Homogeneous Hopf hypersurfaces in a complex hyperbolic space and extrinsic shapes of their trajectories

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Abstract—We study homogeneous real hypersurfaces in a complex hyperbolic space whose characteristic vectors are principle. We characterize some of them by investigating properties of extrinsic shapes of some curves on these hypersurfaces which are associated with their characteristic tensor fields

Keywords—complex hyperbolic space, extrinsic circular trajectories, real hypersurfaces, Sasakian magnetic fields tangentially of order 2.

I. INTRODUCTION

The aim of this paper is to study real hypersurfaces in a complex hyperbolic space by observing a family of curves on these hypersurfaces from the ambient complex hyperbolic space. We explain our standing point by giving an elementary example. If we take a standard sphere in a Euclidean space, all geodesics on this sphere can be seen as circles of same geodesic curvature (or, equivalently, of same radius) in the ambient Euclidean space. Moreover, standard spheres in a Euclidean space are characterized by this property among hypersurfaces in this Euclidean space. Since this result is quite simple, many geometers gave some generalizations. For example, Kimura, Maeda and the second author characterized all homogeneous real hypersurfaces in a complex projective space in [4]. They studied geodesics which can be seen as circles in the ambient space, and characterized them by the property that initial vectors of such geodesics span the tangent subspace orthogonal to the characteristic vector at each point. They also studied ruled real hypersurfaces and totally η-umbilic real hypersurfaces in a nonfla complex space form in [5]. Here, a totally η-umbilic real hypersurface is one of a geodesic sphere, a horosphere, and a tube around a totally geodesic totally complex submanifold. They characterized them by the property that every geodesic whose initial vector is orthogonal to the characteristic vector lies on some totally geodesic real surface in the ambient space.

Since we have three subclasses in the class of totally η-umbilic real hypersurfaces in a complex hyperbolic space, we are interested in giving some characterizations on real hypersurfaces in each class. To do this we investigate trajectories for Sasakian magnetic fields which are generalized objects of geodesics that are closely related with characteristic vector fields We pay attention to second jets of their extrinsic shapes and study principal curvatures of the underlying real hypersurfaces.

The firs author is partially supported by the National Natural Science Foundation of China (No. 11561052 and No. 11661062).

The second author is partially supported by Grant-in-Aid for Scientifi Research (C) (No. 24540075 and No. 16K05126) Japan Society for the Promotion of Science.

II. EXTRINSIC SHAPES OF TRAJECTORIES

Let M be a real hypersurface in a complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature c. The complex structure J on $\mathbb{C}H^n(c)$ induces an almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ on M. If we denote by $\mathcal{N}$ a (local) normal vector fiel on M in $\mathbb{C}H^n$, a vector fiel ξ on M is define by $\xi = -J\mathcal{N}$, a 1-form η by $\eta(v) = \langle v, \xi\rangle$, and a $(1,1)$-tensor fiel ϕ is define by $\phi(v) = Jv - \eta(v)\mathcal{N}$. Here, $\langle\ ,\ \rangle$ denotes the induced metric on M given by the metric on $\mathbb{C}H^n(c)$, which is also denoted by $\langle\ ,\ \rangle$. These vector and tensor field ξ and ϕ are called the characteristic vector fiel and the characteristic tensor, respectively.

In order to study real hypersurfaces from curve theoretic point of view, we defin a family of curves which are deeply concerned with their contact metric structure. Given a constant κ, we say a smooth curve γ parameterized by its arclength a *trajectory* if it satisfie the differential equation $\nabla_{\dot\gamma}\dot\gamma = \kappa\phi\dot\gamma$, where $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ. Dynamical theoretically, if we set a closed 2-form $\mathbb{F}_\kappa$ on M by $\mathbb{F}_\kappa(u,v) = \langle u, \phi v\rangle$ for arbitrary $u, v \in T_pM$ at an arbitrary point $p \in M$, we can interpret such a curve as a trajectory for a magnetic fiel $\mathbb{F}_\kappa$ which gets a Lorentz force $\kappa\phi\big(\dot\gamma(t)\big)$ at each point $\gamma(t)$. This 2-form is called a Sasakian magnetic field When $\kappa = 0$, this curve does not get influenc of magnetic fields and is a geodesic. Thus, trajectories for Sasakian magnetic field are extended objects of geodesics. Since trajectories are related with the characteristic tensors on the underlying real hypersurfaces, it is natural to consider that properties of a real hypersurface reflec to properties of trajectories.

In this paper we pay attention to properties of extrinsic shapes of trajectories. We denote by $\iota : M \to \mathbb{C}H^n(c)$ an isometric embedding. For a smooth curve γ on M, we call the curve $\iota\circ\gamma$ on $\mathbb{C}H^n(c)$ its *extrinsic shape*. For the sake of simplicity we denote $\iota\circ\gamma$ also by γ. The Riemannian connections ∇ and $\widetilde\nabla$ on M and $\mathbb{C}H^n(c)$ are related by Gauss and Weingarten formulae which are given as

$$\widetilde\nabla_X Y = \nabla_X Y + \langle A_M X, Y\rangle\mathcal{N} \quad \text{and} \quad \widetilde\nabla_X\mathcal{N} = -A_M X$$

with the shape operator A_M of M for arbitrary vector field X, Y. Since J is parallel, by use of these formulae we have

$\nabla_X \xi = \phi A_M X$. We compute firs and second covariant differentiations of a trajectory γ for $\mathbb{F}_\kappa$ by using this equality and obtain

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \kappa\phi\dot\gamma + \langle A_M\dot\gamma, \dot\gamma\rangle \mathcal{N}_\gamma, \tag{II.1}$$

$$\widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma = -\kappa^2\dot\gamma - \big(\langle A_M\dot\gamma, \dot\gamma\rangle - \kappa\rho_\gamma\big)\big(A_M\dot\gamma + \kappa\xi_\gamma\big) + \frac{d}{dt}\big(\langle A_M\dot\gamma, \dot\gamma\rangle - \kappa\rho_\gamma\big)\mathcal{N}_\gamma, \tag{II.2}$$

where we set $\rho_\gamma = \langle \dot\gamma, \xi_\gamma\rangle$. We here note that the tangent space T_pM at a point $p \in M$ is decomposed as $T_pM = T_p^0M \oplus \mathbb{R}\xi_p$ with a complex subspace T_p^0M of $T_p\mathbb{C}H^n$. Thus, the function ρ_γ along γ measures the size of the "non-complex" component of $\dot\gamma$. We call this the structure torsion of γ.

A smooth curve σ on a Riemannian manifold N which is parameterized by its arclength is said to be a *circle* if it satisfie the system $\nabla_{\dot\sigma}\dot\sigma = kY,\ \nabla_{\dot\sigma}Y = -k\dot\sigma$ of differential equations, or equivalently, if it satisfie $\nabla_{\dot\sigma}\dot\sigma = -k^2\dot\sigma$, with a nonpositive constant k and a fiel Y of unit vectors along γ. From the viewpoint of Frenet-Serrer formula on curves, circles are simplest curve next to geodesics.

In order to study properties of extrinsic shapes of curves which are related with the underlying real hypersurface, we here pay attention to their tangential components. We say the extrinsic shape of a curve γ parameterized by its arclength to be *tangentially of order* 2 at $\gamma(t_0)$ if $\widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma(t_0) + \|\widetilde{\nabla}_{\dot\gamma}\dot\gamma(t_0)\|^2\dot\gamma(t_0)$ does not have a component tangent to M. Clearly, if the extrinsic shape is a circle, then it is tangentially of order 2 at its arbitrary point. When the extrinsic shape of a curve is a circle, it is said to be *extrinsic circular*. We studied extrinsic circular trajectories in [2], [7] and showed that almost all circles on $\mathbb{C}H^n$ are given as extrinsic circular trajectories on geodesic spheres and tubes around complex hypersurfaces. Our condition that a trajectory on a real hypersurface is tangentially of order 2 at each point is weaker than the condition that it is extrinsic circular. If we use such a terminology, we may say γ is tangentially of order 1 at $\gamma(t_0)$ if it is a geodesic point as a curve on M. But we note that even if it is tangentially of order 1 on an open interval containing $\gamma(t_0)$ it is not necessarily tangentially of order 2 at this point.

For a smooth curve γ on M parameterized by its arclength, we set $k_\gamma = \|\widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma\|$ and call it the firs geodesic curvature of its extrinsic shape. We say a tangent vector $v \in TM$ to be principal if it is an eigenvector of the shape operator A_M. In this case its eigenvalue is said to be its principal curvature. A real hypersurface is said to be *Hopf* if its characteristic vector fiel ξ is principal at each point. For a Hopf hypersurface M we denote by $\delta_M(p)$ the principal curvature of ξ_p at $p \in M$.

For a trajectory γ for $\mathbb{F}_\kappa$ we fin

$$k_\gamma^2 = \kappa^2(1-\rho_\gamma^2) + \langle A_M\dot\gamma, \dot\gamma\rangle^2 \tag{II.3}$$

by (II.1). Thus, by use of (II.2) we see that the extrinsic shape of is tangentially of order 2 at $\gamma(t_0)$ if and only if it satisfie

$$\big(k_\gamma(t_0)^2 - \kappa^2\big)\dot\gamma(0) = \big(\langle A_M\dot\gamma(0), \dot\gamma(0)\rangle - \kappa\rho_\gamma(t_0)\big)\big(A_M\dot\gamma(0) + \kappa\xi_{\gamma(t_0)}\big). \tag{II.4}$$

When M is a Hopf hypersurface, by decomposing the above equality into the components parallel to $\xi_{\gamma(t_0)}$ and parallel to $\dot\gamma(0) - \rho(t_0)\xi_{\gamma(0)}$, we have

$$\big\{k_\gamma(t_0)^2 - \kappa^2\big\}\rho(t_0) = \big(\langle A_M\dot\gamma(0), \dot\gamma(0)\rangle - \kappa\rho_\gamma(t_0)\big)\big(\rho_\gamma(t_0)\delta_M + \kappa\big), \tag{II.5}$$

$$\big\{k_\gamma(t_0)^2 - \kappa^2\big\}\big(\dot\gamma(t_0) - \rho_\gamma(t_0)\xi_{\gamma(t_0)}\big) = \big(\langle A_M\dot\gamma(0), \dot\gamma(0)\rangle - \kappa\rho_\gamma(t_0)\big) A_M\big(\dot\gamma(t_0) - \rho_\gamma(t_0)\xi_{\gamma(t_0)}\big). \tag{II.6}$$

By taking into account of (II.3) we obtain the following.

Lemma 1 ([6]): Let γ be a trajectory for a Sasakian magnetic fiel $\mathbb{F}_\kappa$ on a Hopf hypersurface M in $\mathbb{C}H^n(c)$.

1) If $\rho_\gamma(t_0) = \pm 1$, then the extrinsic shape of γ is tangentially of order 2 at $\gamma(t_0)$ and has $k_\gamma(t_0) = \big|\delta_M\big(\gamma(t_0)\big)\big|$.
2) If $\rho_\gamma(t_0) \neq \pm 1$ and the vector $\dot\gamma(t_0) - \rho_\gamma(t_0)\xi_{\gamma(t_0)}$ is principal, then the extrinsic shape of γ is tangentially of order 2 at $\gamma(t_0)$ if and only if one of the following conditions folds with the principal curvature of λ of $\dot\gamma(t_0) - \rho_\gamma(t_0)\xi_{\gamma(t_0)}$:
   i) $\lambda - \kappa\rho_\gamma(t_0) + \big\{\delta_M\big(\gamma(t_0)\big) - \lambda\big\}\rho_\gamma(t_0)^2 = 0$,
   ii) $\kappa + \big\{\delta_M\big(\gamma(t_0)\big) - \lambda\big\}\rho_\gamma(t_0) = 0$.

   In the former case we have $k_\gamma(t_0) = |\kappa|$ and in the latte case $k_\gamma(t_0)^2 = \kappa^2 - 2\kappa\lambda\rho_\gamma(t_0) + \lambda^2$.
3) Under the condition that $k_\gamma(t_0) \neq |\kappa|$, if $\dot\gamma(t_0) - \rho_\gamma(t_0)\xi_{\gamma(t_0)}$ is principal, then the extrinsic shape of γ is not tangentially of order 2.

III. Characterizations of some real hypersurfaces

We give some characterizations on homogeneous Hopf hypersurfaces in $\mathbb{C}H^n(c)$. To study characteristic vector fiel of a real hypersurface M, we consider the following condition at $p \in M$:

(TC) The extrinsic shape of a trajectory γ_0 for some Sasakian magnetic fiel $\mathbb{F}_{\kappa_0}$ of initial vector ξ_p is tangentially of order 2 at p and satisfie $k_{\gamma_0}(0) \neq |\kappa_0|$.

When the condition (TC) holds, the equalities (II.3), (II.4) turn to

$$k_{\gamma_0}(0)^2 = \kappa_0^2 + \langle A_M\xi_p, \xi_p\rangle^2,$$
$$\big(k_{\gamma_0}(0) - \kappa_0\big)\xi_p = \big(\langle A_M\xi_p, \xi_p\rangle - \kappa_0\big)\big(A_M\xi_p + \kappa_0\xi_p\big),$$

we can characterize Hopf hypersurfaces by the condition (TC).

Proposition 1: A real hypersurface M in $\mathbb{C}H^n(c)$ is Hopf if and only if the condition (TC) holds at each point $p \in M$.

For every Hopf hypersurface, it is known that the principal curvature δ_M associated with its characteristic vector fiel is locally constant. It is also well-known that homogeneous Hopf hypersurfaces are classifie into f ve classes (see [10], for example). They are

1) a horosphere HS,
2) a geodesic sphere $G(r)$ of radius r,
3) a tube $T(r)$ of radius r around totally geodesic $\mathbb{C}H^{n-1}(c)$,

4) a tube $T_\ell(r)$ of radius r around totally geodesic $\mathbb{C}H^\ell(c)$ with $1 \le \ell \le n-2$,
5) a tube $R(r)$ of radius r around totally real and totally geodesic $\mathbb{R}H^n(c/4)$,

where $0 < r < \infty$. Homogeneous real hypersurfaces except those which are congruent to some $R(r)$ are called real hypersurface of type (A), and tubes around $\mathbb{R}H^n(c/4)$ are called real hypersurfaces of type (B). The principal curvature corresponding to ξ is given as

$$\delta_M = \begin{cases} \sqrt{|c|}, & \text{when } M = HS, \\ \sqrt{|c|}\coth\sqrt{|c|}r, & \text{when } M = G(r), T(r), T_\ell(r), \\ \sqrt{|c|}\tanh\sqrt{|c|}r, & \text{when } M = R(r). \end{cases}$$

When M is one of $HS, G(r)$ and $T(r)$, all tangent vector orthogonal to ξ are principal. Their principal curvatures are given as

$$\lambda_M = \begin{cases} \sqrt{|c|}/2, & \text{when } M = HS, \\ \big(\sqrt{|c|}/2\big)\coth\big(\sqrt{|c|}r/2\big), & \text{when } M = G(r), \\ \big(\sqrt{|c|}/2\big)\tanh\big(\sqrt{|c|}r/2\big), & \text{when } M = T(r). \end{cases}$$

When M is either $T_\ell(r)$ or $R(r)$, the bundle T^0M splits into two subbundles $V_M^{(1)} \oplus V_M^{(2)}$ which consist of principal curvature vectors associated to

$$\lambda_M^{(1)} = \big(\sqrt{|c|}/2\big)\coth\big(\sqrt{|c|}r/2\big),$$
$$\lambda_M^{(2)} = \big(\sqrt{|c|}/2\big)\tanh\big(\sqrt{|c|}r/2\big).$$

First we characterize homogeneous Hopf hypersurfaces $HS, G(r), T(r)$ which have two principal curvatures by extrinsic shapes of trajectories. These real hypersurfaces are said to be totally η-umbilic. Given constants κ, ρ with $|\rho| < 1$, we consider the following conditions at $p \in M$:

($\mathbf{T}_{\kappa,\rho}$) There exist linearly independent unit tangent vectors $v_1, \ldots, v_{2n-2} \in U_pM$ with $\langle v_i, \xi_p\rangle = \rho$ which satisfy that the extrinsic shapes of trajectories γ_i $(i = 1, \ldots, 2n-2)$ for $\mathbb{F}_\kappa$ with initial vector v_i are tangentially of order 2 and satisfy $k_{\gamma_i}(0) \neq |\kappa|$.

Theorem 1 ([6]): A connected real hypersurface M in $\mathbb{C}H^n(c)$ is locally congruent to a totally η-umbilic real hypersurface if and only if it satisfie the following two conditions at each point $p \in M$:

i) The condition (TC) holds at p;
ii) There exist constants κ_p, ρ_p with $\kappa_p \neq 0$ and $|\rho_p| < 1$ such that the condition ($\mathrm{T}_{\kappa_p,\rho_p}$) holds at p.

We note that the constants κ_p, ρ_p may depend on p. As we can easily check that real hypersurfaces $HS, G(r), T(r)$ satisfy these conditions, we need to show the converse. The firs condition shows that M is a Hopf hypersurface. Thus the second condition guarantees that $v_i - \rho_p\xi_p$ is principal. Denoting its principal curvature by α_i we obtain

$$k_{\gamma_i}(0)^2 - \kappa_p^2 = \alpha_i\big\{\alpha_i(1-\rho_p^2) + \delta_M\rho_p^2 - \kappa_p\rho_p\big\}$$

by (II.6). As $k_{\gamma_i}(0) \neq |\kappa_p|$, we fin $\kappa = (\alpha_i - \delta_M)\rho_p$ by Lemma 1. Since $\kappa \neq 0$ we obtain $\alpha_i = \delta_M + (\kappa_p/\rho_p)$. This shows M is totally η-umbilic.

In the argument of the above characterization on totally η-umbilic real hypersurfaces, the assumption $\kappa_p \neq 0$ is important. When $\kappa = 0$, we have

$$k_{\gamma_i}(0)^2\rho_{\gamma_i}(0) = \big\{\alpha_i\big(1-\rho_{\gamma_i}(0)^2\big) + \delta_M\rho_{\gamma_i}(0)^2\big\}\rho_{\gamma_i}(0)\delta_M,$$
$$k_{\gamma_i}(0)^2 = \big\{\alpha_i\big(1-\rho_{\gamma_i}(0)^2\big) + \delta_M\rho_{\gamma_i}(0)^2\big\}\alpha_i$$

by (II.5) and (II.6). These show that either $\alpha_i = \delta_M$ or $\rho_{\gamma_i}(0) = 0$. We therefore have the following.

Theorem 2 ([6]): A connected real hypersurface M in $\mathbb{C}H^n(c)$ is locally congruent to either a totally η-umbilic real hypersurface or a tube $R(r_0)$ with $r_0 = \big(1/\sqrt{|c|}\,\big)\log\big((\sqrt{3}+1)/(\sqrt{3}-1)\big)$ if and only if it satisfie the following two conditions at each point $p \in M$:

i) The condition (TC) holds at p;
ii) There exist linearly independent unit tangent vectors $v_1, \ldots, v_{2n-2} \in U_pM$ with $v_i \neq \pm\xi_p$ such that the extrinsic shape of geodesic γ_i with initial vector v_i is tangentially of order 2 at p, and satisfie $k_{\gamma_i}(0) > 0$, and moreover these geodesic curvatures $k_{\gamma_1}(0), \ldots, k_{\gamma_{2n-2}}(0)$ are same for j with $v_j \perp \xi_p$.

If we pose a condition that extrinsic shapes of trajectories have common geodesic curvatures we can eliminate the tube $R(r_0)$ around $\mathbb{R}H^{2n-2}(c/4)$. Given constants κ, k with $k \ge 0$, we consider the following condition:

($\mathbf{C}_{\kappa,k}$) There exist linearly independent unit tangent vectors $v_1, \ldots, v_{2n-2} \in U_pM$ with $v_i \neq \pm\xi_p$ which satisfy that the extrinsic shapes of trajectories γ_i $(i = 1, \ldots, 2n-2)$ for $\mathbb{F}_\kappa$ with initial vector v_i are tangentially of order 2 and satisfy $k_{\gamma_i}(0) = k$ for all i.

Theorem 3 ([6]): A connected real hypersurface M in $\mathbb{C}H^n(c)$ is locally congruent to a totally η-umbilic real hypersurface if and only if it satisfie the following two conditions at each point $p \in M$:

i) The condition (TC) holds at p;
ii) There exist constants κ_p and k_p with $k \neq |\kappa_p|$ such that the condition ($\mathrm{C}_{\kappa_p,k_p}$) holds at p.

Coming back to extrinsic shapes of non-geodesic trajectories, we can characterize tubes around totally geodesic complex hypersurfaces by posing a condition on geodesic curvatures of extrinsic shapes. We here extend some results in [7] where we give characterizations by using extrinsic circular trajectories.

Proposition 2 (cf. [7]): A connected real hypersurface M in $\mathbb{C}H^n(c)$ is locally congruent to a tube around $\mathbb{C}H^{n-1}$ if and only if it satisfie the following two conditions at each point $p \in M$:

i) The condition (TC) holds at p;
ii) There exist constants κ_p, ρ_p with $\kappa_p \neq 0$ and $|\rho_p| < 1$ such that the condition ($\mathrm{T}_{\kappa_p,\rho_p}$) holds at p and geodesic curvatures of extrinsic shapes of trajectories in this condition satisfy $k_{\gamma_i}(0) < \sqrt{|c|}/2$ for all i.

Proposition 3 (cf. [7]): A connected real hypersurface M in $\mathbb{C}H^n(c)$ is locally congruent to a tube around $\mathbb{C}H^{n-1}$ if and only if it satisfie the following two conditions at each point $p \in M$:

i) The condition (TC) holds at p;
ii) There exist constants κ_p and k_p with $k_p \neq |\kappa_p|$ and $0 < k_p < \sqrt{|c|}/2$ such that the condition $(\mathrm{C}_{\kappa_p,k_p})$ holds at p.

We can show these propositions by checking geodesic curvatures of extrinsic shapes of extrinsic circular trajectories. Similarly, if we study extrinsic shapes of geodesics we can characterize horospheres.

Proposition 4 (cf. [7]): A connected real hypersurface M in $\mathbb{C}H^n(c)$ is locally congruent to a horosphere *HS* if and only if it satisfie the following two conditions at each point $p \in M$:

i) The condition (TC) holds at p;
ii) There exists a constant k_p with $k_p > 0$ such that the condition (C_{0,k_p}) holds at p. Moreover, there is a point p_0 with $k_{p_0} = \sqrt{|c|}/2$.

Unfortunately, we can not characterize all geodesic spheres by a property that extrinsic shapes of some trajectories are tangentially of order 2, though we have some characterizations of some class of geodesic spheres by a property on extrinsic shapes of geodesics (see [8] and [7]). This is because every bounded circle on $\mathbb{C}H^n$ can be obtained as the extrinsic shapes of a trajectory on some geodesic sphere and of a trajectory on some tube around totally geodesic complex hypersurface (see [2]).

For a smooth curve $\tilde{\gamma}$ parameterized by its arclength on $\mathbb{C}H^n(c)$ we have another index functions when it has positive firs geodesic curvature function $\|\nabla_{\dot{\tilde{\gamma}}}\dot{\tilde{\gamma}}\|$. We set $\tau = \langle\dot{\tilde{\gamma}}, J\nabla_{\dot{\tilde{\gamma}}}\dot{\tilde{\gamma}}\rangle/\|\nabla_{\dot{\tilde{\gamma}}}\dot{\tilde{\gamma}}\|$ and call it the firs complex torsion. For a trajectory γ for $\mathbb{F}_\kappa$ on a real hypersurface M in $\mathbb{C}H^n(c)$, we fin by (II.1) that the firs complex torsion of its extrinsic shape is given as

$$\tau_\gamma = -\{\kappa(1-\rho_\gamma^2) + \langle A_M\dot{\gamma}, \dot{\gamma}\rangle\rho_\gamma\}/k_\gamma.$$

We here pose a condition on geodesic curvatures and firs complex torsions of extrinsic shapes of trajectories. Given k, τ satisfying $k > 0$ and $|\tau| \leq 1$, we consider the following condition on a unit tangent vector $v \in T_pM$:

$(\mathbf{K}_{k,\tau})$ There exists a Sasakian magnetic fiel $\mathbb{F}_\kappa$ with $\kappa \neq \pm k$ such that the extrinsic shape of the trajectories γ for $\mathbb{F}_\kappa$ with initial vector v is tangentially of order 2 at p and satisfy $k_\gamma(0) = k,\ \tau_\gamma(0) = \tau$.

Theorem 4 ([3]): A connected real hypersurface M in $\mathbb{C}H^n(c)$ is locally congruent to a totally η-umbilic real hypersurface if and only if it satisfie the following conditions:

i) The condition (TC) holds at each point p;
ii) There exist positive constants k, τ with $\tau < 1$ such that at each point $p \in M$ we can choose linearly independent unit tangent vectors $v_1, \ldots, v_{2n-2} \in U_pM$ satisfying
  $a)$ the condition $(\mathrm{K}_{k,\tau})$ holds;.
  $b)$ their components $v_i - \langle v_i, \xi_p\rangle\xi_p$ $(i = 1, \ldots, 2n-2)$ span the subspace T_p^0M orthogonal to ξ_p.

Theorem 5 ([3]): A connected real hypersurface M in $\mathbb{C}H^n(c)$ is locally congruent to a tube around totally geodesic $\mathbb{C}H^\ell(c)$ with some ℓ $(1 \leq \ell \leq n-2)$ if and only if it satisfie the following conditions:

i) The condition (TC) holds at each point p;
ii) There exist positive constants k, τ_1, τ_2 with $k > \sqrt{|c|}$ and $\tau_1 < \tau_2 < 1$ such that at each point $p \in M$ we can choose linearly independent unit tangent vectors $v_1, \ldots, v_{2n-2} \in U_pM$ satisfying
  $a)$ either the condition (K_{k,τ_1}) or the condition (K_{k,τ_2}) holds;.
  $b)$ their components $v_i - \langle v_i, \xi_p\rangle\xi_p$ $(i = 1, \ldots, 2n-2)$ span the subspace T_p^0M orthogonal to ξ_p.

  Moreover, there is a point p_0 such that not all of such $v_1, \ldots, v_{2n-2}$ satisfy one of the conditions (K_{k,τ_1}) and (K_{k,τ_2}).

We should note that in these theorems we need to choose k, τ or k, τ_1, τ_2 so that they do not depend on $p \in M$. Also, we note that those trajectories satisfying the conditions in each theorems and propositions are extrinsic circular. The authors are interested in giving some characterizations of tubes around totally geodesic $\mathbb{R}H^n$ by properties of extrinsic shapes of trajectories.

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