the Darcy drag is independent of viscosity. If, on the other hand, $\frac{1}{V}\int_S \vec{T}\cdot\vec{n}dS$ is of significance (due to importance of momentum exchange of the fluid with the solid matrix), then the Darcy drag could be expressed in terms of both viscosity and pressure.

Clearly, the first of these two cases would be more in line with the idea that viscosity $\mu(p)$ is a measure of the frictional resistance in between fluid layers while $\alpha(p)$ is a measure of the friction between the fluid and the solid, at the pore. In either case, the surface integrals, $\frac{1}{V}\int_S p^{\circ}\vec{\boldsymbol{n}}dS$ and $\frac{1}{V}\int_S \vec{T}\cdot\vec{\boldsymbol{n}}dS$ can be combined to form a surface filter $\frac{1}{V}\int_S (\overrightarrow{T}\cdot\vec{n}-p^{\circ}\vec{n})dS$, in the sense introduced by Whitaker, [28, 29], that involves the normal derivative of $\vec{u}$.

Since the solid porous matrix affects the fluid through the portion of the surface area of the solid that is in contact with it, this surface filter contains the information necessary to quantify the forces exerted on the flowing fluid by the porous matrix. This form of surface filter has been abundantly analyzed in the literature for the case of constant viscosity, and has been identified with the force that gives rise to Darcy resistance. Since viscosity is a function of pressure, we can write this integral in terms of a function of pressure in order to emphasize significance of pressure fluctuations at pore level:

$$\frac{1}{V}\int_S -(p^{\circ}\vec{\boldsymbol{n}}-\vec{\boldsymbol{T}}\cdot\vec{\boldsymbol{n}})dS = -f\lambda(p)\varphi < \vec{\boldsymbol{u}} >_{\varphi} \qquad (24)$$

wherein $f$ is a friction factor and $\lambda(p)$ is a function of pressure that is not necessarily related to the functional form of viscosity as a function of pressure.

The above surface integral, or the friction factor, $f$, is a shear force integral which accounts for the viscous drag effects (Darcy resistance) that predominate in the Darcy regime, that is, for small Reynolds number flow. It is the *velocity-independent* viscous shear geometric factor that depends on the geometry of the porous medium and gives rise to the Darcy resistance.

Assuming that viscous drag effect is the only frictional force present, and defining hydrodynamic permeability as, [25]:

$$\eta = \frac{\varphi}{f} \qquad (25)$$

then as a first approximation, the following expression can be used for surface integral (24):

$$\frac{1}{V}\int_S -(p^{\circ}\vec{\boldsymbol{n}}-\vec{\boldsymbol{T}}\cdot\vec{\boldsymbol{n}})dS = -\frac{\varphi}{\eta}\lambda(p)\varphi < \vec{u} >_{\varphi} = -\alpha(p,\vec{x})\varphi < \vec{u} >_{\varphi} \qquad (26)$$

where the Darcy drag term in (26) is given by:

$$\alpha(p,\vec{x}) = f\lambda(p) = \frac{\varphi}{\eta(\vec{x})}\lambda(p) \qquad (27)$$

Using (26), equation (19) can be written in the following form:

$$\rho\nabla\cdot\varphi < \vec{u} >_{\varphi} < \vec{u} >_{\varphi} = -\varphi\nabla < p >_{\varphi} + \nabla\cdot\ \varphi < \vec{T} >_{\varphi} - \alpha(p,\vec{x})\varphi < \vec{u} >_{\varphi} + \rho\varphi < \vec{g} >_{\varphi} \qquad (28)$$

Equation (28) is the intrinsically averaged momentum equation (10). Porosity is left in this equation as a variable function of position. The momentum equation corresponding to constant porosity is easily recovered from (28) by factoring out $\varphi$, as follows. Denoting $< \vec{u} >_{\varphi}$, $< \vec{T} >_{\varphi}$, $< \vec{p} >_{\varphi}$ and $< \vec{g} >_{\varphi}$ by $\vec{u}$, $\vec{T} = \mu(\nabla\vec{u} + \nabla\vec{u}^T)$, $\vec{p}$ and $\vec{g}$, respectively, equation (28) reduces to:

$$\rho\nabla\cdot\vec{u}\,\vec{u} = -\nabla p + \nabla\cdot\ \vec{T} - \alpha(p,\vec{x})\vec{u} + \rho\vec{g} \qquad (29)$$

while continuity equation is obtained from (20) as:

$$\nabla\cdot\vec{u} = 0 \qquad (30)$$

For variable porosity, we let $\vec{q} = \varphi < \vec{u} >_{\varphi}$ be the specific discharge, $\vec{G} = \varphi < \vec{g} >_{\varphi}$, $\mu^* = \varphi < \mu >_{\varphi}$ we write (20) in the following final form:

$$\nabla\cdot\ \vec{q} = 0 \qquad (31)$$

and equation (28) in the following final form:

$$\rho\nabla\cdot\vec{q}\vec{q}/\varphi = -\varphi\nabla < p >_{\varphi} + +\nabla\cdot\ \mu^*\left\{\nabla\left(\frac{\vec{q}}{\varphi}\right) + \nabla\left(\frac{\vec{q}}{\varphi}\right)^T\right\} - \alpha(p,\vec{x})\vec{q} + \rho\vec{G} \qquad (32)$$

where $\alpha(p,\vec{x})$ is as given by (27).

## III. CHOICES FOR THE DRAG COEFFICIENT

Equation (27) shows that we need $\lambda(p)$, $\mu^*(p)$ and friction factor $f$ in order to determine a drag coefficient that takes into account the effects of the porous microstructure. Once a choice is made for $\alpha(p,\vec{x})$, model equations (31) and (32) are updated. In what follows, we list choices for $\lambda(p)$ and $\mu^*(p)$, and expressions for the friction factor and its values for different porosities.

*Choices for $\lambda(p)$ and $\mu^*(p)$*

Many choices exist for $\lambda(p)$ and $\mu^*(p)$ and have been used by various authors (*cf.* Kannan and Rajagopal [18], Fusi *et.al*. [4] and the references therein). Typical among these are the Barus' relations for viscosity, [5], namely

$$\mu(p) = \mu_0 e^{\delta(p-p_0)} \quad (33)$$

which is approximated for small $\delta$, or small pressure differences, by

$$\mu(p) = \mu_0[1+\delta(p-p_0)] \quad (34)$$

and the following for $\lambda(p)$, [18]

$$\lambda(p) = \lambda_0;\ \lambda(p) = \lambda_0(1+\gamma p);\ \ \lambda(p) = \lambda_0 e^{\gamma p} \quad (35)$$

where $\lambda_0 > 0;\ \ \gamma > 0$

*Porous Microstructure and Friction Factor*

Expressions for $f$ require a mathematical description of the porous matrix and its microstructure. Du Plessis and Diedericks, [25], carried out extensive analysis on evaluating these geometric factors for isotropic porous media, based on Du Plessis and Masliyah's concept of a Representative Unit Cell (RUC), [22, 23], which they defined as the minimal REV in which the average properties of the porous medium are embedded. For granular and consolidated isotropic porous media, the following expressions, [25], summarized and computed in **Tables 1-7**, are adopted in this work for $f$ and for the hydrodynamic permeability, $\eta$. We note that as $\varphi \to 1$, the porous medium approaches free-space and $f \to 0$, for all microstructures considered.

| **Description** | **Geometric Factor** $f$ |
|---|---|
| Granular Matter | $\dfrac{36(1-\varphi)^{\frac{2}{3}}}{d^2[1-(1-\varphi)^{\frac{1}{3}}][1-(1-\varphi)^{\frac{2}{3}}]}$ |
| Consolidated Matter | $\dfrac{42.69(1-\tau)}{(\tau d)^2\varphi}$ |
| Unidirectional Fibre Bed | $\dfrac{24\sqrt{1-\varphi}}{d^2(1-\sqrt{1-\varphi})^2}$ |
| Ergun's Equation | $\dfrac{150(1-\varphi)^2}{\varphi^2(d_p)^2}$ |
| Kozeny-Carman Relation | $\dfrac{180(1-\varphi)^2}{\varphi^2(d_m)^2}$ |

**Table 1 Geometric Factor Expressions**

| **Description** | **Hydrodynamic Permeability** $\eta = \dfrac{\varphi}{f}$ |
|---|---|
| Granular Matter | $\dfrac{d^2\varphi[1-(1-\varphi)^{\frac{1}{3}}][1-(1-\varphi)^{\frac{2}{3}}]}{36(1-\varphi)^{\frac{2}{3}}}$ |
| Consolidated Matter | $\dfrac{(\varphi\tau d)^2}{42.69(1-\tau)}$ |
| Unidirectional Fibre Bed | $\dfrac{\varphi d^2(1-\sqrt{1-\varphi})^2}{24\sqrt{1-\varphi}}$ |
| Ergun's Equation | $\dfrac{\varphi^3(d_p)^2}{150(1-\varphi)^2}$ |
| Kozeny-Carman Relation | $\dfrac{\varphi^3(d_m)^2}{180(1-\varphi)^2}$ |

**Table 2 Hydrodynamic Permeability Expressions**

| $\varphi$ | **Kozeny-Carman** $f = \dfrac{180(1-\varphi)^2}{\varphi^2(d_m)^2}$ |
|---|---|
| 0.5 | $180/d_m^2$ |
| 0.9 | $2.222222/d_m^2$ |

| | |
|---|---|
| 0.95 | $0.498615//d_m^2$ |
| 0.99 | $0.02222222/d_m^2$ |
| 0.995 | $0.00454534/d_m^2$ |
| 0.999 | $0.00018036/d_m^2$ |
| 1 | 0 |

**Table 3 Geometric Factor Values Vs. Porosity**

| $\varphi$ | **Ergun's Equation** $f=\frac{150(1-\varphi)^2}{\varphi^2(d_p)^2}$ |
|---|---|
| 0.5 | $150/d_p^2$ |
| 0.9 | $1.85185167/d_p^2$ |
| 0.95 | $0.4155125/d_p^2$ |
| 0.99 | $0.01851851/d_p^2$ |
| 0.995 | $0.00378778/d_p^2$ |
| 0.999 | $0.00015030/d_p^2$ |
| 1 | 0 |

**Table 4 Geometric Factor Values Vs. Porosity**

| $\varphi$ | **Unidirectional Fibre Bed** $f=\frac{24\sqrt{1-\varphi}}{d^2(1-\sqrt{1-\varphi})^2}$ |
|---|---|
| 0.5 | $197.824447/d^2$ |
| 0.9 | $16.232637/d^2$ |
| 0.95 | $8.902941/d^2$ |
| 0.99 | $2.9629630/d^2$ |
| 0.995 | $1.96514386/d^2$ |
| 0.999 | $0.80932342/d^2$ |
| 1 | 0 |

**Table 5 Geometric Factor Values Vs. Porosity**

| $\varphi$ | **Granular Matter** $f=\frac{36(1-\varphi)^{\frac{2}{3}}}{d^2[1-(1-\varphi)^{\frac{1}{3}}][1-(1-\varphi)^{\frac{2}{3}}]}$ |
|---|---|
| 0.5 | $297.0774570/d^2$ |
| 0.9 | $18.44906/d^2$ |
| 0.95 | $8.95067/d^2$ |
| 0.99 | $2.233504000/d^2$ |
| 0.995 | $1.308021611/d^2$ |
| 0.999 | $0.404040404/d^2$ |
| 1 | 0 |

**Table 6 Geometric Factor Values Vs. Porosity**

| $\varphi$ | **Consolidated Matter** $f=\frac{42.69(1-\tau)}{(\tau d)^2\varphi}$, $\tau\approx\frac{1+2\varphi}{3}$ $\tau$ | $f$ |
|---|---|---|
| 0.5 | 0.5 | $170.76/d^2$ |
| 0.9 | 0.719 | $25.782926/d^2$ |
| 0.95 | 0.967 | $1.585855/d^2$ |
| 0.99 | 0.993 | $0.306119/d^2$ |
| 0.995 | 0.9967 | $0.1439733/d^2$ |
| 0.999 | 0.9993 | $0.0285265/d^2$ |
| 1 | 1 | 0 |

**Table 7 Geometric Factor Values Vs. Porosity**

In the above Tables, tortuosity is approximated by:

$$\tau\approx\frac{1+2\varphi}{3}$$

and $d$ is a microscopic characteristic length, $d_p$ is the average pore diameter in a channel-like porous material, and $d_m$ is a median diameter of spherical particles

## IV. CONCLUSION

In this work, intrinsic volume averaging was implemented in deriving equations governing the flow of an incompressible fluid with variable, pressure-dependent viscosity through isotropic porous media. Five porous microstructures were incorporated in the drag coefficient in order to distinguish between porous structures. This work might prove to be of value in the determination of which pressure function to use for a given porous configuration.

For flow through a constant porosity medium, governing equations are (29) and (30), with drag coefficient given by (27). For flow through variable porosity media, governing equations are (31) and (32), with drag coefficient given by (27). In both momentum equations (29) and (32), we left in convective terms.

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**Contribution of individual authors:**

All authors reviewed the literature and conducted averaging process.

M.S. Abu Zaytoon and S. Jayyousi Dajani provided the tables and calculations, and contributed to the writing of some parts of the mathematical analysis.

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