# Doubly Warped Product Manifolds With Respect to a Semi-symmetric Non-Metric Connection

Sibel Sular

***Abstract*—**The main goal of the present paper is to get new equalities with respect to a special connection titled ''Semi-symmetric non-metric connection''. Whoever wants to study manifol or submanifold theory with this type of special connection, our new equalities can be used in the future studies. In this study, we calculate Koszul formulas on doubly warped product manifolds endowed with the semi-symmetric non-metric connection and we give relations between Levi-Civita connection and semi-symmetric non-metric connection of a doubly warped product manifold $M={}_f B\times_b F$. We also get some results of Einstein doubly warped product manifolds with a semi-symmetric non-metric connection.

***Keywords*—**Doubly warped product manifold, semi-symmetric non-metric connection, Einstein manifold, quasi-Einstein manifold.

## I. INTRODUCTION

IN [6], H. A. Hayden gave the definition of a semi-symmetric metric. In 1970, K. Yano studied semi-symmetric metric connection and he proved that a Riemannian manifold admitting the semi-symmetric metric connection has vanishing curvature tensor if and only if it is conformally flat [15].

On the other hand, the idea of a semi-symmetric non-metric connection was introduced by N. S. Agashe and M. R. Chafle in [1]. They also studied some of its properties and submanifolds of a Riemannian manifold with semi-symmetric non-metric connections in [2].

The above author and C. Özgür studied semi-symmetric non-metric connection on a warped product manifold and gave relations between the Levi-Civita connection and the semi-symmetric non-metric connection of a warped product manifold in [11]. They also considered Einstein warped product manifolds endowed with a semi-symmetric non-metric connection.

Furthermore, the present author considered semi-symmetric metric connection on a doubly warped product manifolds in [13]. The curvature tensor, Ricci tensor and the scalar curvature were obtained respectively.

By the above studies, we study doubly warped product manifolds with semi-symmetric non-metric connection and find relations between the Levi-Civita connection and the semi-symmetric non-metric connection.

Moreover, in [5], A. Gebarowski studied Einstein warped product manifolds. As an application, in this study we consider Einstein doubly warped product manifolds endowed with a semi-symmetric non-metric connection.

There are also various studies on doubly warped product manifolds as [4], [10], [12]. We have examined these studies and have comparisons of the features of doubly warped product manifolds with Levi-Civita connection and semi-symmetric non-metric connection.

## II. SEMI-SYMMETRIC NON-METRIC CONNECTION

Let $\nabla$ be a Levi-Civita connection of an n-dimensional Riemannian manifold M. A linear connection ${}^{\circ}\nabla$ is defined by

$$ {}^{\circ}\nabla_X Y = \nabla_X Y + \pi(Y)X \tag{1}$$

where $\pi$ is a 1-form associated with the vector field P on M defined by

$$\pi(X) = g(X,P) \qquad \text{and} \qquad P = P_B + P_F , \tag{2}$$

where $P_B$ (resp. $P_F$) is the component of P on B (resp. on F).

${}^{\circ}\nabla$ is said to be a *semi-symmetric connection,* if its torsion tensor T

$$T(X,Y) = {}^{\circ}\nabla_X Y - {}^{\circ}\nabla_Y X - [X,Y] \tag{3}$$

satisfies

$$T(X,Y) = \pi(Y)X - \pi(X)Y, \tag{4}$$

A semi-symmetric connection ${}^{\circ}\nabla$ is said to be a *semi-symmetric non-metric connection* if

$${}^{\circ}\nabla g \neq 0.$$

From (1) it is easy to see that

$$\begin{aligned} {}^{\circ}\nabla_X g(Y, Z) &= ({}^{\circ}\nabla_X g)(Y, Z)+g({}^{\circ}\nabla_X Y, Z)+g(Y, {}^{\circ}\nabla_X Z) \\ &= ({}^{\circ}\nabla_X g)(Y, Z)+ \nabla_X g(Y, Z) \\ &\quad + \pi(Y)g(X, Z)+ \pi(Z)g(X, Y), \end{aligned}$$

which means that

$$({}^{\circ}\nabla_X g)(Y, Z) = - \pi(Y)g(X, Z)+ \pi(Z)g(X, Y), \tag{5}$$

for any vector fields X, Y, Z on M.

Furthermore, by the use of (1), we can write the relation between R and ${}^{\circ}R$ as follows

$$^{\circ}R(X,Y)Z = R(X,Y)Z+g(Z,\nabla_X P)Y- g(Z,\nabla_Y P) +\pi(Z)[\pi(Y)X - \pi(X)Y], \quad (6)$$

for any vector fields X,Y,Z on M [16].

## III. DOUBLY WARPED PRODUCT MANIFOLDS

Let $(B, g_B)$ and $(F, g_F)$ be two Riemannian manifolds and $b:B\rightarrow(0,\infty)$ and $f:F\rightarrow(0,\infty)$ smooth functions. Consider the product manifold $B\times F$ with its projections $\pi:B\times F\rightarrow B$ and $\sigma:B\times F\rightarrow F$. The *doubly warped product* $_f B\times_b F$ is the manifold $B\times F$ with the Riemannian structure such that

$$g = (f\circ \sigma)^2\pi_* (g_B)\oplus(b\circ \pi)^2\sigma_* (g_F),$$

which implies that

$$g = f^2 g_B + b^2 g_F \quad (7)$$

The functions $b:B\rightarrow(0,\infty)$ and $f:F\rightarrow(0,\infty)$ are called *warping functions* of the doubly warped product [9].

If either $b \equiv 1$ or $f \equiv 1$, but not both then we obtain a *singly warped product*. If both $b \equiv 1$ and $f \equiv 1$, then we have a product manifold. If neither b nor f is constant, then we have a *non-trivial doubly warped product.*

We need the following three lemmas from [9], for the later use :

**Lemma 3.1:** Let us consider $M={}_f B\times_b F$ and denote by $\nabla$, $^B\nabla$ and $^F\nabla$ the Riemannian connections on M, B and F, respectively. If X,Y are vector fields on B and V,W on F, then:

**(i)** $\nabla_X Y = {}^B\nabla_X Y - (1/(fb^2))g(X,Y)\mathrm{grad}_F f$,

**(ii)** $\nabla_X V = \nabla_V X = ((V(f))/f)X+((X(b))/b)V$,

**(iii)** $\nabla_V W = {}^F\nabla_X Y - (1/(bf^2))g(V,W)\mathrm{grad}_B b$.

**Lemma 3.2:** Let $M={}_f B\times_b F$ be a doubly warped product, with Riemannian curvature $^MR$. Given fields X,Y,Z on B and U,V,W on F, then:

**(i)** $^MR(X,Y)Z = {}^BR(X,Y)Z +(1/(fb^3))[g(Y,Z)X(b) - g(X,Z)Y(b)]\, \mathrm{grad}_F f - (1/(b^2))[g_B(Y,Z)X - g_B(X,Z)Y](\mathrm{grad}_F f)(f)$,

**(ii)** $^MR(X,V)Y = ((H_B{}^b(X,Y))/b)V- ((H^{f\circ\sigma}(Y,V))/f)X +((g_B(X,Y))/b)[((f^F)/b)\nabla_V \mathrm{grad}_F f-((V(f))/f)\mathrm{grad}_B b]$,

**(iii)** $^MR(X,Y)V = ((H^{f\circ\sigma}(Y,V))/f)X+((H^{f\circ\sigma}(X,V))/f)Y$,

**(iv)** $^MR(V,W)X =-((H^{b\circ\pi}(X,W))/b)V+((H^{b\circ\pi}(X,V))/b)W$,

**(v)** $^MR(X,V)W = -((H_F{}^f(V,W))/f)X+((H^{b\circ\pi}(X,W))/b)V - ((g_F(V,W))/f)[((b^B)/f)\nabla_X \mathrm{grad}_B b-((X(b))/b)\mathrm{grad}\, f]$

**(vi)** $^MR(V,W)U = {}^FR(V,W)U +(1/(bf^3))[g(V,W)U(f)-g(U,W)V(f)]\mathrm{grad}_B b -(1/(f^2))[g_F(V,W)U-g_F(U,W)V](\mathrm{grad}_B b)(b)$.

**Lemma 3.3:** Let $M={}_f B\times_b F$ be a doubly warped product with Ricci tensor $^MS$. Given fields X,Y on B and V,W on F, then:

**(i)** $^MS(X,Y) = {}^BS(X,Y) -(1/(b^2))[(r-1)(\mathrm{grad}_F f)(f)+f\Delta_F(f)]g_B(X,Y) -(s/b)H_B{}^b(X,Y)$,

where r = dimB and s = dimF,

**(ii)** $^MS(X,V) = (n-2)((V(f)X(b))/(fb))$,

**(iii)** $^MS(V,W) = {}^FS(V,W) -(1/(f^2))[(s-1)(\mathrm{grad}_B b)(b)+b\Delta_B(b)]g_F(V,W) -(r/f)H_F{}^f(V,W)$.

Moreover, the scalar curvature $^M\tau$ of M satisfies the condition

$$^M\tau = ({}^B\tau)/(b^2)+({}^B\tau)/(f^2) - 2s((\Delta_B(b))/(bf^2))-2r((\Delta_F(f))/(fb^2)) - s(s-1)(((\mathrm{grad}_B b)(b))/(f^2b^2)) - r(r-1)(((\mathrm{grad}_F f)(f))/(f^2b^2)), \quad (7)$$

where $^B\tau$ and $^F\tau$ are scalar curvatures of B and F, respectively.

## IV. DOUBLY WARPED PRODUCT MANIFOLDS WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

In this section, we consider doubly warped product manifolds with respect to the semi-symmetric non-metric connection and find the expressions of curvature tensors, Ricci tensors and scalar curvatures with this connection where $P\in\chi(M)$.

Let's begin with the following lemma:

**Lemma 4.1:** Let us consider $M={}_f B\times_b F$ and denote by $\nabla$ the semi-symmetric non-metric connection on M, $^M\nabla$ and $^F\nabla$ be connections on B and F, respectively. If $X,Y\in\chi(B)$, $V,W\in\chi(F)$, then:

**(i)** $^{\circ}\nabla_X Y = {}^{B\circ}\nabla_X Y - (1/(fb^2))g(X,Y)(grad_F f)(f)$,

**(ii)** $^{\circ}\nabla_X V = ((V(f))/f)X+((X(b))/b)V+\pi(V)X$,

**(iii)** $^{\circ}\nabla_V X = ((V(f))/f)X+((X(b))/b)V+\pi(X)V$,

**(iv)** $^{\circ}\nabla_V W = {}^{F\circ}\nabla_X Y-(1/(bf^2))g(V,W)(grad_B b)(b)$.

**Proof :** In view of Koszul formula from [9] we have

$$2g(\nabla_X Y,Z) = Xg(Y,Z)+Yg(X,Z) - Zg(X,Y) - g(X,[Y,Z]) - g(Y,[X,Z])+g(Z,[X,Y]), \quad (9)$$

for all vector fields X,Y,Z on M, where $\nabla$ is the Levi-Civita connection of M. By virtue of (1), the equation (9) turns into

$$2g(^{\circ}\nabla_X Y,V) = Xg(Y,V)+Yg(X,V) -Vg(X,Y) - g(X,[Y,V]) - g(Y,[X,V])+g(V,[X,Y]) +2\pi(Y)g(X,V), \quad (10)$$

for any vector fields $X,Y\in\chi(B)$ and $V\in\chi(F)$.

Since X,Y and [X,Y] are lifts from B and V is vertical, we know from [9] we can write

$$g(Y,V)=g(X,V)=0 \quad (11)$$

and

$$[X,V]=[Y,V]=0. \quad (12)$$

Thus, (10) reduces to

$$2g(^{\circ}\nabla_X Y,V) = -Vg(X,Y) \quad (13)$$

By the use of(7), we have

$$g(X,Y) = (f\circ\sigma)^2 g_B(X,Y).$$

Taking $f = f\circ\sigma$, we can write

$$g(X,Y) = f^2(g_B(X,Y)\circ\pi).$$

Thus, we get

$$Vg(X,Y) = V[f^2(g_B(X,Y)\circ\pi)] = 2fV(f)(g_B(X,Y)\circ\pi)+f^2V(g_B(X,Y)\circ\pi).$$

Since $(g_B(X,Y)\circ\pi)$ is constant on fibers, by the use of (7), the last equation turns into

$$Vg(X,Y) = 2((V(f))/f)g(X,Y). \quad (14)$$

By virtue use of (14) in (13), we obtain

$$g(^{\circ}\nabla_X Y,V) = -[((V(f))/f)+\pi(V)]g(X,Y). \quad (15)$$

Since $V(f) = (1/(b^2))g(grad_F f,V)$ on F, we get (i).

By the definition of the covariant derivative with respect to the semi-symmetric non-metric connection, we can write

$$g(^{\circ}\nabla_X V,Y) = Xg(Y,V) - (^{\circ}\nabla_X g)(V, Y) - g(V, {}^{\circ}\nabla_X Y),$$

for all vector fields X,Y on B and V on F. By making use of (5), (11) and (15), the above equation turns into

$$g(^{\circ}\nabla_X V,Y) = [((V(f))/f)+\pi(V)]g(X,Y). \quad (16)$$

On the other hand, from Koszul formula with respect to the semi-symmetric non-metric connection it follows that

$$2g(^{\circ}\nabla_X V,W) = Xg(V,W)+Vg(X,W) - Wg(X,V) - g(X,[V,W]) - g(V,[X,W])+g(W,[X,V]) +2\pi(V)g(X,W),$$

for any vector fields X on B and V,W on F. In view of (11) and (12), the last equation reduces to

$$2g(^{\circ}\nabla_X V,W) = Xg(V,W) - g(X,[V,W]).$$

Since X is horizontal and [V,W] is vertical, g(X,[V,W])=0. Then we find

$$2g(^{\circ}\nabla_X V,W) = Xg(V,W). \quad (17)$$

In view of (7), we have

$$g(V,W) = (b\circ\pi)^2 g_F(V,W).$$

Taking $b = b\circ\pi$, we get

$$g(V,W) = b^2(g_F(V,W)\circ\sigma).$$

So, we obtain

$$Xg(V,W) = X[b^2(g_F(V,W)\circ\sigma)] = 2bX(b)(g_F(V,W)\circ\sigma)+b^2X(g_F(V,W)\circ\sigma).$$

Since $(g_F(V,W)\circ\sigma)$ is constant on leaves, by the use of (7), the last equation reduces to

$$Xg(V,W) = 2((X(b))/b)g(V,W). \quad (18)$$

By making use of (18) in (17), we obtain

$$g(\overset{\circ}{\nabla}_X V,W) = ((X(b))/b)g(V,W). \quad (19)$$

Then in view of the equations (16) and (19), we get (ii).

Now, by the use of (3) we can write

$$\overset{\circ}{\nabla}_V X = \overset{\circ}{\nabla}_X V - [X,V] - T(X,V).$$

Then, from (4) and (12), the above equation turns into

$$\overset{\circ}{\nabla}_V X = \overset{\circ}{\nabla}_V V - \pi(V)X + \pi(X)V. \quad (20)$$

By virtue of the equation (ii), we get

$$\overset{\circ}{\nabla}_V X = ((X(b))/b)V + ((V(f))/f)X + \pi(X)V. \quad (21)$$

Hence we obtain (iii).

On the other hand, by the definition of the covariant derivative endowed with the semi-symmetric non-metric connection, it follows that

$$Vg(X,W) = (\overset{\circ}{\nabla}_V g)(X, W) + g(\overset{\circ}{\nabla}_V X,W) + g(\overset{\circ}{\nabla}_V W,X),$$

for any vector fields X on B and V,W on F. From (5) and (11), the above equation turns into

$$g(\overset{\circ}{\nabla}_V W,X) = \pi(X)g(V,W) - g(\overset{\circ}{\nabla}_V X,W). \quad (22)$$

In view of (21), we get

$$g(\overset{\circ}{\nabla}_V W,X) = -((X(b))/b)g(V,W),$$

which means that

$$\overset{\circ}{\nabla}_V W = {}^{F}\overset{\circ}{\nabla}_V W - (1/(bf^2))g(V,W)\operatorname{grad}_B b\ ,$$

where $X(b) = (1/(f^2))g(\operatorname{grad}_B b,X)$ for any vector field X on B. Therefore, we complete the proof of the lemma.

**Lemma 4.2 :** Let $M={}_f B\times_b F$ be a doubly warped product and R and $\overset{\circ}{R}$ denote the Riemannian curvature tensors of M with respect to the Levi-Civita connection and the semi-symmetric non-metric connection, respectively . If $X,Y,Z \in \chi(B)$ and $U,V,W\in \chi(F)$, then:

**(i)** $${}^{B\circ}R(X,Y)Z = {}^{B}R(X,Y)Z - (1/(b^2))[g_B(Y,Z)X-g_B(X,Z)Y](\operatorname{grad}_F f)(f) + g(Z,{}^{B}\nabla_X P_B)Y - g(Z,{}^{B}\nabla_Y P_B)X + ((P_F(f))/f)[g(X,Z)Y-g(Y,Z)X] + \pi(Z)[\pi(Y)X-\pi(X)Y],$$

**(ii)** $${}^{F\circ}R(X,Y)Z = (1/(fb^2))[g(Y,Z)((X(b))/b) - g(X,Z)((Y(b))/b)]\operatorname{grad}_F f\ ,$$

**(iii)** $${}^{B\circ}R(V,X)Y = ((H^{f\circ\sigma}(Y,V))/f)X + ((V(f))/(fb))g_B(X,Y)\operatorname{grad}_B b + ((V(f))/f)\pi(Y)X - ((Y(b))/b)\pi(V)X - \pi(Y)\pi(V)X,$$

**(iv)** $${}^{F\circ}R(V,X)Y = ((H_B{}^{b}(X,Y))/b)V - (f/(b^2))g_B(X,Y){}^{F}\nabla_V \operatorname{grad}_F f - g(Y,{}^{B}\nabla_X P_B)V - ((P_F(f))/f)\, g(X, Y)V + \pi(X)\pi(Y)V,$$

**(v)** $${}^{B\circ}R(X,Y)V = -((H^{f\circ\sigma}(Y,V))/f)X + ((H^{f\circ\sigma}(X,V))/f)Y - ((V(f))/f)[\pi(X)Y-\pi(Y)X] + [((X(b))/b)Y-((Y(b))/b)X]\pi(V) - [\pi(X)Y-\pi(Y)X]\pi(V),$$

**(vi)** $${}^{F\circ}R(X,Y)V=0,$$

**(vii)** $${}^{B\circ}R(V,W)X=0,$$

**(viii)** $${}^{F\circ}R(V,W)X=((H^{b\circ\pi}(X,W))/b)V+((H^{b\circ\pi}(X,V))/b)W - ((X(b))/b)[\pi(V)W-\pi(W)V] + \pi(X)[((V(f))/f)W-((W(f))/f)V] - \pi(X)[\pi(V)W-\pi(W)V],$$

**(ix)** $${}^{B\circ}R(X,V)W = -((H_F{}^{f}(V,W))/f)X - (b/(f^2))g_F(V,W){}^{B}\nabla_X \operatorname{grad}_B b - ((P_B(b))/b)g(V,W)X - g(W,{}^{F}\nabla_V P_F)X + \pi(V)\pi(W)X,$$

**(x)** $${}^{F\circ}R(X,V)W = ((H^{b\circ\pi}(X,W))/b)V + ((X(b))/(bf))g_F(V,W)\operatorname{grad}_F f - ((W(f))/f)\pi(X)V + ((X(b))/b)\pi(W)V - \pi(X)\pi(W)V,$$

**(xi)** $${}^{B\circ}R(U,V)W = (1/(bf^2))[g(V,W)((U(f))/f) - g(U,W)((V(f))/f)]\operatorname{grad}_B b,$$

**(xii)** $${}^{F\circ}R(U,V)W = {}^{F}R(U,V)W - (1/(f^2))[g_F(V,W)U - g_F(U,W)V](\operatorname{grad}_B b)(b) + g(W,{}^{F}\nabla_U P_F)V - g(W,{}^{F}\nabla_V P_F)U + ((P_B(b))/b)[g(U,W)V-g(V,W)U] + [\pi(V)U-\pi(U)V]\pi(W).$$

**Proof :** Let $M={}_f B\times_b F$ be a doubly warped product and R and $\overset{\circ}{R}$ denote the curvature tensors of the Levi-Civita connection and the semi-symmetric non-metric connection, respectively. In view of (6), we can write

$$\overset{\circ}{R}(X,Y)Z = R(X,Y)Z+g(Z,\nabla_X P)Y-g(Z,\nabla_Y P)X +\pi(Z)[\pi(Y)X-\pi(X)Y],$$

for any vector fields X,Y,Z on B.
In view of the equation (2), Lemma 3.1 and Lemma 3.2, we get

$$\begin{aligned}\overset{\circ}{R}(X,Y)Z = {}& {}^B R(X,Y)Z \\ &+(1/(fb^2))[g(Y,Z)((X(b))/b) \\ &-g(X,Z)((Y(b))/b)]grad_F f \\ &- (1/(b^2))[g_B(Y,Z)X-g_B(X,Z)Y](grad_F f)(f) \\ &+g(Z,{}^B\nabla_X P_B)Y-g(Z,{}^B\nabla_Y P)X \\ &+((P_F(f))/f)[g(X,Z)Y-g(Y,Z)X] \\ &+\pi(Z)[\pi(Y)X-\pi(X)Y].\end{aligned}$$

Thus, we obtain (i) and (ii).
By the use of (6) again, we have

$$\overset{\circ}{R}(V,X)Y = R(V,X)Y+g(Y,\nabla_V P)X-g(Y,\nabla_X P)V +\pi(Y)[\pi(X)V-\pi(V)X],$$

for all vector fields $X,Y\in\chi(B)$ and $V\in\chi(F)$, respectively. Then, using (2), Lemma 3.1 and Lemma 3.2 again we obtain (iii) and (iv).
Putting Z = V in equation (6), we get

$$\overset{\circ}{R}(X,Y)V = R(X,Y)V+g(V,\nabla_X P)Y-g(V,\nabla_Y P)X +\pi(V)[\pi(Y)X-\pi(X)Y],$$

where $X,Y\in\chi(B)$ and $V\in\chi(F)$. In view of (8), Lemma 3.1 and Lemma 3.2, the last equation can be written as follows

$$\begin{aligned}\overset{\circ}{R}(X,Y)V = {}& - ((H^{f\circ\,\sigma}(Y,V))/f)X+((H^{f\circ\,\sigma}(X,V))/f)Y \\ &- ((V(f))/f)[\pi(X)Y-\pi(Y)X] \\ &+[((X(b))/b)Y-((Y(b))/b)X]\pi(V) \\ &- [\pi(X)Y-\pi(Y)X]\pi(V),\end{aligned}$$

which shows us (v) and (vi).
By virtue of (6), we can write

$$\overset{\circ}{R}(V,W)X = R(V,W)X+g(X,\nabla_V P)W-g(X,\nabla_W P)V +\pi(X)[\pi(W)V-\pi(V)W],$$

for any vector fields X on B and V,W on F, respectively. Similarly from (2), Lemma 3.1 and Lemma 3.2, we get

$$\begin{aligned}\overset{\circ}{R}(V,W)X = {}& - ((H^{b\circ\,\pi}(X,W))/b)V+((H^{b\circ\,\pi}(X,V))/b)W \\ &- ((X(b))/b)[\pi(V)W-\pi(W)V] \\ &+\pi(X)[((V(f))/f)W-((W(f))/f)V] \\ &- \pi(X)[\pi(V)W-\pi(W)V].\end{aligned}$$

Thus, we obtain (vii) and (viii).
In view of (6), we get

$$\overset{\circ}{R}(X,V)W = R(X,V)W+g(W,\nabla_X P)V-g(W,\nabla_V P)X +\pi(W)[\pi(V)X-\pi(X)V],$$

for any vector fields $X\in\chi(B)$ and $V,W\in\chi(F)$. By the use of Lemma 3.1 and Lemma 3.2 and from (2), we obtain (ix) and (x), respectively.
From (6) again, we can write

$$\overset{\circ}{R}(U,V)W = R(U,V)W+g(W,\nabla_U P)V-g(W,\nabla_V P)U +\pi(W)[\pi(V)U-\pi(U)V],$$

for any vector fields U,V,W on F. Similarly from (2), Lemma 3.1 and Lemma 3.2 we get (xi) and (xii). Therefore, we finish the proof of the lemma.

From Lemma 4.2, by a contraction of the curvature tensors we have the following corollary:

**Corollary 4.3:** Let $M={}_f B\times_b F$ be a doubly warped product and S and $\overset{\circ}{S}$ denote the Ricci tensors of M with respect to the Levi-Civita connection and the semi-symmetric non-metric connection, respectively, where dimB=r and dimF=s. If $X,Y\in\chi(B)$, $V,W\in\chi(F)$, then:

**(i)** $$\begin{aligned}\overset{\circ}{S}(X,Y) = {}& {}^{B\circ}S(X,Y) -((r-1))/(b^2f^2))g(X,Y)(grad_F f)(f) \\ &- \sum[g(Y,{}^B\nabla_{e_i}P_B)g(X,e_i) - ng(X,Y)g({}^B\nabla_{e_i}P_B,e_i)] \\ &- (n-1)((P_F(f))/f)+(n-1)\pi(X)\pi(Y), \\ &- s((H_B{}^b(X,Y))/b) - ((\Delta_F(f))/(fb^2))g(X, Y)\end{aligned}$$

**(ii)** $$\begin{aligned}\overset{\circ}{S}(X,V) = {}& -(r-1)((H^{f\circ\,\sigma}(X,V))/f -(s-1)((H^{b\circ\,\pi}(X,V))/b) \\ &+(n-1)((V(f))/f)\pi(X)- (n-1)((X(b))/b)\pi(V) \\ &+(n-1)\pi(V)\pi(X),\end{aligned}$$

**(iii)** $$\begin{aligned}\overset{\circ}{S}(V,X) = {}& -(r-1)((H^{f\circ\,\sigma}(X,V))/f)-(s-1)((H^{b\circ\,\pi}(X,V))/b) \\ &- (n-1)((V(f))/f)\pi(X)+(n-1)((X(b))/b)\pi(V) \\ &+(n-1)\pi(V)\pi(X),\end{aligned}$$

**(iv)** $$\begin{aligned}\overset{\circ}{S}(V,W) = {}& {}^{F\circ}S(V,W)- ((s-1))/(b^2f^2))g(V,W)(grad_B b)(b) \\ &- \sum[g(W,{}^F\nabla_{e_i}P_F)g(V,e_i) - ng(V,W)g({}^F\nabla_{e_i}P_F,e_i)] \\ &- (n-1)((P_B(b))/b)g(V,W)+(n-1)\pi(V)\pi(W) \\ &- r((H_F{}^f(V,W))/f) - ((\Delta_B(b))/(bf^2))g(V, W).\end{aligned}$$

From Corollary 4.3, by a contraction of the Ricci tensors we have:

**Corollary 4.4 :** Let $M={}_f B\times_b F$ be a doubly warped product and $\tau$ and $\overset{\circ}{\tau}$ denote the scalar curvatures of M with respect to the Levi-Civita connection and the semi-symmetric non-metric connection, respectively. Then, we have

$$\overset{\circ}{\tau} = ({}^{B\circ}\tau)/(f^2)+(({}^{F\circ}\tau)/(b^2)) - ((r(r-1))/(b^2f^2))(grad_F f)(f) - ((s(s-1))/(b^2f^2))(grad_B b)(b) - (n-1)\sum g({}^B\nabla_{e_i}P_B,e_i) - (n-1)\sum g({}^F\nabla_{e_i}P_F,e_i) - s(n-1)((P_B(b))/b) - r(n-1)((P_F(f))/f) - (n-1)\pi(P) - (r/f)[1+(1/(b^2))]\Delta_F(f) - (s/b)[1+(1/(f^2))]\Delta_B(b).$$

## V. EINSTEIN DOUBLY WARPED PRODUCT MANIFOLDS WITH THE SEMI-SYMMETRIC NON-METRIC CONNECTION

An n-dimensional Riemannian manifold $(M,g)$, $(n\geq 2)$, is said to be an Einstein manifold if its Ricci tensor S satisfies the condition $S=(\tau/n)g$, where $\tau$ denotes the scalar curvature of M.

An n-dimensional Riemannian manifold $(M,g)$, $(n\geq 2)$, is defined to be an quasi-Einstein manifold [3] if the condition

$$S(X,Y) = \lambda g(X,Y)+\beta A(X)A(Y),$$

is fulfilled on M, where $\lambda$ and $\beta$ are scalar functions on M with $\beta\neq 0$ and A is non-zero 1-form such that

$$g(X,U) = A(X),$$

for any vector field $X,U\in\chi(M)$ where U is a unic vector field. If $\beta=0$, then the manifold reduces to an Einstein manifold.

In this section we consider Einstein doubly warped products endowed with the semi-symmetric non-metric connection.

Let's begin with the following theorem:

**Theorem 5.1:** Let $(M,g)$ be a doubly warped product ${}_f I\times_b F$, where $\dim I=1$ and $\dim F=n-1$ $(n\geq 3)$. Then $(M,g)$ is an Einstein manifold endowed with a semi-symmetric non-metric connection, $P_F\in\chi(F)$ is parallel on F with respect to the Levi-Civita connection and f is constant on F, then b is constant on I and F is a quasi-Einstein manifold with respect to the Levi-Civita connection.

**Proof :** Let denote by $g_I$ the metric on I. By making use of Corollary 4.3, we can write

$$\overset{\circ}{S}((\partial/(\partial t)),(\partial/(\partial t))) = (n-1)f^4 - (n-1)((b'')/b), \quad (23)$$

$$\overset{\circ}{S}((\partial/(\partial t)),V) = -(n-2)((H^{b\circ\pi}((\partial/(\partial t)),V))/b) - (n-1)[((b')/b)-f^2]\pi(V), \quad (24)$$

$$\overset{\circ}{S}(V,(\partial/(\partial t))) = -(n-2)((H^{b\circ\pi}((\partial/(\partial t)),V))/b) +(n-1)[((b')/b)+f^2]\pi(V) \quad (25)$$

and

$$\overset{\circ}{S}(V,W) = {}^FS(V,W)+(n-1)\pi(V)\pi(W), \quad (26)$$

for any vector fields $\partial/\partial t$ on I and V,W on F.

Since M is an Einstein manifold with respect to the semi-symmetric non-metric connection, we have

$$\overset{\circ}{S}((\partial/(\partial t)),(\partial/(\partial t))) = \alpha g((\partial/(\partial t)),(\partial/(\partial t))), \quad (27)$$

$$\overset{\circ}{S}((\partial/(\partial t)),V) = \overset{\circ}{S}(V,(\partial/(\partial t))) = \alpha g(V,(\partial/(\partial t))) \quad (28)$$

and

$$\overset{\circ}{S}(V,W) = \alpha g(V,W). \quad (29)$$

Comparing the right hand sides of the equations (24) and (25) and by the use of (28), we get

$$2(n-2)((b')/b)\pi(V)=0,$$

which gives us $b'=0$ $(n\geq 3)$. So, b is constant on I.

On the other hand from (7), the equations (27) and (29) reduce to

$$\overset{\circ}{S}((\partial/(\partial t)),(\partial/(\partial t))) = \alpha f^2 \quad (30)$$

and

$$\overset{\circ}{S}(V,W) = \alpha b^2 g_F(V,W). \quad (31)$$

Comparing the right hand sides of (23) and (30), we get

$$\alpha = (n-2)f^2. \quad (32)$$

Similarly, comparing the right hand sides of (26) and (29) and by making use of (32), we obtain

$${}^FS(V,W) = (n-1)b^2f^2g_F(V,W)-(n-1)\pi(V)\pi(W),$$

which implies that F is a quasi-Einstein manifold with respect to the Levi-Civita connection. Hence, we complete the proof of the theorem.

**Theorem 5.2 :** Let $(M,g)$ be a doubly warped product ${}_f B\times_b I$, where $\dim I=1$ and $\dim B=n-1$ $(n\geq 3)$, $P_B\in\chi(B)$ is parallel on B with respect to the Levi-Civita connection on B and b and f are both constant on B and I, respectively. Then

**(i)** If $(M,g)$ is an Einstein manifold with respect to the semi-symmetric non-metric connection, then:

$${}^B\tau = f^2(n-1)[(n-1)\pi(P) - ng(P_B,P_B)].$$

**(ii)** If B is an Einstein manifold with respect to the the Levi-Civita connection, then M is a quasi-Einstein manifold endowed with a semi-symmetric non-metric connection.

**Proof : (i)** Let (M,g) be an Einstein manifold with respect to the semi-symmetric non-metric connection. Then we can write

$$\overset{\circ}{S}(X,Y) = (\overset{\circ}{\tau}/n)g(X,Y), \qquad (33)$$

for any vector fields $X,Y\in\chi(B)$. By the use of the equation (7) and Corollary 4.4, the equation (33) reduces to

$$\overset{\circ}{S}(X,Y)=(1/n)[((^B\tau)/(f^2)) - (n-1)\pi(P)]g(X,Y).$$

Contracting the above equation over X and Y, we get

$$\overset{\circ}{\tau} = ( (n-1)/n)[((^B\tau)/(f^2))-(n-1)\pi(P)]. \qquad (34)$$

On the other hand, by making use of Corollary 4.3, we can write

$$\overset{\circ}{S}(X,Y) = {}^{B}\overset{\circ}{S}(X,Y)+(n-1)\pi(X)\pi(Y).$$

Similarly, by a contraction from the last equation over X and Y, it can be easily seen that

$$\overset{\circ}{\tau}=((^B\tau)/(f^2))+(n-1)g(P_B,P_B). \qquad (35)$$

Comparing the right hand sides of the equations (34) and (35), we get

$$( (n-1)/n)[((^B\tau)/(f^2))-(n-1)\pi(P)] = ((^B\tau)/(f^2))(n-1)\, g(P_B,P_B),$$

which gives us

$$^B\tau = f^2(n-1)[(n-1)\pi(P) - ng(P_B,P_B)].$$

**(ii)** Let B be an Einstein manifold with respect to the Levi-Civita connection. Then, we can write

$$^BS(X,Y)=\alpha g_B(X,Y), \qquad (36)$$

for any vector fields X,Y on B. In view of (7), the last equation turns into

$$^BS(X,Y)=(\alpha/(f^2))g(X,Y). \qquad (37)$$

On the other hand, in view of Corollary 4.3, we can write

$$\overset{\circ}{S}(X,Y) = {}^BS(X,Y)+(n-1)\pi(X)\pi(Y).$$

By the use of (37) in the last equation, we obtain

$$\overset{\circ}{S}(X,Y) = (\alpha/(f^2))g(X,Y)+(n-1)\, \pi(X)\pi(Y),$$

which means that $_f B\times_b I$ is a quasi-Einstein manifold with respect to the semi-symmetric non-metric connection. Therefore, we complete the proof of the theorem.

## References


[1] N. S. Agashe and M. R. Chafle, A semi-symmetric non-metric connection on a Riemannian manifold, Indian J. Pure and Appl. Math. 23 (1992), no. 6, 399-409.

[2] N. S. Agashe and M. R. Chafle, On submanifolds of a Riemannian manifold with a semi-symmetric non-metric connection, Tensor (N. S.) 55 (1994), no. 2, 120-130.

[3] M. C. Chaki and R. K. Maity, On quasi-Einstein manifolds, Publ. Math. Debrecen 57 (2000), no. 3-4, 297-306.

[4] F. Morteza and G. Narges, On doubly warped product submanifolds of generalized (κ,μ)-space forms, Afrika Matematika 24 (2014), 10-23.

[5] A. Gebarowski, On Einstein warped products, Tensor (N.S.) 52 (1993), no. 3, 204-207.

[6] H. A. Hayden, Subspace of a space with torsion, Proceedings of the London Mathematical Society II Series 34 (1932), 27-50.

[7] T. Imai, Notes on semi-symmetric metric connections, Commemoration volumes for Prof. Dr. Akitsugu Kawaguchi's seventieth birthday, Vol. I. Tensor (N.S.) 24 (1972), 293-296.

[8] S. Nölker, Isometric immersions of warped products. Differential Geom. Appl. 6, 1 (1996), 1--30.

[9] B. O'Neill, Semi-Riemannian geometry with applications to relativity, Academic Press, N-Y, London 1983.

[10] A. Olteanu, Cr-doubly warped product submanifolds in Sasakian manifolds, Bulletin of the Transilvania University of Braşov 50 (2008), no. 1, 269-278.

[11] C. Özgür and S. Sular, Warped product manifolds with a semi-symmetric non-metric connection, Arabian Journal for Science and Engineering, 36 (2011), no. 3, 461-473.

[12] S. Sular and C. Özgür, Doubly warped product submanifolds of (κ,μ)-contact metric manifolds. Ann. Polon. Math. 100, 3 (2011), 223--236.

[13] S. Sular, Semi-symmetric metric connection on doubly warped product manifolds, in press.

[14] B. Ünal, Doubly warped products, Differential Geom. Appl. 15 (2001), no. 3, 253-263.

[15] K. Yano, On semi-symmetric metric connections, Rev. Roumaine Math. Pures Appl. 15(1970), 1579-1586.