System Parameters Identification in a General Class of Non-linear Mechanical Systems

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***Abstract*—** The problem of system identification and parameters monitoring for a general class of non-linear systems is discussed together with the introduction of a new method based on Lie series expansion. In order to use this approach, the system features must be modeled by analytic or sufficiently smooth functions of the state variables, including the time parameter.

The method uses the Lie differential operator representations. The solution obtained are expressed in the form of analytical power series including the system parameters. The information carried by these solutions is sufficiently complete and provides good estimates of the system parameters.

In this paper, a mechanical system made by a reverse pendulum jointed to a sliding mass is studied. The identification of parameters of this system is the main aim of this simple application. Since the motion equations can be numerically solved, a comparison between theoretical and experimental values of parameters is performed. This comparison is based on the minimization of the difference between numerical and approximated solution, the last obtained by Lie series.

***Keywords*—**Lie Series Expansion, PDE, System Parameters Identification.

I. Introduction

IN general terms, the aim of system identification is to find system properties through investigations of experimental data records. Therefore, the choice of appropriate methods for system identification depends on both types of experimental records and the objectives of the identification. There are various identification tools well documented in literature, also concerning non-linear approaches to the identification problems. This is very important because the presence of non-linearities essentially complicates the identification problems since the linear superposition principle becomes inapplicable and therefore explicit analytical solutions are usually unavailable. However, it should be noted that the level of complexity of a non-linear identification problem depends on the assumptions regarding the availability of information obtained from experimental measurements. For example, if all the state variables and external excitations are known, or can be determined from experimental measurements, then the differential equations of motion, rather than their solutions, can be used for parameters identification purposes. However, success of non-linear identification formulations depends strongly on the availability of direct methods for the solutions of the non-linear dynamics.

II. Lie Series

To illustrate the idea of Lie series, let us consider the standard initial value problem for a single non-linear first-order differential equation described by:

$$\begin{aligned} \dot{x}(t) &= f\left(x(t), a\right), \\ x(0) &= x^0, \end{aligned} \tag{1}$$

where the function $f(x,a)$ is assumed to be sufficiently smooth such that, in some neighbourhood of the initial point x^0, and for any admissible value of the parameter(s) a, it possesses as many derivatives with respect to x as may be needed. The Taylor's expansion of the function $x(t)$ is

$$x(t) = x(0) + \dot{x}(0)t + \frac{1}{2!}\ddot{x}(0)t^2 + \ldots \tag{2}$$

In order to obtain the corresponding Lie series, one should enforce expression (1) for calculating the coefficients of series (2), as follows:

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$$
\begin{aligned}
&x(0) = x^0 \\
&\dot{x}(0) = f\left(x^0, a\right) \\
&\ddot{x}(0) = \dot{f}\left(x(t), a\right)\big|_{t=0} = f'_{x^0}\left(x^0, a\right)\dot{x}(0) = \\
&\quad = f'_{x^0}\left(x^0, a\right) f\left(x^0, a\right) \\
&\ldots
\end{aligned} \tag{3}
$$

Substituting Eq. (3) into Eq. (2), one obtains the Lie series solution of the initial value problem (1) in the form of local expansion around the initial time $t = 0$. Following the idea of Lie operators, such a solution can be represented in the form:

$$
\begin{aligned}
&x(t) = e^{tA(a)}x^0 = \\
&= x^0 + tA(a)x^0 + \frac{t^2}{2!}A^2(a)x^0 + \ldots
\end{aligned} \tag{4}
$$

where

$$
A(a) = f\left(x^0, a\right)\frac{\partial}{\partial x^0} \tag{5}
$$

is a linear differential Lie operator, associated with the dynamical system (1). As shown by the form of the series (4), the initial condition coordinate x^0 and the right-hand side of Eq. (1) determine all the coefficients of the series.

One of the advantages of the Lie series solution is that its extension to a multi-dimensional (vector) case is quite simple. For example, let us consider the initial-value problem

$$
\begin{aligned}
&\dot{\vec{x}}(t) = \vec{f}\left(\vec{x}(t), \vec{a}\right) \\
&\vec{x}(0) = \vec{x}^0
\end{aligned} \tag{6}
$$

where $\vec{x}(t) = \left(x_1(t), \ldots, x_n(t)\right)$ and $\vec{x}^0(t) = \left(x_1^0(t), \ldots, x_n^0(t)\right)$ are respectively the unknown vector function and initial vector, $\overline{a} = \left(a_1, \ldots, a_n\right)$ is a vector of the system parameters, and $\vec{f} = \left(f_1, \ldots, f_n\right)$ is a vector function of the right-hand side whose components depend on the components of vectors $\vec{x}$ and $\vec{a}$. In this case, the Lie series solution of the initial value problem (6) is represented in vector form as

$$
\begin{aligned}
&\vec{x}(t) = e^{tA(\vec{a})}\vec{x}^0 = \\
&= \vec{x}^0 + tA(\vec{a})\vec{x}^0 + \frac{t^2}{2!}A^2(\vec{a})\vec{x}^0 + \ldots
\end{aligned} \tag{7}
$$

where

$$
\begin{aligned}
A(\vec{a}) &= \vec{f}\left(\vec{x}^0, \vec{a}\right)\frac{\partial}{\partial \vec{x}^0} \\
&\equiv f_1\left(x_1^0, \ldots, x_n^0; a_1, \ldots, a_n\right)\frac{\partial}{\partial x_1^0} + \ldots \\
&\ldots + f_n\left(x_1^0, \ldots, x_n^0; a_1, \ldots, a_n\right)\frac{\partial}{\partial x_n^0}
\end{aligned} \tag{8}
$$

is the Lie operator associated with the dynamical system (6). If $t_0 \neq 0$; then series (7) should be modified by shifting the time as follows:

$$
\begin{aligned}
x(t) &= e^{(t-t_0)A(a)}x^0 = \\
&x^0 + (t - t_0)A(a)x^0 + \\
&+ \frac{(t - t_0)^2}{2!}A^2(a)x^0 + \ldots
\end{aligned} \tag{9}
$$

III. Problem Solution

The motion equation of the system shown in Fig. 1 can be obtained by means of Lagrange's equations.

$$
\begin{cases}
(m_1 + m_2)\ddot{z} = m_2 L_2 \sin\varphi\, \ddot{\varphi} - m_2 L_2 \cos\varphi\, \dot{\varphi}^2 \\
\left(m_2 L_2^2 + I_{zz}\right)\ddot{\varphi} = m_2 L_2 \sin\varphi\, \ddot{z} - m_2 L_2 g \cos\varphi
\end{cases} \tag{10}
$$

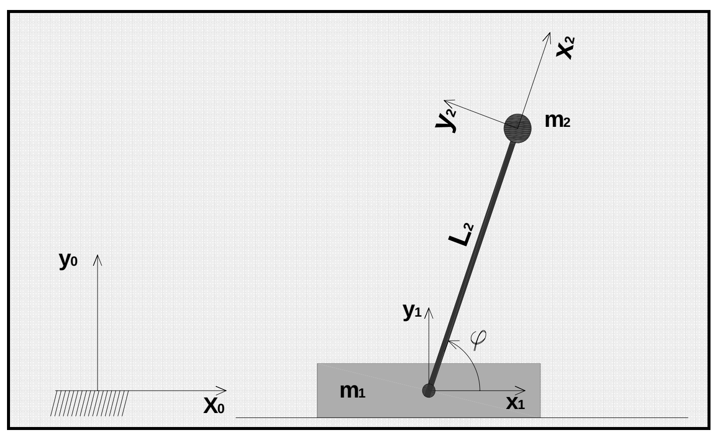

Fig. 1: Mechanical System

Equations (10) can be rewritten in the following form:

$$
\begin{cases}
\ddot{z} = \dfrac{m_2 L_2}{m_1 + m_2}\left(\sin\varphi\, \ddot{\varphi} - \cos\varphi\, \dot{\varphi}^2\right) \\
\ddot{\varphi} = \dfrac{m_2 L_2}{m_2 L_2^2 + I_{zz}}\left(\sin\varphi\, \ddot{z} - g\, \cos\varphi\right)
\end{cases} \tag{11}
$$

If we put:

$$A = \frac{m_2 L_2}{m_1 + m_2};$$
$$B = \frac{m_2 L_2}{m_2 L_2^2 + I_{zz}}; \qquad (12)$$
$$C = \frac{m_2 L_2 g}{m_2 L_2^2 + I_{zz}};$$

equations (11) can be written as follow:

$$\ddot{z} = A \sin\varphi\, \ddot{\varphi} - A \cos\varphi\, \dot{\varphi}^2 \qquad (13a)$$
$$\ddot{\varphi} = B \sin\varphi\, \ddot{z} - C \cos\varphi \qquad (13b)$$

Substituting in (13b) $\ddot{z}$, the following pair of equation is obtained:

$$\ddot{z} = A \sin\varphi\, \ddot{\varphi} - A \cos\varphi\, \dot{\varphi}^2 \qquad (14a)$$
$$\ddot{\varphi} = AB \sin^2\varphi\, \ddot{\varphi} - AB \sin\varphi \cos\varphi\, \dot{\varphi}^2 - C \cos\varphi \qquad (14b)$$

where (14b) can be re-written in explicit form:

$$\ddot{\varphi} = \frac{AB \sin\varphi \cos\varphi}{AB \sin^2\varphi - 1} \dot{\varphi}^2 + \frac{C \cos\varphi}{AB \sin^2\varphi - 1} \qquad (15)$$

Putting $\varphi = z_1$, $\dot{\varphi} = z_2$, equation (15) is transformed in the following first order differential system:

$$\begin{cases} \dot{z}_1 = z_2 & = f_1(z_1, z_2) \\ \dot{z}_2 = \dfrac{AB \sin z_1 \cos z_1}{AB \sin^2 z_1 - 1} z_2^2 + \dfrac{C \cos z_1}{AB \sin^2 z_1 - 1} & = f_2(z_1, z_2) \end{cases} \qquad (16)$$

Here z_1 and z_2 are respectively the angular position and the angular velocity of the pendulum. They may be considered components of the vector:

$$\vec{z} = (z_1, z_2) \qquad (17)$$

In this case the Lie operator assumes the form:

$$D = z_2 \frac{\partial}{\partial z_1} + \left(\frac{AB \sin z_1 \cos z_1}{AB \sin^2 z_1 - 1} z_2^2 + \frac{C \cos z_1}{AB \sin^2 z_1 - 1} \right) \frac{\partial}{\partial z_2} \qquad (18)$$

and vector $\vec{z}$ is given by:

$$\vec{z} = e^{tD_0} z \Big|_{z \to z_0} \qquad (19)$$

or in component form by:

$$\begin{cases} z_1 = e^{tD_0} z_1 \Big|_{\substack{z_1 \to z_{10} \\ z_2 \to z_{20}}} & (20a) \\ z_2 = e^{tD_0} z_2 \Big|_{\substack{z_1 \to z_{10} \\ z_2 \to z_{20}}} & (20b) \end{cases}$$

Expanding the exp operator, (20a) can be written as follows:

$$z_1 = z_{10} + tD(a) z_{10} + \frac{t^2}{2!} D^2(a) z_{10} + \frac{t^3}{3!} D^3(a) z_{10} + \frac{t^4}{4!} D^4(a) z_{10} + \ldots \qquad (21)$$

where:

$$D(a)\varphi_0 = \left[\eta \frac{\partial}{\partial\varphi} + \left(\frac{AB \sin\varphi \cos\varphi}{AB \sin^2\varphi - 1} \eta^2 + \frac{C \cos\varphi}{AB \sin^2\varphi - 1} \right) \frac{\partial}{\partial\eta} \right] (\varphi_0) = \eta + 0 \qquad (22)$$

$$\begin{aligned} D^2(a)\varphi_0 &= D(a) \left[\eta \frac{\partial}{\partial\varphi} + \left(\frac{AB \sin\varphi \cos\varphi}{AB \sin^2\varphi - 1} \eta^2 + \frac{C \cos\varphi}{AB \sin^2\varphi - 1} \right) \frac{\partial}{\partial\eta} \right] (\varphi_0) = \\ &= D(a)(\eta) = \left[\eta \frac{\partial}{\partial\varphi} + \left(\frac{AB \sin\varphi \cos\varphi}{AB \sin^2\varphi - 1} \eta^2 + \frac{C \cos\varphi}{AB \sin^2\varphi - 1} \right) \frac{\partial}{\partial\eta} \right] (\eta) = \\ &= 0 + \left(\frac{AB \sin\varphi \cos\varphi}{AB \sin^2\varphi - 1} \eta^2 + \frac{C \cos\varphi}{AB \sin^2\varphi - 1} \right) \\ &\ldots \end{aligned} \qquad (23)$$

and so on.

Truncating the expansion series (21) at fourth order, the value of the unknown parameter m_2 can be calculated by minimizing the error:

$$e = \sum_{i=1}^{n} \left(z_i - \hat{z}_i(m_2) \right)^2 \qquad (24)$$

The value of m_2 so obtained, resulted in good agreement with the actual value of this parameter.

IV. Conclusion

In this paper a method by using Lie series development has been proposed in order to identify the parameters of a mechanical system composed of a pendulum jointed to a sliding mass. To authors' opinion the proposed procedure is very useful in non linear identification problems arising in the multibody systems dynamics.

References

[1] V. N. Pilipchuk, C. A. Nan, Non linear system identification based on Lie series solutions, *Mechanical Systems and Signal Processing*, 19 (2005) 71-86.

[2] D. Guida, J. Quartieri, S. Steri, A Procedure for Calculating an Approximate Analytical Response in a Large Class of Mechanical Systems, *WSEAS International Conference on Non-Linear Analysis*, Tenerife 2006.

[3] Quartieri J., Steri S., A New Method for Solving 2D Nonlinear PDEs Relevant in Physics, *Int. J. Nonlin. Sci. Num. Sim.,* 4, n.4 (2003), 423-427

[4] Jer-Nan Juang, Minh Q. Phan, Identification and Control of Mechanical Systems, Cambridge University Press (2001).

[5] Jer-Nan Juang, System Identification, Prentice Hall (1994).

[6] J. Quartieri, S. Steri, G. Volzone, Int. J. Nonlin. Scien. And Num. Simul. 2001.V. 2.

[7] Guida M., Quartieri J., Steri S., Groebner's integration of a special class of nonlinear Cauchy problems, Int. J. Nonlin. Sci. Num. Sim., 4 (2003), 99-102.

[8] Quartieri J., Steri S., Volzone G., Lie series and nonlinear evolution problems. Int. J. Nonlin. Sci. Num. Sim., 2 (2001), 167-168.

[9] Di Bartolomeo A., Quartieri J., Steri,S., Perturbed non-linear evolution problem solved by a generalized Lie series method. Int. J. Nonlin. Sci. Num. Sim., 1, 3 (2002), 75-76.

[10] Di Bartolomeo A., Quartieri J., Steri S., A class of non-linear implicit Cauchy problems integrated by Groebner method. Int. J. Nonlin. Sci. Num. Sim., 4 (2003), 103-104.

[11] Di Bartolomeo A., Quartieri J., Steri S., Lie and Lagrange series in nonlinear equations with controlled evolution, Int. J. Nonlin. Sci. Num. Sim., 4 (2003), 95-97.

[12] Di Bartolomeo A., Quartieri J., Steri S., Second order nonlinear Cauchy problem in a four dimensional space. Nonlinear Oscillations, Vol.6, N.2 (2003), 166-170.

[13] Quartieri J., Steri S., A New Method for Solving 2D Nonlinear PDEs Relevant in Physics, Int. J. Nonlin. Sci. Num. Sim., 4, n.4 (2003), 423-427.

[14] S. Steri, J. Quartieri, G. Volzone, C. Guarnaccia, On Picard Iteration and Lie Series in Analytic Evolutionary Problems of Physics, Proceeding of the 9th WSEAS *International Conference on Mathematics & Computers in Biology & Chemistry* (MCBC '08), Bucharest, Romania, 2008. ISBN: 978-960-6766-75-6 / ISSN: 1790-5125, pp 157-163.

[15] J. Quartieri, S. Steri, M. Guida, C. Guarnaccia, S. D'Ambrosio, A Biomathematical Study of a Controlled Birth and Death Process Describing Malignancy, *Advanced Topics on Mathematical Biology and Ecology*, Proceeding of the 4th WSEAS Int. Conf. MABE '08. Acapulco, Mexico, January 25-27, 2008. ISBN: 978-960-6766-32-9 / ISSN: 1790-2769.

[16] W. Gröbner, Gruppi anelli e algebre di Lie, Cremonese,Roma 1975.

[17] W. Gröbner, Serie di Lie e loro applicazioni, Cremonese,Roma 1973.

[18] B.C. Goodwin, Temporal organization in cells, Accademic Press, London and New York, 1963.

[19] W. Gröbner, H. Knapp, Contribution to the method of Lie series, Bibliographisches Institut, Univ. Mannheim 1967.

[20] M. Guida, J. Quartieri, E. Quarto, S. Steri, G. Volzone, Feedback control with delay in biological problems: a new approach. *WSEAS Transactions on Mathematics* Issue 5, Volume 6, p. 688, May 2007.