)-u(t-s))dsdx \\ &- \int_\Omega u_t \int_0^\infty h'(s)(u(t)-u(t-s))dsdx \\ &- \left(\int_0^\infty h(s)ds\right)\int_\Omega u_t^2 dx. \end{aligned} \quad (41)$$

Similarly to (39), *we estimate the right-hand side terms of* (41). *So, by using Young's inequality, the first term gives, for any* $\delta > 0$,

$$\int_\Omega \Delta u \int_0^\infty h(s)(\Delta u(t)-\Delta u(t-s))dsdx \leq \frac{\delta}{4}||\Delta u||_2^2 + \frac{c}{\delta}(ho\Delta u)(t). \quad (42)$$

Using Lemma 4, Young's and Poincaré's inequalities, the second and fifth terms lead to

$$\int_\Omega u \int_0^\infty h(s)(u(t)-u(t-s))dsdx \leq \frac{\delta}{4}||\Delta u||_2^2 + \frac{c}{\delta}(ho\Delta u)(t) \quad (43)$$

and

$$-\int_\Omega u_\infty \int_0^t h'(s)(u(t)-u(t-s))dsdx \leq \delta||u_t||_2^2 - \frac{c}{\delta}(h'o\Delta u)(t). \quad (44)$$

Similarly, the third term can be estimated as follows

$$\int_\Omega \int_0^\infty h(s)(\Delta u(t)-\Delta u(t-s))ds \int_0^\infty h(s)\Delta u(t-s)dsdx \leq \frac{\delta}{4}||\Delta u||_2^2 + c\left(1+\frac{1}{\delta}\right)(ho\Delta u)(t). \quad (45)$$

Let $\epsilon_0 \in (0,1)$, *so the following inequality holds:*

$$s|\ln s| \leq s^2 + d_{\epsilon_0}s^{1-\epsilon_0}, \ \forall s > 0. \quad (46)$$

Applying (46) *to* $u\ln|u|$, *using Cauchy-Schwarz' inequality, the embedding of* $H_0^2(\Omega)$ *in* $L^\infty(\Omega)$ *and performing the same calculations as before, we get, for any* $\delta_1 > 0$,

$$\begin{aligned}&\int_\Omega u\ln|u|^\alpha \int_0^\infty h(s)(u(t)-u(t-s))dsdx \\ &\leq c\delta_1||\Delta u||_2^2 + \frac{c}{\delta_1}\int_\Omega \left|\int_0^\infty h(s)(u(t)-u(t-s))ds\right|^2 dx \\ &+ c_{\epsilon_0,\delta_1}\int_\Omega \left|\int_0^\infty h(s)(u(t)-u(t-s))ds\right|^{\frac{2}{1+\epsilon_0}} dx,\end{aligned}$$

then, putting $\frac{\delta}{4} = c\delta_1$ *and using Holder's inequality and Lemma 4, we find*

$$\int_\Omega u\ln|u|^\alpha \int_0^\infty h(s)(u(t)-u(t-s))dsdx \leq \frac{\delta}{4}||\Delta u||_2^2 + \frac{c}{\delta}(ho\Delta u)(t) + c_{\epsilon_0,\delta}(ho\Delta u)^{\frac{1}{1+\epsilon_0}}(t). \quad (47)$$

The above inequalities imply (38).

Lemma 6 *Assume that* $(A1)-(A3)$ *and* (33) *hold and let* $\varepsilon_0 \in (0,1)$. *Assume that*

$$0 < E(0) < \frac{e\ell\pi}{4c_p}. \quad (48)$$

Then, for α *small enough, there exist positive constants* ε *and* N *such that the functional*

$$L := E(t) + \varepsilon_1\psi(t) + \varepsilon_2\chi(t),$$

satisfies

$$L \sim E \quad (49)$$

and, for any $t \geq 0$, *there exists a positive constant* m *such that*

$$L'(t) \leq -mE(t) + c(h\circ\Delta u)(t) + c_{\epsilon_0}(h\circ\Delta u)^{\frac{1}{1+\epsilon_0}}(t). \quad (50)$$

Proof *For the proof of* (49), *we see that, using similar calculations as before,*

$$\begin{aligned}|L(t)-E(t)| &= |\varepsilon_1\psi(t)+\varepsilon_2\chi(t)| \\ &\leq c(\varepsilon_1+\varepsilon_2)\left(||u_t||_2^2 + ||\Delta u||_2^2 + (ho\Delta u)(t)\right),\end{aligned}$$

therefore, from (34) *and* (32), *we obtain*

$$|L(t)-E(t)| \leq c(\varepsilon_1+\varepsilon_2)\left(\frac{1}{2}||u_t||_2^2 + J(t)\right) = c(\varepsilon_1+\varepsilon_2)E(t),$$

then

$$(1-c(\varepsilon_1+\varepsilon_2))E(t) \leq L(t) \leq (1+c(\varepsilon_1+\varepsilon_2))E(t).$$

Hence, for ε_1, $\varepsilon_2 > 0$ *satisfying*

$$1 - c(\varepsilon_1+\varepsilon_2) > 0, \quad (51)$$

the equivalence (49) holds. To prove (50), we let $\int_0^{+\infty} h(s)ds =: h_0$ and use (25), (37), (38) and the definition of $E(t)$, therefor for any $t \geq 0$ and $m > 0$ we have

$$\begin{aligned}
L'(t) &\leq -mE(t) - \left(\varepsilon_2(h_0-\delta) - \varepsilon_1 - \frac{m}{2}\right)\|u_t\|_2^2 \\
&- \left(\frac{\ell}{2}\varepsilon_1 - \varepsilon_2\delta - \frac{m}{2}\right)\|\Delta u\|_2^2 - \left(\varepsilon_1 - \frac{(\alpha+2)m}{4}\right)\|u\|_2^2 \\
&+ \left(\alpha\varepsilon_1 - \alpha\frac{m}{2}\right)\int_\Omega u^2 \ln|u|dx + \left(c\varepsilon_1 + \varepsilon_2\frac{c}{\delta} + \frac{m}{2}\right)(ho\Delta u)(t) \\
&+ \left(\frac{1}{2} - \frac{c\varepsilon_2}{\delta}\right)(h'o\Delta u)(t) + \varepsilon_2 c_{\epsilon_0,\delta}(ho\Delta u)^{\frac{1}{1+\epsilon_0}}(t).
\end{aligned} \tag{52}$$

Using the Logarithmic Sobolev inequality, for $0 < m < 2\varepsilon_1$, we get

$$\begin{aligned}
L'(t) &\leq -mE(t) - \left(\varepsilon_2(h_0-\delta) - \varepsilon_1 - \frac{m}{2}\right)\|u_t\|_2^2 \\
&- \left(\frac{\ell}{2}\varepsilon_1 - \varepsilon_2\delta - \frac{m}{2} - \alpha\left(\varepsilon_1 - \frac{m}{2}\right)\frac{c_p a^2}{2\pi}\right)\|\Delta u\|_2^2 \\
&- \left(\varepsilon_1 - \frac{m(\alpha+2)}{4} + \alpha\left(\varepsilon_1 - \frac{m}{2}\right)(1+\ln a)\right)\|u\|_2^2 \ln\|u\|_2^2 \\
&- \left(\alpha\left(\frac{m}{4} - \frac{\varepsilon_1}{2}\right)\right)\|u\|_2^2 + \left(c\varepsilon_1 + \varepsilon_2\frac{c}{\delta} + \frac{m}{2}\right)(ho\Delta u)(t) \\
&+ \left(\frac{1}{2} - \frac{c\varepsilon_2}{\delta}\right)(h'o\Delta u)(t) + \varepsilon_2 c_{\epsilon_0,\delta}(ho\Delta u)^{\frac{1}{1+\epsilon_0}}(t).
\end{aligned} \tag{53}$$

At this point we choose δ so small that

$$h_0 - \delta > \frac{1}{2}h_0 \quad \text{and} \quad \delta < \frac{\ell h_0}{16}.$$

Whence δ is fixed, the choice of any two positive constants ε_1 and ε_2 satisfying

$$\frac{h_0}{4}\varepsilon_2 < \varepsilon_1 < \frac{h_0}{2}\varepsilon_2 \tag{54}$$

will make

$$k_1 := \varepsilon_2(h_0-\delta) - \varepsilon_1 > 0 \quad \text{and} \quad k_2 := \frac{\ell}{2}\varepsilon_1 - \varepsilon_2\delta > 0.$$

Then, we choose ε_1 and ε_2 so small so that (51) and (54) remain valid and, further,

$$\frac{1}{2} - \frac{c\varepsilon_2}{\delta} > 0.$$

Consequently, we get (49) and

$$\begin{aligned}
L(t) &\leq -mE(t) - \left(k_1 - \frac{m}{2}\right)\|u_t\|_2^2 \\
&- \left(k_2 - \frac{m}{2} - \alpha\left(\varepsilon_1 - \frac{m}{2}\right)\frac{c_p a^2}{2\pi}\right)\|\Delta u\|_2^2 \\
&- \left(\varepsilon_1 - \frac{m(\alpha+2)}{4} + \alpha\left(\varepsilon_1 - \frac{m}{2}\right)(1+\ln a)\right)\|u\|_2^2 \ln\|u\|_2^2 \\
&- \left(\alpha\left(\frac{m}{4} - \frac{\varepsilon_1}{2}\right)\right)\|u\|_2^2 + c(ho\Delta u)(t) + c_{\epsilon_0,\delta}(ho\Delta u)^{\frac{1}{1+\epsilon_0}}(t).
\end{aligned} \tag{55}$$

By choosing a satisfying (28) and α so small so that

$$\alpha_1 = k_1 - \frac{m}{2} > 0, \;\; \alpha_2 = k_2 - \frac{m}{2} - \alpha\left(\varepsilon_1 - \frac{m}{2}\right)\frac{c_p a^2}{2\pi} > 0$$

and

$$\begin{aligned}
\alpha_3 &= \varepsilon_1 - \frac{m(\alpha+2)}{4} + \alpha\left(\varepsilon_1 - \frac{m}{2}\right)(1+\ln a) \\
&+ \alpha\left(\frac{m}{4} - \frac{\varepsilon_1}{2}\right)\ln\|u\|_2^2 > 0.
\end{aligned}$$

Therefore, we arrive at the desired result (50).

Remark 5 *Recalling (24), (25), (30) and (34) , we have*

$$E(0) \geq E(t) = J(t) + \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} \geq J(t) \geq \frac{1}{3}(h\circ\Delta u)(t),$$

which gives

$$(h\circ\Delta u)(t) \leq 3E(0). \tag{56}$$

Using (56), we obtain the following

$$\begin{aligned}
(h\circ\Delta u)(t) &= (h\circ\Delta u)^{\frac{\epsilon_0}{1+\epsilon_0}}(t)(h\circ\Delta u)^{\frac{1}{1+\epsilon_0}}(t) \\
&\leq c(h\circ\Delta u)^{\frac{1}{1+\epsilon_0}}(t)
\end{aligned} \tag{57}$$

Lemma 7 *If (A1)−(A2) are satisfied, then we have, for all $t > 0$, the following estimate*

$$\int_0^t h(s)||\Delta u(t) - \Delta(t-s)||_2^2 ds \leq \frac{t}{q}H^{-1}\left(\frac{q\mu(t)}{t\xi(t)}\right) \tag{58}$$

where q small enough, H is defined in Remark (2) and

$$\mu(t) := -\int_0^t h'(s)||\Delta u(t) - \Delta(t-s)||_2^2 ds \leq -cE'(t), \tag{59}$$

.

Proof *To establish (58), we introduce the following functional*

$$\lambda(t) := \frac{q}{t}\int_0^t ||\Delta u(t) - \Delta(t-s)||_2^2 ds. \tag{60}$$

Then, using the fact that E is nonincreasing and (24) to get

$$\begin{aligned}
\lambda(t) &\leq \frac{2q}{t}\left(\int_0^t ||\Delta u(t)||_2^2 + \int_0^t ||\Delta(t-s)||_2^2 ds\right). \\
&\leq \frac{4q}{\ell t}\left(\int_0^t \big(E(t) + E(t-s)\big)ds\right) \\
&\leq \frac{8q}{\ell t}\int_0^t E(s)ds \\
&\leq \frac{8q}{\ell t}\int_0^t E(0)ds \\
&< +\infty.
\end{aligned} \tag{61}$$

Thus, q can be chosen so small so that, for all $t > 0$,

$$\lambda(t) < 1. \tag{62}$$

Without loss of the generality, for all $t > 0$, we assume that $\mu(t) > 0$, otherwise we get an exponential decay from

(50). *The use of Jensen's inequality and using* (59), (3) *and* (62) *gives*

$$\begin{aligned}\mu(t) &= \frac{1}{q\lambda(t)}\int_0^t \lambda(t)(-h'(s))\int_\Omega q|\Delta u(t)-\Delta(t-s)|^2 dxds\\ &\geq \frac{1}{q\lambda(t)}\int_0^t \lambda(t)\xi(s)H(h(s))\int_\Omega q|\Delta u(t)-\Delta(t-s)|^2 dxds\\ &\geq \frac{\xi(t)}{q\lambda(t)}\int_0^t H(\lambda(t)h(s))\int_\Omega q|\Delta u(t)-\Delta(t-s)|^2 dxds\\ &\geq \frac{t\xi(t)}{q}H\left(\frac{q}{t}\int_0^t h(s)\int_\Omega |\Delta u(t)-\Delta(t-s)|^2 dxds\right)\\ &= \frac{t\xi(t)}{q}\overline{H}\left(\frac{q}{t}\int_0^t h(s)\int_\Omega |\Delta u(t)-\Delta u(t-s)|^2 dxds\right),\end{aligned} \tag{63}$$

hence (58) *is established.*

IV. Energy decay

In this section, we state and prove our main general decay result. Our decay result is in the following theorem.

Theorem 2 *Let* $(u_0,u_1)\in H_0^2(\Omega)\times L^2(\Omega)$. *Assume that* $(A1)-(A3)$ *and* (33) *hold. Then, there exist positive constants* C_1, C_2 *such that the solution of* (17) *satisfies, for all* $t\geq t_1$,

$$E(t)\leq \frac{C_1}{\gamma(t)}\Phi_2^{-1}\left(\frac{C_2\gamma(t)}{\int_{t_1}^t \xi(s)ds}\right). \tag{64}$$

where the functions $\gamma(s), \Phi_2(s)$ *are functions will be defined in the proof.*

Proof *Combining* (50), (57) *and* (58), *then for any* $t\geq t_0$, *we get*

$$L'(t)\leq -mE(t)+c\left(\frac{t}{q}\right)^{\frac{1}{1+\epsilon_0}}\left(H^{-1}\left(\frac{q\mu(t)}{t\xi(t)}\right)\right)^{\frac{1}{1+\epsilon_0}}(t). \tag{65}$$

Combining the strictly increasing property of H *and the fact* $\frac{1}{t}<1$ *whenever* $t>1$, *we obtain*

$$H^{-1}\left(\frac{q\mu(t)}{t\xi(t)}\right)\leq H^{-1}\left(\frac{q\mu(t)}{(t)^{\frac{1}{1+\epsilon_0}}\xi(t)}\right), \tag{66}$$

and, then, (65) *becomes, for* $t_1=\max\{t_0,1\}$ *and for any* $t\geq t_1$,

$$L'(t)\leq -mE(t)+c_{\epsilon_0}\frac{(t)^{\frac{1}{1+\epsilon_0}}}{q}\left(H^{-1}\left(\frac{q\mu(t)}{(t)^{\frac{1}{1+\epsilon_0}}\xi(t)}\right)\right)^{\frac{1}{1+\epsilon_0}}. \tag{67}$$

For simplicity, we let $\gamma(t)=:\frac{q}{(t)^{\frac{1}{1+\epsilon_0}}}$. *Then,* (67) *becomes*

$$L'(t)\leq -mE(t)+\frac{c_{\epsilon_0}}{\gamma(t)}\left(H^{-1}\left(\frac{\gamma(t)\mu(t)}{\xi(t)}\right)\right)^{\frac{1}{1+\epsilon_0}}. \tag{68}$$

Further, letting $\chi(t)=\left(\frac{\gamma(t)\mu(t)}{\xi(t)}\right)$ *and*

$$\Phi^{-1}(\chi(t)) =: \left(H^{-1}(\chi(t))\right)^{\frac{1}{1+\epsilon_0}}.$$

Then, (68) *reduces to*

$$L'(t)\leq -mE(t)+\frac{c_{\epsilon_0}}{\gamma(t)}\Phi^{-1}(\chi(t)),\ \forall t\geq t_1. \tag{69}$$

Now, for $\varepsilon<r$ *and using the fact that* $E'\leq 0$, $\Phi'>0, \Phi''>0$ *on* $(0,r]$, *we find that the functional* L_1, *defined by*

$$L_1(t):=\Phi'\left(\varepsilon\cdot\frac{\gamma(t)E(t)}{E(0)}\right)L(t),\quad \forall t\geq t_1, \tag{70}$$

satisfies, for some $\alpha_1,\alpha_2>0$,

$$\alpha_1 L_1(t)\leq E(t)\leq \alpha_2 L_1(t). \tag{71}$$

Using the fact $(\gamma E)'(t)\leq 0$ *and* $\Phi'>0$, *therefore, combining* (70) *with* (69), *we arrive at*

$$\begin{aligned}L_1'(t)\leq &-mE(t)\Phi'\left(\varepsilon\cdot\frac{\gamma(t)E(t)}{E(0)}\right)\\ &+\frac{c_{\epsilon_0}}{\gamma(t)}\Phi'\left(\varepsilon\cdot\frac{\gamma(t)E(t)}{E(0)}\right)\Phi^{-1}(\chi(t)).\end{aligned} \tag{72}$$

Let Φ^* *be the convex conjugate of* Φ *in the sense of Young (see [37]), then*

$$\Phi^*(s)=s(\Phi')^{-1}(s)-\Phi\left[(\Phi')^{-1}(s)\right],\quad \text{if } s\in(0,\Phi'(r)] \tag{73}$$

and Φ^* *satisfies the following generalized Young inequality*

$$AB\leq \Phi^*(A)+\Phi(B),\quad \text{if } A\in(0,\Phi'(r)],\ B\in(0,r]. \tag{74}$$

Using (73) *and the generalized Young inequality* (74) *on the last term in* (72) *with* $A=\Phi'\left(\varepsilon\cdot\frac{\gamma(t)E(t)}{E(0)}\right)$ *and* $B=\Phi^{-1}(\chi(t))$, *we get*

$$\begin{aligned}L_1'(t)\leq &-mE(t)\Phi'\left(\varepsilon\cdot\frac{\gamma(t)E(t)}{E(0)}\right)+\frac{c_{\epsilon_0}}{\gamma(t)}\Phi^*\left(\Phi'\left(\varepsilon\cdot\frac{\gamma(t)E(t)}{E(0)}\right)\right)\\ &+\frac{c_{\epsilon_0}}{\gamma(t)}\chi(t)\\ \leq &-mE(t)\Phi'\left(\varepsilon\cdot\frac{\gamma(t)E(t)}{E(0)}\right)+c_{\epsilon_0}\varepsilon\frac{E(t)}{E(0)}\Phi'\left(\varepsilon\cdot\frac{\gamma(t)E(t)}{E(0)}\right)\\ &+\frac{c_{\epsilon_0}}{\gamma(t)}\chi(t).\end{aligned} \tag{75}$$

Multiplying both sides of (75) *by* $\xi(t)$ *and using the fact that* $\frac{\xi(t)}{\gamma(t)}\chi(t)\leq -cE'(t)$, *we obtain*

$$\begin{aligned}\xi(t)L_1'(t)\leq &-m\xi(t)E(t)\Phi'\left(\varepsilon\cdot\frac{\gamma(t)E(t)}{E(0)}\right)\\ &+c_{\epsilon_0}\varepsilon\xi(t)\frac{E(t)}{E(0)}\Phi'\left(\varepsilon\cdot\frac{\gamma(t)E(t)}{E(0)}\right)\\ &-c_{\epsilon_0}E'(t).\end{aligned} \tag{76}$$

Using the non-increasing property of $\xi(t)$, *we obtain, for all* $t\geq t_1$,

$$\begin{aligned}(\xi L_1+c_{\epsilon_0}E)'(t)\leq &-m\xi(t)E(t)\Phi'\left(\varepsilon\cdot\frac{\gamma(t)E(t)}{E(0)}\right)\\ &+c_{\epsilon_0}\varepsilon\xi(t)\frac{E(t)}{E(0)}\Phi'\left(\varepsilon\cdot\frac{\gamma(t)E(t)}{E(0)}\right).\end{aligned} \tag{77}$$

Therefore, by setting $L_2 := \xi L_1 + c_{\epsilon_0} E \sim E$*, we get*

$$\begin{aligned} L_2'(t) &\leq -m\xi(t)E(t)\Phi'\left(\varepsilon \cdot \frac{\gamma(t)E(t)}{E(0)}\right) \\ &+ c_{\epsilon_0}\varepsilon\xi(t)\frac{E(t)}{E(0)}\Phi'\left(\varepsilon \cdot \frac{\gamma(t)E(t)}{E(0)}\right). \end{aligned} \tag{78}$$

This gives, for a suitable choice of ε *so that* $k = m - c_{\epsilon_0}\varepsilon > 0$,

$$L_2'(t) \leq -k\xi(t)\frac{E(t)}{E(0)}\Phi'\left(\varepsilon \cdot \frac{\gamma(t)E(t)}{E(0)}\right), \ t \geq t_1. \tag{79}$$

Integrating (79) *over* (t_1, t) *gives*

$$\int_{t_1}^{t} k\xi(s)\frac{E(t)}{E(0)}\Phi'\left(\varepsilon \cdot \frac{\gamma(s)E(s)}{E(0)}\right) \leq -\int_{t_1}^{t} L_2'(s)ds \leq L_2(t_1). \tag{80}$$

Using the facts that $\Phi', \Phi'' > 0$ *and the non-increasing property of* E*, we deduce that the map* $t \mapsto E(t)\Phi'\left(\varepsilon \cdot \frac{\gamma(s)E(s)}{E(0)}\right)$ *is non-increasing and consequently, we have for any* $t \geq s \geq t_1$

$$\begin{aligned} &k\frac{E(t)}{E(0)}\Phi'\left(\varepsilon \cdot \frac{\gamma(t)E(t)}{E(0)}\right)\int_{t_1}^{t}\xi(s)ds \\ &\leq \int_{t_1}^{t} k\xi(s)\frac{E(t)}{E(0)}\Phi'\left(\varepsilon \cdot \frac{\gamma(s)E(s)}{E(0)}\right) \\ &\leq L_2(t_1). \end{aligned} \tag{81}$$

Multiplying each side of (81) *by* $\gamma(t)$*, we have for* $k_1 > 0$,

$$k\frac{\gamma(t)E(t)}{E(0)}\Phi'\left(\varepsilon \cdot \frac{\gamma(t)E(t)}{E(0)}\right)\int_{t_0}^{t}\xi(s)ds \leq k_1\gamma(t). \tag{82}$$

Next, we set $\Phi_2(s) = s\Phi'(\varepsilon s)$ *which is strictly increasing, and then we obtain for any* $t \geq t_1$ *and some constant* $k_2 > 0$,

$$\Phi_2\left(\varepsilon \cdot \frac{\gamma(t)E(t)}{E(0)}\right)\int_{t_1}^{t}\xi(s)ds \leq k_2\gamma(t). \tag{83}$$

This gives, for any $t \geq t_0$ *and some constant* $k_3 > 0$,

$$E(t) \leq \frac{k_3}{\gamma(t)}\Phi_2^{-1}\left(\frac{k_2\gamma(t)}{\int_{t_1}^{t}\xi(s)ds}\right). \tag{84}$$

This finishes the proof.

Example 1 *The following example illustrates our results:*
Let $h(t) = \frac{a}{(1+t)^q}$*, where* $q > 1 + \epsilon_0$ *and* a *is chosen so that hypothesis* (A_1) *remains valid. Then*

$$h'(t) = -bH(h(t)), \quad \text{with} \quad H(s) = s^{\frac{q+1}{q}},$$

where b *is a fixed constant. Since* $\Phi(s) = s^{\frac{(\epsilon_0+1)(q+1)}{q}}$. *Then, the estimate* (64) *gives,* $\forall t \geq t_1$

$$E(t) \leq ct^{\frac{-(q-1-\epsilon_0)}{(1+\epsilon_0)^2(q+1)}}. \tag{85}$$

V. Conclusions and future works

We succeed to prove the decay results known for the case of finite history to the case of infinite history where the relaxation function satisfies a wider class of relaxation functions. Note that our decay result (64) generalizes the once of [22], [24], [38], [39], [40], [41] and [25]. For the future works, we plan to find a class of relaxation functions for which we obtain an optimal decay.

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