Numerical approximation of coupled 1D and 2D non-linear Burgers’ equations by employing Modified Quartic Hyperbolic B-spline Differential Quadrature Method

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Abstract—In this paper, the numerical solution of coupled 1D and coupled 2D Burgers’ equation is provided with the appropriate initial and boundary conditions, by implementing "modified quartic Hyperbolic B-spline DQM". In present method, the required weighting coefficients are computed using modified quartic Hyperbolic B-spline as a basis function. These coupled 1D and coupled 2D Burgers' equations got transformed into the set of ordinary differential equations, tackled by SSP-RK43 scheme. Efficiency of the scheme and exactness of the obtained numerical solutions is declared with the aid of 8 numerical examples. Numerical results obtained by modified quartic Hyperbolic B-spline are efficient and it is easy to implement.

Keywords—Coupled 1D and 2D non-linear Burgers’ equations, Modified Quartic Hyperbolic B-spline, Differential Quadrature Method, SSP-RK43 scheme.

I. INTRODUCTION

1.1. Coupled 1D non-linear Burgers’ equation

Coupled 1D Burgers’ equation is defined as:

\[ U_t + \delta U_{xx} + \eta U U_x + \alpha (UV)_x = 0 \]  
\[ V_t + \mu V_{xx} + \xi V V_x + \beta (UV)_x = 0 \]  

Initial conditions:

\[ U(x, 0) = g_1(x) \]  
\[ V(x, 0) = g_2(x) \]  

Boundary conditions:

\[ U(x, t) = h_1(x, t) \]  
\[ V(x, t) = h_2(x, t) \]  

1.2. Coupled 2D non-linear Burgers’ equation

Coupled 2D Burgers' equation is given as:

\[ U_t + U U_x + V U_y = \nu [U_{xx} + U_{yy}] \]  
\[ V_t + U V_x + V V_y = \nu [V_{xx} + V_{yy}] \]  

Initial conditions:

\[ U(x, y, 0) = \psi_1(x, y) \]  
\[ V(x, y, 0) = \psi_2(x, y) \]  

\[ D = \{(x, y); x \in [a, b], y \in [c, d]\} \]

Boundary conditions:

\[ U(x, y, t) = \phi_1(x, y, t) \]  
\[ V(x, y, t) = \phi_2(x, y, t) \]  

where \( (x, y) \in \mathbb{R} \) and \( t > 0 \).

Some relevant studies regarding Burgers’ equation could be found ahead. Coupled 1D Burgers’ equation was derived by Esipov [1]. The system of coupled Burgers’ equation is very important from the numerical aspect, as in most of the cases, analytical solutions are not available. Kaya [2] used the Adomian Decomposition Method for getting the exact solution of the coupled 1D Burgers’ equation. Soliman [3] used modified extended tanh function approach. Several researchers have solved the coupled 1D Burgers’ equation from the numerical point of view. Esipov [1] gave the numerical solution. Wei and Gu [4] used the conjugate filter approach. Abdou and Soliman [5] implemented the Variational Iteration Method for 1D Burgers’ equation and coupled Burgers’ equation. Rashid and Ismail [6] implemented Fourier pseudo spectral method. Mittal and Arora [7] employed cubic B-spline collocation approach for coupled viscous Burgers’ equation. Fletcher [8] used the Hopf-Cole transformation in order to find the analytical solution of coupled 2D Burgers’ equation. The numerical solution of coupled Burgers’ equation is obtained by many researchers due to its demand in different fields of engineering and sciences. Some of their work is presented. Tamsir et al. [9] used the notion of extended modified cubic B-spline DQM to approximate the solution of coupled 2D Burgers’ equation, mentioned paper extended modified cubic B-spline DQM was implemented in space and strong stability preserving Runge-Kutta stages 5 and order 4 (SSP-RK 54) was employed in time, stability analysis of the method was also provided. Tamsir et al. [10] employed the technique of DQM built by exponential modified cubic B-spline for the solution of coupled 2D non-linear Burgers’ equation and also provided the stability analysis of the matrix stability analysis method.
1.3. Differential Quadrature Method
DQM is a numerical discretization tool. DQM was initially proposed by Bellman and his associates [11] in 1972. DQM has widely come in to notice and emerged as a preferable method in previous decades due to its ease of application. Numerous researchers have provided the different numerical approximations based upon DQM. These different numerical regimes are mostly done by using the different test functions, like, Legendre polynomial functions, spline function [11][12], Lagrange interpolation polynomial function [13][14][15], radial basis function [16], Hermite polynomials [17], Sinc function [18][19], B-spline functions [20][21][22][23] [24][25][36][37] and many others.

Present paper is divided into different sections. In Section II, the numerical scheme (Modified Quartic Hyperbolic B-spline DQM) is elaborated completely, moreover formation of quartic Hyperbolic B-spline is provided as well as the derivative value of the quartic Hyperbolic B-spline is also evaluated. Tabular values of quartic Hyperbolic B-spline and its derivative are calculated at the different node points. Present scheme is completely novel and has never been implemented to solve coupled 1D and coupled 2D Burgers’ equations as per literature. In this work quartic Hyperbolic B-spline is developed and modified values of the mentioned Hyperbolic B-splines are implemented to solve coupled 1D and 2D coupled Burgers’ equations. Results obtained by this scheme are acceptable. This work will surely help others researchers in the solution of complex non-linear partial differential equations.

II. NUMERICAL METHOD (MODIFIED QUARTIC HYPERBOLIC B-SPLINE DIFFERENTIAL QUADRATURE METHOD)

2.1 Formation of Quartic Hyperbolic B-spline

\[
H_m(x) = \frac{1}{M_1} \times \begin{cases} 
1, & [\sinh(x-x_n), x_{n+1}, x_{n+2}, x_{n+3}] \\
2, & [\sinh(x-x_n), x_{n+1}, x_{n+2}, x_{n+3}] \\
+\sinh(x-x_n)\sinh(x_{n+1}-x)\sinh(x_{n+2}-x) \sinh(x_{n+3}-x) & [x_{n+1}, x_{n+2}, x_{n+3}] \\
+\sinh(x-x_n)\sinh(x_{n+2}-x)\sinh(x_{n+3}-x) & [x_{n+2}, x_{n+3}] \\
+\sinh(x-x_n)\sinh(x_{n+3}-x) & [x_{n+3}] \\
\end{cases}
\]

Where

\[
M_1 = \sinh(h)\sinh(2h)\sinh(3h)\sinh(4h)
\]

Derivative value of Quartic Hyperbolic B-spline is provided as,

\[
\frac{B_m(x)}{B_m'(x)} = \begin{bmatrix} 0 & b_1 & b_2 & b_3 & b_1 & 0 \\ 0 & b_3 & b_4 & -b_4 & -b_3 & 0 \end{bmatrix}
\]

\[
\phi_1(x) = H_1(x) + 2H_0(x)
\]
2.2. Determination of weighting coefficients

Weighting coefficients can be easily obtained by implementing modified values of Quartic Hyperbolic B-spline in the discretization formula of DQM.

\[ \phi_k(x) = \sum_{j=1}^{n} b_{ij} \phi_i(x), k = 1, 2, 3, \ldots, n \]  

\[ \begin{bmatrix} \phi_1(x_1) \\ \phi_2(x_1) \\ \vdots \\ \phi_n(x_1) \end{bmatrix} = \begin{bmatrix} \phi_1(x_2) \\ \phi_2(x_2) \\ \vdots \\ \phi_n(x_2) \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{12} \\ \vdots \\ b_{n1} \end{bmatrix} \]

At grid point \( x_1: \)

\[ \begin{bmatrix} \phi_1(x_1) \\ \phi_2(x_1) \\ \vdots \\ \phi_n(x_1) \end{bmatrix} = \begin{bmatrix} \phi_1(x_2) \\ \phi_2(x_2) \\ \vdots \\ \phi_n(x_2) \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{12} \\ \vdots \\ b_{n1} \end{bmatrix} \]

At grid point \( x_2: \)

\[ \begin{bmatrix} \phi_1(x_1) \\ \phi_2(x_1) \\ \vdots \\ \phi_n(x_1) \end{bmatrix} = \begin{bmatrix} \phi_1(x_2) \\ \phi_2(x_2) \\ \vdots \\ \phi_n(x_2) \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{12} \\ \vdots \\ b_{n1} \end{bmatrix} \]

At grid point \( x_3: \)

\[ \begin{bmatrix} \phi_1(x_1) \\ \phi_2(x_1) \\ \vdots \\ \phi_n(x_1) \end{bmatrix} = \begin{bmatrix} \phi_1(x_2) \\ \phi_2(x_2) \\ \vdots \\ \phi_n(x_2) \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{12} \\ \vdots \\ b_{n1} \end{bmatrix} \]

At grid point \( x_n: \)
Where

\[
\begin{bmatrix}
\phi_1'(x_1) \\
\phi_1'(x_2) \\
\phi_1'(x_3) \\
\vdots \\
\phi_1'(x_n)
\end{bmatrix}
= 
\begin{bmatrix}
-b_1 \\
b_1 + b_3 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\phi_2'(x_1) \\
\phi_2'(x_2) \\
\phi_2'(x_3) \\
\vdots \\
\phi_2'(x_n)
\end{bmatrix}
= 
\begin{bmatrix}
-b_2 \\
b_3 + b_4 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\phi_3'(x_1) \\
\phi_3'(x_2) \\
\phi_3'(x_3) \\
\vdots \\
\phi_3'(x_n)
\end{bmatrix}
= 
\begin{bmatrix}
-b_3 \\
b_4 \\
b_3 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

III. IMPLEMENTATION OF SCHEME

In this section developed scheme is implemented upon coupled 1D and coupled 2D equations using the differential quadrature formula. Spatial partial derivatives are dealt by the DQM formulae given as per equations (21), (22), (25), (26), (27) and (28).

3.1. Upon Coupled 1D Burgers' equation

Discretization formula for first order partial derivatives:
Discretization formula for second order partial derivatives:
\[
U_x^{(2)} = \sum_{j=1}^{n} w_{ij}^{(2)} U(x_j) \quad \text{and} \quad V_x^{(2)} = \sum_{j=1}^{n} w_{ij}^{(2)} V(x_j)
\]
(22)

By applying the DQM approximations (21) and (22) in coupled 1D Burgers’ equations, following set of equations will be obtained.

\[
\frac{dU}{dt} = -\delta \sum_{j=1}^{n} w_{ij}^{(2)} U(x_j) - \eta U_t \sum_{j=1}^{n} w_{ij}^{(1)} U(x_j) - \alpha (UV)_{x_1} = 0
\]
(23)

\[
\frac{dV}{dt} = -\mu \sum_{j=1}^{n} w_{ij}^{(2)} V(x_j) - \xi V_t \sum_{j=1}^{n} w_{ij}^{(1)} V(x_j) - \alpha (UV)_{x_1} = 0
\]
(24)

3.2. Upon Coupled 2D Burgers’ equation

\[
\frac{\partial U(x_i, y_j)}{\partial x} = \sum_{k=1}^{n} w_{ik}^{(1)} U(x_k, y_j, t)
\]

\[
\frac{\partial^2 U(x_i, y_j)}{\partial x^2} = \sum_{k=1}^{n} w_{jk}^{(2)} U(x_i, y_k, t)
\]

\[
\frac{\partial V(x_i, y_j)}{\partial y} = \sum_{k=1}^{n} w_{ik}^{(1)} V(x_i, y_k, t)
\]

\[
\frac{\partial^2 V(x_i, y_j)}{\partial y^2} = \sum_{k=1}^{n} w_{jk}^{(2)} V(x_i, y_k, t)
\]
(25)

By the means of DQM approximation formulae (25)(26)(27)(28), implementation of the scheme upon coupled 2D Burgers’ equations, is given as follows:

\[
\frac{dU_{ij}}{dt} = -U_{ij} \sum_{k=1}^{n} w_{ik}^{(2)} U(x_k, y_j, t) - V_{-i,j} \sum_{k=1}^{n} w_{jk}^{(1)} U(x_i, y_k, t) - \alpha \left[\sum_{k=1}^{n} w_{ik}^{(2)} U(x_i, y_k, t) + \sum_{k=1}^{n} w_{jk}^{(2)} U(x_i, y_k, t)\right]
\]
(29)

\[
\frac{dV_{ij}}{dt} = -U_{ij} \sum_{k=1}^{n} w_{ik}^{(1)} V(x_k, y_j, t) - V_{-i,j} \sum_{k=1}^{n} w_{jk}^{(2)} V(x_i, y_k, t) - \alpha \left[\sum_{k=1}^{n} w_{ik}^{(2)} V(x_i, y_k, t) + \sum_{k=1}^{n} w_{jk}^{(2)} V(x_i, y_k, t)\right]
\]
(30)

The obtained system of ordinary differential equations is tackled by the means of the SSP-RK43 scheme [29][30][31]. The higher order weighting coefficients [32] are evaluated in MATLAB by the help of program.

IV. NUMERICAL EXPERIMENTS AND DISCUSSION

In present section 8 numerical examples are discussed. First three examples are associated to coupled 1D Burgers’ equation and rest five examples are associated to the concept of coupled 2D Burgers’ equation. \(L_2\) and \(L_{\alpha\beta}\) errors norms are provided for these examples. Moreover exact solutions are matched with the numerical solutions. Via graphical representation of the results it got noticed that in all cases numerical and exact solutions are compatible. Accuracy of the scheme is verified with the aid of RMS and Relative error norms also. It is obvious with all these details the developed scheme is quite acceptable and easy to implement.

Example 1:

In this example coupled 1D Burgers’ equations (1) and (2) are considered with the following exact solutions from [2], which are given as,

\[
U(x, t) = a_0 - 2A \left(\frac{2\alpha - 1}{4\alpha^2 - 1}\right) \tanh[A(x - 2At)], -10 \leq x \leq 10, t > 0
\]
(31)

\[
V(x, t) = a_0 - \frac{2\beta - 1}{2\alpha - 1} - 2A \left(\frac{2\alpha - 1}{4\alpha^2 - 1}\right) \tanh[A(x - 2At)], -10 \leq x \leq 10, t > 0
\]
(32)

Computational Domain: \([-L, L] = [-10, 10]\)

Initial conditions:

\[
U(x, 0) = a_0 - 2A \left(\frac{2\alpha - 1}{4\alpha^2 - 1}\right) \tanh[Ax], -10 \leq x \leq 10
\]
(33)
\[ V(x, 0) = a_0 - \frac{(2 \beta - 1)}{(2 \alpha - 1)} \frac{1}{2A} \frac{(2 \alpha - 1)}{(4 \alpha \beta - 1)} \tanh[Ax], -10 \leq x \leq 10 \]  
(34)

**Boundary conditions:**

\[ U(-L, t) = a_0 - \frac{(2 \alpha - 1)}{2A} \frac{1}{(4 \alpha \beta - 1)} \tanh[A(-L - 2At)], t > 0 \]  
(35)

\[ U(L, t) = a_0 - \frac{(2 \alpha - 1)}{2A} \frac{1}{(4 \alpha \beta - 1)} \tanh[A(L - 2At)], t > 0 \]  
(36)

\[ V(-L, t) = a_0 - \frac{(2 \beta - 1)}{(2 \alpha - 1)} \frac{1}{2A} \frac{(2 \alpha - 1)}{(4 \alpha \beta - 1)} \tanh[A(-L - 2At)], t > 0 \]  
(37)

\[ V(L, t) = a_0 - \frac{(2 \beta - 1)}{(2 \alpha - 1)} \frac{1}{2A} \frac{(2 \alpha - 1)}{(4 \alpha \beta - 1)} \tanh[A(L - 2At)], t > 0 \]  
(38)

In Table 2, \( L_2 \) and \( L_\infty \) errors are provided at time level \( t = 0.001, \Delta t = 0.0001, N = 21 \) at the different values of \( a_0, \alpha \) and \( \beta \). In Table 3, RMS and Relative errors for both u and v components are given at the mentioned time levels for \( N = 31, \Delta t = 0.001, a_0 = 0.005, \alpha = 0.3 \) and \( \beta = 0.3 \). In Table 4, comparison of Exact and Numerical approximations is provided at \( t = 0.001 \) and \( t = 0.005 \) for different values of \( x \). In Figure 1, Exact and Numerical u and v components are graphically matched at \( t = 0.001, 0.003 \) and \( 0.005 \) respectively.

### Table 2: \( L_2 \) and \( L_\infty \) error at time level \( t = 0.001, \Delta t = 0.0001, N = 21 \), for different values of \( a_0, \alpha \) and \( \beta \)

<table>
<thead>
<tr>
<th>( a_0 )</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( L_2 ) u</th>
<th>( L_\infty ) u</th>
<th>( L_2 ) v</th>
<th>( L_\infty ) v</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.1</td>
<td>0.1</td>
<td>2.25E-05</td>
<td>1.06E-05</td>
<td>2.25E-05</td>
<td>1.06E-05</td>
</tr>
<tr>
<td>0.01</td>
<td>0.1</td>
<td>0.3</td>
<td>2.05E-05</td>
<td>9.72E-06</td>
<td>2.18E-05</td>
<td>1.01E-05</td>
</tr>
<tr>
<td>0.01</td>
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<td>0.1</td>
<td>4.19E-05</td>
<td>1.97E-05</td>
<td>3.68E-05</td>
<td>1.81E-05</td>
</tr>
<tr>
<td>0.01</td>
<td>0.3</td>
<td>0.3</td>
<td>3.02E-05</td>
<td>1.43E-05</td>
<td>3.02E-05</td>
<td>1.43E-05</td>
</tr>
<tr>
<td>0.001</td>
<td>0.1</td>
<td>0.1</td>
<td>2.27E-07</td>
<td>1.07E-07</td>
<td>2.27E-07</td>
<td>1.07E-07</td>
</tr>
<tr>
<td>0.001</td>
<td>0.1</td>
<td>0.3</td>
<td>2.08E-07</td>
<td>9.79E-08</td>
<td>2.20E-07</td>
<td>1.02E-07</td>
</tr>
<tr>
<td>0.001</td>
<td>0.3</td>
<td>0.1</td>
<td>4.23E-07</td>
<td>1.98E-07</td>
<td>3.74E-07</td>
<td>1.83E-07</td>
</tr>
<tr>
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<td>0.3</td>
<td>3.06E-07</td>
<td>1.44E-07</td>
<td>3.06E-07</td>
<td>1.44E-07</td>
</tr>
<tr>
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<td>0.1</td>
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<td>2.59E-04</td>
<td>5.44E-04</td>
<td>2.59E-04</td>
</tr>
<tr>
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<td>0.3</td>
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<td>2.38E-04</td>
<td>5.34E-04</td>
<td>2.49E-04</td>
</tr>
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<td>0.1</td>
<td>9.70E-04</td>
<td>4.58E-04</td>
<td>8.41E-04</td>
<td>4.16E-04</td>
</tr>
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<td>0.3</td>
<td>7.22E-04</td>
<td>3.43E-04</td>
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<td>3.43E-04</td>
</tr>
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<td>0.1</td>
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<tr>
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<td>1.05E-05</td>
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</tr>
<tr>
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<td>7.59E-06</td>
<td>3.57E-06</td>
<td>7.59E-06</td>
<td>3.57E-06</td>
</tr>
</tbody>
</table>

### Table 3: Root mean square and Relative error norms for u and v components for \( N = 31, \Delta t = 0.001, a_0 = 0.005, \alpha = 0.3, \beta = 0.3 \) at different time levels

<table>
<thead>
<tr>
<th>t</th>
<th>RMS u</th>
<th>RMS v</th>
<th>Relative u</th>
<th>Relative v</th>
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<td></td>
<td></td>
<td></td>
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<tr>
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<td></td>
<td></td>
</tr>
<tr>
<td>0.003</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.005</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x</td>
<td>Num. u</td>
<td>Exact u</td>
<td>Num. v</td>
<td>Exact v</td>
</tr>
<tr>
<td>-----</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
</tr>
<tr>
<td>-8</td>
<td>5.12E-03</td>
<td>5.12E-03</td>
<td>5.12E-03</td>
<td>5.12E-03</td>
</tr>
<tr>
<td>-6</td>
<td>5.09E-03</td>
<td>5.09E-03</td>
<td>5.09E-03</td>
<td>5.09E-03</td>
</tr>
<tr>
<td>-4</td>
<td>5.06E-03</td>
<td>5.06E-03</td>
<td>5.06E-03</td>
<td>5.06E-03</td>
</tr>
<tr>
<td>-2</td>
<td>5.03E-03</td>
<td>5.03E-03</td>
<td>5.03E-03</td>
<td>5.03E-03</td>
</tr>
<tr>
<td>2</td>
<td>4.97E-03</td>
<td>4.97E-03</td>
<td>4.97E-03</td>
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<td>4.88E-03</td>
<td>4.88E-03</td>
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</tbody>
</table>

**Table 4:** Comparison of Numerical and Exact approximations of $u$ and $v$ components for $N = 31$, $\Delta t = 0.001$, $\alpha = 0.1$, $\beta = 0.1$ at time levels $0.001$ and $0.005$ mentioned, for different values of $x$

**Example 2:**
In this example following couple 1D Burgers’ equations are considered.

\[
\begin{align*}
U_t - U_{xx} - 2U U_x + \frac{5}{2} (UV)_x &= 0, \quad -20 \leq x \leq 20, \quad t > 0 \\
V_t - V_{xx} - 2V V_x + \frac{5}{2} (UV)_x &= 0, \quad -20 \leq x \leq 20, \quad t > 0
\end{align*}
\]

(40)

$\alpha = 0.1$, $\beta = 0.1$ at time levels $t = 0.0001$, $0.0003$ and $0.0005$

Computational Domain $=[-L, L] = [-20, 20]$

Exact solution is provided as [33],

(39)
Initial conditions:
\[ U(x, 0) = \lambda \left[ 1 - \tanh \left( \frac{3}{2} \frac{\lambda}{x} \right) \right]; \quad -L \leq x \leq L; \quad t > 0 \] (41)
\[ V(x, 0) = \lambda \left[ 1 - \tanh \left( \frac{3}{2} \frac{\lambda}{x} \right) \right]; \quad -L \leq x \leq L \] (42)

Boundary conditions:
\[ U(-L, t) = \lambda \left[ 1 - \tanh \left( \frac{3}{2} \frac{\lambda}{x} \right) \left( -L - 3 \frac{\lambda}{t} \right) \right]; \quad t > 0 \] (43)
\[ U(L, t) = \lambda \left[ 1 - \tanh \left( \frac{3}{2} \frac{\lambda}{x} \right) \left( L - 3 \frac{\lambda}{t} \right) \right]; \quad t > 0 \] (44)
\[ V(-L, t) = \lambda \left[ 1 - \tanh \left( \frac{3}{2} \frac{\lambda}{x} \right) \left( -L - 3 \frac{\lambda}{t} \right) \right]; \quad t > 0 \] (45)
\[ V(L, t) = \lambda \left[ 1 - \tanh \left( \frac{3}{2} \frac{\lambda}{x} \right) \left( L - 3 \frac{\lambda}{t} \right) \right]; \quad t > 0 \] (46)

In Figure 2, comparison of Exact and Numerical $u$ and $v$ components is given at $t = 0.0001$, 0.0003 and 0.0005 for $\lambda = 0.2$. In Table 5, $L_2$ and $L_{\infty}$ error norms are provided at $t = 0.0001$ and $t = 0.0003$ for $\lambda = 0.1$ and $\lambda = 0.2$ respectively. In Table 6, Numerical and Exact components are evaluated at $t = 0.0001$ and $t = 0.0005$ for the different values of $x$. In Table 7, RMS and Relative error are provided at $t = 0.001$, 0.002, 0.003, 0.004 and 0.005 respectively.

![Figure 2: Comparison of Exact and Numerical u and v components for N = 21, $\Delta t = 0.0001$, $\lambda = 0.2$ at time levels $t = 0.0001$, 0.0003 and 0.0005](image)

<table>
<thead>
<tr>
<th>$L_2$ $u$</th>
<th>$L_{\infty}$ $u$</th>
<th>$L_2$ $v$</th>
<th>$L_{\infty}$ $v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 0.0001$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda = 0.1$</td>
<td>5.95E-04</td>
<td>1.59E-04</td>
<td>5.95E-04</td>
</tr>
<tr>
<td>$\lambda = 0.2$</td>
<td>1.29E-03</td>
<td>3.20E-04</td>
<td>1.29E-03</td>
</tr>
<tr>
<td>$t = 0.0003$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda = 0.1$</td>
<td>1.80E-03</td>
<td>4.85E-04</td>
<td>1.80E-03</td>
</tr>
<tr>
<td>$\lambda = 0.2$</td>
<td>3.90E-03</td>
<td>9.73E-04</td>
<td>3.90E-03</td>
</tr>
</tbody>
</table>
Table 6: Comparison of Numerical and Exact u and v components for \( N = 31 \), \( \Delta t = 0.001 \), \( \lambda = 0.1 \) for different values of \( x \) at time levels \( t = 0.001 \) and \( t = 0.005 \) respectively

<table>
<thead>
<tr>
<th>( x )</th>
<th>Num U</th>
<th>Exact U</th>
<th>Num V</th>
<th>Exact V</th>
<th>Num U</th>
<th>Exact U</th>
<th>Num V</th>
<th>Exact V</th>
</tr>
</thead>
<tbody>
<tr>
<td>-20</td>
<td>2.00E-01</td>
<td>2.00E-01</td>
<td>2.00E-01</td>
<td>2.00E-01</td>
<td>2.00E-01</td>
<td>2.00E-01</td>
<td>2.00E-01</td>
<td>2.00E-01</td>
</tr>
<tr>
<td>-16</td>
<td>1.99E-01</td>
<td>1.98E-01</td>
<td>1.99E-01</td>
<td>1.98E-01</td>
<td>2.00E-01</td>
<td>1.98E-01</td>
<td>2.00E-01</td>
<td>1.98E-01</td>
</tr>
<tr>
<td>-12</td>
<td>1.94E-01</td>
<td>1.95E-01</td>
<td>1.94E-01</td>
<td>1.95E-01</td>
<td>1.94E-01</td>
<td>1.95E-01</td>
<td>1.94E-01</td>
<td>1.95E-01</td>
</tr>
<tr>
<td>-8</td>
<td>1.83E-01</td>
<td>1.83E-01</td>
<td>1.83E-01</td>
<td>1.83E-01</td>
<td>1.84E-01</td>
<td>1.83E-01</td>
<td>1.84E-01</td>
<td>1.83E-01</td>
</tr>
<tr>
<td>-4</td>
<td>1.54E-01</td>
<td>1.54E-01</td>
<td>1.54E-01</td>
<td>1.54E-01</td>
<td>1.54E-01</td>
<td>1.54E-01</td>
<td>1.54E-01</td>
<td>1.54E-01</td>
</tr>
<tr>
<td>0</td>
<td>9.93E-02</td>
<td>1.00E-01</td>
<td>9.93E-02</td>
<td>1.00E-01</td>
<td>9.68E-02</td>
<td>1.00E-01</td>
<td>9.68E-02</td>
<td>1.00E-01</td>
</tr>
<tr>
<td>4</td>
<td>4.74E-02</td>
<td>4.63E-02</td>
<td>4.74E-02</td>
<td>4.63E-02</td>
<td>5.19E-02</td>
<td>4.63E-02</td>
<td>5.19E-02</td>
<td>4.63E-02</td>
</tr>
<tr>
<td>8</td>
<td>1.52E-02</td>
<td>1.66E-02</td>
<td>1.52E-02</td>
<td>1.66E-02</td>
<td>8.66E-03</td>
<td>1.66E-02</td>
<td>8.66E-03</td>
<td>1.66E-02</td>
</tr>
<tr>
<td>12</td>
<td>6.85E-03</td>
<td>5.32E-03</td>
<td>6.85E-03</td>
<td>5.32E-03</td>
<td>1.43E-02</td>
<td>5.32E-03</td>
<td>1.43E-02</td>
<td>5.32E-03</td>
</tr>
<tr>
<td>16</td>
<td>1.88E-05</td>
<td>1.63E-03</td>
<td>1.88E-05</td>
<td>1.63E-03</td>
<td>-8.98E-03</td>
<td>1.63E-03</td>
<td>-8.98E-03</td>
<td>1.63E-03</td>
</tr>
</tbody>
</table>

Table 7: Different error norms (Root Mean Square, Relative and Average Error Norms) for \( N = 21 \), \( \Delta t = 0.001 \), \( \lambda = 0.1 \) at time levels \( t = 0.001 \), \( 0.002 \), \( 0.003 \), \( 0.004 \) and \( 0.005 \) respectively

<table>
<thead>
<tr>
<th>( t )</th>
<th>RMS U</th>
<th>RMS V</th>
<th>Relative U</th>
<th>Relative V</th>
<th>AVG. Error U</th>
<th>AVG. Error V</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>2.54E-03</td>
<td>2.54E-03</td>
<td>3.83E-04</td>
<td>3.83E-04</td>
<td>1.83E-03</td>
<td>1.83E-03</td>
</tr>
<tr>
<td>0.002</td>
<td>5.57E-03</td>
<td>5.57E-03</td>
<td>1.84E-03</td>
<td>1.84E-03</td>
<td>3.97E-03</td>
<td>3.97E-03</td>
</tr>
<tr>
<td>0.003</td>
<td>9.15E-03</td>
<td>9.15E-03</td>
<td>4.97E-03</td>
<td>4.97E-03</td>
<td>6.43E-03</td>
<td>6.43E-03</td>
</tr>
<tr>
<td>0.004</td>
<td>1.34E-02</td>
<td>1.34E-02</td>
<td>1.06E-02</td>
<td>1.06E-02</td>
<td>9.26E-03</td>
<td>9.26E-03</td>
</tr>
<tr>
<td>0.005</td>
<td>1.83E-02</td>
<td>1.83E-02</td>
<td>1.98E-02</td>
<td>1.98E-02</td>
<td>1.25E-02</td>
<td>1.25E-02</td>
</tr>
</tbody>
</table>

**Example 3:**
In present example, considered coupled 1D Burgers’ equations are presented as follows:

\[
U_t - U_{xx} + \eta \ U \ U_x + \alpha \ (U \ V)_x = 0 \quad (49) \\
V_t - V_{xx} + \xi \ V \ V_x + \beta \ (U \ V)_x = 0 \quad (50)
\]

Where \( \eta, \xi, \alpha \) and \( \beta \) all are treated as arbitrary constants.

**Initial conditions:**

\[
U(x, 0) = \begin{cases} 
\sin(2\pi x), & x \in [0, 0.5) \\
0, & x \in [0.5, 1) 
\end{cases} \quad (51)
\]

\[
V(x, 0) = \begin{cases} 
0, & [0, 0.5) \\
-\sin(2\pi x), & [0.5, 1) 
\end{cases} \quad (52)
\]

**Boundary conditions:**
In this example all boundary conditions are considered as zero.

In Figure 3, numerical approximation of \( u \) and \( v \) components is presented graphically at the different time levels for the mentioned parameters. In Figure 4, Numerical \( u \) is provided at the mentioned time levels for \( \eta = 1, \xi = 1, \alpha = 1 \) and \( \beta = 1 \). In Figure 5, Numerical profiles of \( v \) component is given at the mentioned time level for \( \eta = 1, \xi = 1, \alpha = 1 \) and \( \beta = 1 \).
Figure 3 Numerical $U(x, t)$ and $V(x, t)$ for $N = 21$, $\Delta t = 0.0001$, $\eta = 1$, $\zeta = 1$, $\alpha = 1$ and $\beta = 1$

Figure 4: Numerical approximation of $U(x, t)$ for $N = 25$, $\eta = 1$, $\zeta = 1$, $\alpha = 1$, $\beta = 1$, $\Delta t = 0.0001$ at different time levels
Example 4:
Considered the equations (1) and (2) with analytical solutions given by Fletcher in 1983 [8] as follows,

\[
\begin{align*}
u(x, y, t) &= \frac{3}{4} - \frac{1}{[4 \left(1 + \exp\left(-4x+4y-\frac{Re}{32}\right)\right)]} \tag{53} \\
v(x, y, t) &= \frac{3}{4} + \frac{1}{[4 \left(1 + \exp\left(-4x+4y-\frac{Re}{32}\right)\right)]} \tag{54}
\end{align*}
\]

Computational domain: \([a, b] \times [c, d] = [0, 1] \times [0, 1]\]

Initial conditions:

\[
\begin{align*}
u(x, y, 0) &= \frac{3}{4} - \frac{1}{[4 \left(1 + \exp\left(-4x+4y-\frac{Re}{32}\right)\right)]} \\
v(x, y, 0) &= \frac{3}{4} + \frac{1}{[4 \left(1 + \exp\left(-4x+4y-\frac{Re}{32}\right)\right)]} \tag{55} \tag{56}
\end{align*}
\]

Boundary conditions:

\[
\begin{align*}
u(a, y, t) &= \frac{3}{4} - \frac{1}{[4 \left(1 + \exp\left(-4a+4y-\frac{Re}{32}\right)\right)]} \\
v(b, y, t) &= \frac{3}{4} - \frac{1}{[4 \left(1 + \exp\left(-4b+4y-\frac{Re}{32}\right)\right)]} \\
v(x, c, t) &= \frac{3}{4} - \frac{1}{[4 \left(1 + \exp\left(-4x+4c-t-\frac{Re}{32}\right)\right)]} \\
v(x, d, t) &= \frac{3}{4} - \frac{1}{[4 \left(1 + \exp\left(-4x+4d-t-\frac{Re}{32}\right)\right)]} \\
v(a, y, t) &= \frac{3}{4} + \frac{1}{[4 \left(1 + \exp\left((-4a+4y-\frac{Re}{32})\right)]} \\
v(b, y, t) &= \frac{3}{4} + \frac{1}{[4 \left(1 + \exp\left((-4b+4y-\frac{Re}{32})\right)]} \\
v(x, c, t) &= \frac{3}{4} + \frac{1}{[4 \left(1 + \exp\left((-4x+4c-t-\frac{Re}{32})\right)]} \\
v(x, d, t) &= \frac{3}{4} + \frac{1}{[4 \left(1 + \exp\left((-4x+4d-t-\frac{Re}{32})\right)]}
\end{align*}
\]

In Figures 6, Numerical and Exact profiles of \(u\) and \(v\) components are given at \(t = 0.0001\) for \(Re = 100\) and \(Re = 200\) respectively. In Figures 7, Numerical and Exact profiles of \(u\) and \(v\) components are given at \(t = 0.0003\) for \(Re = 500\) and \(Re = 1000\) respectively. In Table 8, \(L_2\) and \(L_{09}\) error norms are given for different grid points at the mentioned time levels with \(Re = 100\) and \(500\). In Table 9, Exact and Numerical \(u\) and \(v\) components are matched at \(t = 0.0001\). In Table 10, RMS and Relative errors are provided at the mentioned time levels for \(Re = 100\).
Figure 7: Numerical and Exact profile of U and V components for \(N = 21 \times 21, \Delta t = 0.00001, \text{Re} = 500, 1000\) at time level \(t = 0.0003\)

Table 8: Details of \(L_2\) and \(L_{\infty}\) error norms at different grid points for \(\text{Re} = 100\) and \(\text{Re} = 500\) at the time levels \(t = 0.001\) and \(t = 0.005\) respectively

<table>
<thead>
<tr>
<th>Grid Points</th>
<th>(L_2 U)</th>
<th>(L_{\infty} U)</th>
<th>(L_2 V)</th>
<th>(L_{\infty} V)</th>
<th>(L_2 U)</th>
<th>(L_{\infty} U)</th>
<th>(L_2 V)</th>
<th>(L_{\infty} V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(11 \times 11)</td>
<td>6.24E-04</td>
<td>5.50E-04</td>
<td>9.84E-04</td>
<td>8.12E-04</td>
<td>2.92E-03</td>
<td>2.83E-03</td>
<td>4.59E-03</td>
<td>3.64E-03</td>
</tr>
<tr>
<td>(21 \times 21)</td>
<td>1.83E-03</td>
<td>1.11E-03</td>
<td>2.91E-03</td>
<td>1.62E-03</td>
<td>8.46E-03</td>
<td>5.03E-03</td>
<td>1.34E-02</td>
<td>7.05E-03</td>
</tr>
<tr>
<td>(41 \times 41)</td>
<td>5.21E-03</td>
<td>2.34E-03</td>
<td>8.07E-03</td>
<td>3.33E-03</td>
<td>2.24E-02</td>
<td>9.67E-03</td>
<td>3.43E-02</td>
<td>1.40E-02</td>
</tr>
<tr>
<td>(51 \times 51)</td>
<td>7.29E-03</td>
<td>3.02E-03</td>
<td>1.11E-02</td>
<td>4.26E-03</td>
<td>2.96E-02</td>
<td>1.21E-02</td>
<td>4.42E-02</td>
<td>1.75E-02</td>
</tr>
<tr>
<td>(71 \times 71)</td>
<td>1.22E-02</td>
<td>4.65E-03</td>
<td>1.77E-02</td>
<td>6.43E-03</td>
<td>4.28E-02</td>
<td>1.68E-02</td>
<td>6.21E-02</td>
<td>2.43E-02</td>
</tr>
</tbody>
</table>

Table 9: Comparison of Exact and Numerical profiles of U and V components for \(N = 11 \times 11, \Delta t = 0.00001, \text{Re} = 100\) at time level \(t = 0.0001\)

<table>
<thead>
<tr>
<th>Mesh Points</th>
<th>Exact U</th>
<th>Num. U</th>
<th>Exact V</th>
<th>Num. V</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.1, 0.1)</td>
<td>0.62498</td>
<td>0.62486</td>
<td>0.87502</td>
<td>0.87485</td>
</tr>
<tr>
<td>(0.2, 0.2)</td>
<td>0.62498</td>
<td>0.624769</td>
<td>0.87502</td>
<td>0.874713</td>
</tr>
<tr>
<td>(0.2, 0.3)</td>
<td>0.694311</td>
<td>0.694473</td>
<td>0.805689</td>
<td>0.806013</td>
</tr>
<tr>
<td>(0.3, 0.5)</td>
<td>0.73103</td>
<td>0.73155</td>
<td>0.76897</td>
<td>0.769639</td>
</tr>
<tr>
<td>(0.7, 0.8)</td>
<td>0.694311</td>
<td>0.694356</td>
<td>0.805689</td>
<td>0.805561</td>
</tr>
</tbody>
</table>

Table 10: RMS and Relative error norms for U and V components for \(N = 11 \times 11, \Delta t = 0.00001, \text{Re} = 100\) at time levels \(t = 0.0001, 0.0002, 0.0003, 0.0004\) and \(0.0005\) respectively

<table>
<thead>
<tr>
<th>(t)</th>
<th>RMS U</th>
<th>RMS V</th>
<th>Relative U</th>
<th>Relative V</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0001</td>
<td>2.38E-04</td>
<td>2.38E-04</td>
<td>2.38E-04</td>
<td>2.38E-04</td>
</tr>
</tbody>
</table>
Example 5:
In present Example considered coupled 2D Burgers’ equations are having the analytical solutions as following from [34].

\[ u(x, y, t) = \frac{(x+y-2xt)}{(1 - 2t^2)} \]  
\[ v(x, y, t) = \frac{(x-y-2yt)}{(1 - 2t^2)} \]  

\[ (57) \quad (58) \]

 Computational Domain: \([a, b] \times [c, d] = [0, 0.5] \times [0, 0.5]\)

**Initial conditions:**

\[ u(x, y, 0) = (x + y) \]  
\[ v(x, y, 0) = (x - y) \]  

\[ (59) \quad (60) \]

**Boundary conditions:**

\[ u(a, y, t) = \frac{(a+y-2at)}{(1 - 2t^2)} \]  
\[ u(b, y, t) = \frac{(b+y-2bt)}{(1 - 2t^2)} \]  
\[ u(x, c, t) = \frac{(x+c-2xt)}{(1 - 2t^2)} \]  
\[ u(x, d, t) = \frac{(x+d-2xt)}{(1 - 2t^2)} \]  
\[ v(a, y, t) = \frac{(a-y-2yt)}{(1 - 2t^2)} \]  
\[ v(b, y, t) = \frac{(b-y-2yt)}{(1 - 2t^2)} \]  
\[ v(x, c, t) = \frac{(x-c-2ct)}{(1 - 2t^2)} \]  
\[ v(x, d, t) = \frac{(x-d-2dt)}{(1 - 2t^2)} \]  

\[ (61) \quad (62) \quad (63) \quad (64) \quad (65) \quad (66) \quad (67) \quad (68) \]

In Figure 8, Numerical and Exact profiles of both \(u\) and \(v\) components are given at \(t = 0.0001\) for \(Re = 500\). In Figure 9, Numerical and Exact solutions are matched at \(t = 0.0001\) for \(Re = 1000\). In Figure 10, Numerical and Exact profiles of both components are matched at time level \(t = 0.0001\) for \(Re = 1500\). In Table 11, \(L_2\) and \(L_\infty\) errors are given at the mentioned time levels for \(Re = 100, 500\) and \(1000\). In Table 12, comparison of Exact and Numerical profiles is done for both components at \(t = 0.0001\) with \(Re = 500\).
Figure 9: Numerical and Exact profiles of U and V components for $N = 20 \times 20$, $\Delta t = 0.00001$, $Re = 1000$ at time level $t = 0.0001$

Figure 10: Numerical and Exact profiles of U and V components for $N = 10 \times 10$, $\Delta t = 0.00001$, $Re = 1500$ at time level $t = 0.0001$

Table 11: $L_2$ and $L_{\infty}$ error norms for $N = 10 \times 10$, $\Delta t = 0.00001$, $Re = 100$, 500 and 1000 at the time levels $t = 0.0001$, 0.0002 and 0.0003 respectively

Table 12: Comparison of Exact and Numerical profiles of U and V components for $N = 10 \times 10$, $\Delta t = 0.00001$, $Re = 500$ at the time level $t = 0.0001$ for different mesh points

Example 6: In the following example Burgers’ equations are given with

Computational domain: $[a, b] \times [c, d] = [0, 0.5] \times [0, 0.5]$

Initial conditions:
\[ u(x, y, 0) = \sin(\pi x) + \cos(\pi y) \quad (69) \]
\[ v(x, y, 0) = x + y \quad (70) \]

**Boundary conditions:**
- \[ u(0, y, t) = \cos(\pi y) \]
- \[ u(0.5, y, t) = 1 + \cos(\pi y) \]
- \[ u(x, 0, t) = 1 + \sin(\pi x) \]
- \[ u(x, 0.5, t) = \sin(\pi x) \]
- \[ v(0, y, t) = y \]
- \[ v(0.5, y, t) = 0.5 + y \]
- \[ v(x, 0, t) = x \]
- \[ v(x, 0.5, t) = x + 0.5 \]

In Figure 11, Numerical profiles of both \( u \) and \( v \) components are given at \( t = 0.0001 \) and \( t = 0.0003 \) for \( Re = 1000 \).

\[ u(x, y) = \sin(\pi x) \sin(\pi y) \quad (71) \]
\[ v(x, y) = (\sin(\pi x) + \sin(2 \pi x))(\sin(\pi y) + \sin(2\pi y)) \quad (72) \]

Domain = \([0,1] \times [0,1]\).

In Figure 12, Numerical profiles of \( u \) and \( v \) components are given at \( t = 0.0001 \) with \( Re = 1000 \) and \( Re = 1500 \) respectively.

**Example 7**
In this example considered coupled Burgers’ equations has the exact solution [35] as follows:

**Example 8**

**Figure 11:** Numerical profiles of \( U \) and \( V \) components for \( N = 10, \Delta t = 0.000001, \; Re = 1000 \) at the time level \( t = 0.0001 \) and \( 0.0003 \) respectively

**Figure 12:** Numerical profile of \( U \) and \( V \) components for \( N = 40, \Delta t = 0.00001 \) at time level \( t = 0.0001 \) for \( Re = 1000 \) and \( Re = 1500 \) respectively
In this example considered coupled equations are given with following Exact solutions \[ \text{\text{\[3 \quad (73)\]}}, \]

\[ u(x, y, t) = \exp(-2\nu t) \sin(x + y) \quad (73) \]

\[ v(x, y, t) = \exp(-2\nu t) \sin(x + y) \quad (74) \]

**Domain** = \([a, b] \times [c, d] = [-\pi, \pi] \times [-\pi, \pi]\)

**Initial conditions:**

\[ u(x, y, 0) = \sin(x + y); \quad (x, y) \in [a, b] \times [c, d] \]

\[ v(x, y, 0) = \sin(x + y); \quad (x, y) \in [a, b] \times [c, d] \]

**Boundary conditions:**

\[ u(a, y, t) = -\exp(-2\nu t) \sin(a + y), \quad t \geq 0 \]

\[ u(b, y, t) = -\exp(-2\nu t) \sin(b + y), \quad t \geq 0 \]

\[ u(x, c, t) = -\exp(-2\nu t) \sin(x + c), \quad t \geq 0 \]

\[ u(x, d, t) = -\exp(-2\nu t) \sin(x + d), \quad t \geq 0 \]

\[ v(a, y, t) = \exp(-2\nu t) \sin(a + y), \quad t \geq 0 \]

\[ v(b, y, t) = \exp(-2\nu t) \sin(b + y), \quad t \geq 0 \]

\[ v(x, c, t) = \exp(-2\nu t) \sin(x + c), \quad t \geq 0 \]

\[ v(x, d, t) = \exp(-2\nu t) \sin(x + d), \quad t \geq 0 \]

In Figure 13, comparison of Numerical and Exact solutions is given graphically for both components at \(t = 0.0001\) for \(\text{Re} = 500\). In Figure 14, Numerical and Exact solutions are matched for \(u\) and \(v\) components at \(t = 0.0001\) for \(\text{Re} = 1000\). In Figure 15, a comparison of Numerical and Exact profiles of \(u\) and \(v\) components is given at \(t = 0.0001\) for \(\text{Re} = 1500\). In Table 13, \(L_2\) and \(L_{\infty}\) errors are given at the mentioned time levels for \(\text{Re} = 500, 800\) and \(1500\) respectively. In Table 14, RMS and Relative errors for both components are provided at the different time levels for \(\text{Re} = 1500\).

**Figure 13:** Comparison of Numerical and Exact profiles of \(U\) and \(V\) components for \(N = 50, \Delta t = 0.00001, \text{Re} = 500\) at time level \(t = 0.0001\)
Figure 14: Comparison of Numerical and Exact profiles of U and V components for $N = 50$, $\Delta t = 0.00001$, $Re = 1000$ at time level $t = 0.0001$

Figure 15: Comparison of Numerical and Exact profiles of U and V components for $N = 50$, $\Delta t = 0.00001$, $Re = 1500$ at time level $t = 0.0001$

Table 13: $L_2$ and $L_{\infty}$ error norms for $N = 10$, $\Delta t = 0.00001$, $Re = 500$, 800 and 1500 at time levels $t = 0.0001$, 0.0002 and 0.0003 respectively

<table>
<thead>
<tr>
<th></th>
<th>$L_2 U$</th>
<th>$L_{\infty} U$</th>
<th>$L_2 V$</th>
<th>$L_{\infty} V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Re = 500$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t = 0.0001$</td>
<td>1.16E-03</td>
<td>3.12E-04</td>
<td>1.16E-03</td>
<td>3.12E-04</td>
</tr>
<tr>
<td>$t = 0.0002$</td>
<td>2.22E-03</td>
<td>5.95E-04</td>
<td>2.22E-03</td>
<td>5.95E-04</td>
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<tr>
<td>$t = 0.0003$</td>
<td>3.27E-03</td>
<td>8.78E-04</td>
<td>3.27E-03</td>
<td>8.78E-04</td>
</tr>
<tr>
<td>$Re = 800$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t = 0.0001$</td>
<td>1.16E-03</td>
<td>3.12E-04</td>
<td>1.16E-03</td>
<td>1.16E-03</td>
</tr>
<tr>
<td>$t = 0.0002$</td>
<td>2.22E-03</td>
<td>5.95E-04</td>
<td>2.22E-03</td>
<td>5.95E-04</td>
</tr>
<tr>
<td>$t = 0.0003$</td>
<td>3.27E-03</td>
<td>8.78E-04</td>
<td>3.27E-03</td>
<td>8.78E-04</td>
</tr>
<tr>
<td>$Re = 1500$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t = 0.0001$</td>
<td>1.16E-03</td>
<td>3.12E-04</td>
<td>1.16E-03</td>
<td>3.12E-04</td>
</tr>
</tbody>
</table>
V. CONCLUSION

In this work, modified quartic Hyperbolic B-spline DQM is implemented upon coupled 1D and coupled 2D Burgers' equations. 8 test problems are discussed in this work. $L_2$ and $L_\infty$ errors along with Root mean square and Relative errors are discussed at different parameters. Numerical approximation and Exact solutions are matched for the different values. A compatible nature of numerical and Exact values is obtained. This compatibility of the Numerical and Exact solutions declares that the results obtained from the present scheme are acceptable. This research work will help researchers in their future research work to solve some complex linear and non-linear partial differential equations. In this paper quartic Hyperbolic B-spline of fourth order is developed. But higher order Hyperbolic B-splines can also be developed to solve higher order partial differential equations, especially when the analytical solution of the partial differential equation is not available.

REFERENCES


M. Kapoor and V. Joshi, “Numerical regime uniform algebraic hyperbolic tension B-spline DQM for the solution of fisher’s reaction-diffusion equation”.


