Reliability Functions Based on Fuzzy Probability Densities

Reinhard Viertl and Owat Sunanta

Abstract— In general, real life time data are imprecise numbers, also called fuzzy. This kind of imprecision is a result of measurement and/or observation, including time observation, which is a characteristic of interest in the area of reliability. Therefore, estimation methods for reliability characteristics have to be adapted to such fuzzy life time data for realistic results. In this contribution, definitions of fuzzy numbers and related foundations are explained. Then, a generalized reliability function is defined based on fuzzy probability densities. The generalized function is generated by fuzzy stochastic information and, hence, a fuzzy valued function describing the adapted life time distribution.

Keywords—fuzzy number, fuzzy probability density, fuzzy reliability, reliability function

I. Introduction

In studying life time data, reliability characteristics is of interest and important. Standard reliability functions $R(\cdot)$ for life times T are defined by

$$R(t) := Pr\{T > t\} \quad \forall\, t \geq 0.$$

Based on traditional probability densities $f(\cdot)$, with support $[f(\cdot)] \subseteq [0, \infty)$, the function can be defined as

$$R(t) := \int_t^\infty f(x)dx \quad \forall\, t \geq 0.$$

The imprecise (fuzzy) life time data are not usually assumed in standard reliability functions. This challenge has to be approached with caution, however. The most suitable mathematical model to describe the fuzziness is by fuzzy numbers and their characterizing functions [8] and [9]. The extension principle [10] in fuzzy set theory has been used in applications which call for an extension of the domain of a relation. These concepts are vital and make the necessary generalization possible.

For a given partition of the observation space in some cases, it is not possible to decide to which class a fuzzy observation belongs. This makes it necessary to generalize the concept of histograms. Motivated by such histograms for the so-called fuzzy data, a more general concept of probability is simply the concept of fuzzy probability densities. Based on fuzzy probability densities, a generalized concept of probability for an event can be defined, i.e. these generalized probabilities are basically fuzzy intervals. In other words, a fuzzy probability density is based on the concept of fuzzy numbers, especially fuzzy intervals, and is a special fuzzy valued function. For more details compare [6] and [7].

In this contribution, a generalized reliability function, which is a fuzzy valued function describing the adapted life time distribution is described. In section II, the foundations of fuzzy numbers, intervals, and fuzzy probability densities are explained. For clarity, an example of fuzzy probability densities is also provided in section III. The definition of fuzzy reliability functions is described in section IV. The crucial part, which is the fuzzy extension of the standard reliability function along with its estimation based on fuzzy life time data, is explained, along with examples, in section V. In section VI, the use of fuzzy data in databases is introduced. Finally, the contribution is concluded with final remarks in section VII.

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II. Fuzzy numbers and fuzzy probability densities

Measurement data from continuous quantities are always more or less imprecise, i.e. they cannot be represented by precise numbers. Therefore, a more general concept than real numbers is necessary. For such case, the best up-to-date models are the so-called fuzzy numbers.

Definition 2.1:
A *fuzzy number* x^* is a fuzzy subset of real-number set $\mathbb{R}$, whose membership function $\xi(\cdot)$ has special properties and is called *characterizing function*, where the followings hold:

(1) $\{x \in \mathbb{R}: \xi(\cdot) > 0\}$ is contained in a compact interval

(2) $\forall \delta \in (0,1]$, the so-called δ-cut $C_\delta[\xi(\cdot)] = \{x \in \mathbb{R}: \xi(x) \geq \delta\}$, is a finite union of compact intervals, i.e.

$$C_\delta[\xi(\cdot)] = \bigcup_{j=1}^{k_\delta} [a_{\delta,j}, b_{\delta,j}] \neq \emptyset,\ k_\delta \in \mathbb{N}$$

If all δ-cuts of a fuzzy number x^* are compact intervals $C_\delta[\xi(\cdot)] = [a_\delta, b_\delta]$, the corresponding fuzzy number is called *fuzzy interval*. The set of all fuzzy numbers is denoted by $\mathcal{F}(\mathbb{R})$ and the set of all fuzzy intervals is denoted by $\mathcal{F}_I(\mathbb{R})$.

As an extension to handle multivariate data and their statistical inferences, the following concept of fuzzy vectors is necessary:

Definition 2.2:
A fuzzy subset $\mathbf{x}^*$ of the Euclidean space $\mathbb{R}^n$ is called *n*-dimensional *fuzzy vector* if the membership function $\zeta(\cdot)$ of $\mathbf{x}^*$ fulfils the following:

(1) supp[$\zeta(\cdot)$] is a bounded set, i.e. it is contained in an *n*-dimensional interval $\times_{i=1}^{n}[a_i,b_i]$ of finite volume.

(2) $C_\delta[\zeta(\cdot)]$ is non-empty for all $\delta\in(0,1]$, and it is a finite union of simply connected and closed subsets of $\mathbb{R}^n$.

Remark 1: A vector $(x_1^*,\dots,x_n^*)$ of fuzzy numbers is not a fuzzy vector. In this case, it is necessary to combine the fuzzy numbers $x_1^*,\dots,x_n^*$ to obtain a fuzzy element $(x_1,\dots,x_n)^*$ of the sample space $M_x^n \in \mathbb{R}^n$ and, then, the following Lemma 1 holds:

Lemma 1: Let $x_1^*,\dots,x_n^*$ be fuzzy numbers with corresponding characterizing functions $\xi_1(\cdot),\dots,\xi_n(\cdot)$. Then, the function $\zeta(\cdot,\dots,\cdot)$, defined by

$$\zeta(x_1,\dots,x_n) = min\{\xi_1(x_1),\dots,\xi_n(x_n)\} \quad \forall(x_1,\dots,x_n)\in\mathbb{R}^n,$$

is the vector-characterizing function of an *n*-dimensional fuzzy vector $\mathbf{x}^* = (x_1,\dots,x_n)^*$.

Remark 2: A fuzzy vector $\mathbf{x}^*$ is obtained via minimum t-norm when the individual values of the variables x_i are fuzzy numbers x_i^*. Through the minimum-t-norm, the combination of n fuzzy numbers with characterizing functions $\xi_i(\cdot)$, $i = 1(1)n$, a fuzzy vector $\mathbf{x}^* = (x_1,\dots,x_n)^*$ is obtained. In this case, the following holds:

$$C_\delta[\zeta(\cdot,\dots,\cdot)] = \times_{i=1}^{n} C_\delta[\xi_i(\cdot)] \;\forall\, \delta\in(0,1]$$

In words, the δ-cuts of the fuzzy vector $\mathbf{x}^* = (x_1,\dots,x_n)^*$ are the Cartesian products of the δ-cuts of the fuzzy numbers x_i^*, $i = 1(1)n$.

The concept of combined fuzzy samples is useful for succinct multivariate statistical analysis of fuzzy data, including fuzzy life times accordingly.

Definition 2.3:
A function $f^*: M \to \mathcal{F}(\mathbb{R})$ is called fuzzy-valued function. If $f^*: M \to \mathcal{F}_I(\mathbb{R})$, then $\forall\delta>0$, the so-called δ-level functions $\underline{f}_\delta(\cdot)$ and $\overline{f}_\delta(\cdot)$ are defined by the endpoints of the δ-cut of $f^*(x)\ \forall\, x\in M$, i.e.

$$C_\delta[f^*(x)] = [\underline{f}_\delta(x), \overline{f}_\delta(x)].$$

Remark 3: δ-level functions are real valued functions, i.e. $\underline{f}_\delta: M\to\mathbb{R}$ and $\overline{f}_\delta: M\to\mathbb{R}$.

Integration of fuzzy-valued functions is defined in the following way:

The fuzzy-valued integral is based on the classical integrals of the δ-level functions and the following so-called generation of fuzzy numbers.

Definition 2.4:
Let $(A_\delta;\ \delta\in(0,1])$ be a family of non-empty nested compact intervals which are uniformly bounded, then the characterizing function $\xi(\cdot)$ of the generated fuzzy interval is defined by

$$\xi(x) := sup\,\{\delta.\mathbb{1}_{A_\delta}(x) : \delta\in[0,1]\} \quad \forall x\in\mathbb{R},\ \text{where } A_0 := \mathbb{R}.$$

Remark 4: $\xi(\cdot)$ is the characterizing function such that the δ-cuts of $\xi(\cdot)$ are as close as possible to the sets A_δ as explained in the following lemma 2.

Lemma 2:
Under the condition of Definition 2.4, the following property holds (see also [1] for an extended proof):

$$C_\delta[\xi(\cdot)] = A_\delta \text{ iff } A_\delta = \bigcap_{\beta<\delta} A_\beta$$

Then, the generalized integral $I^* = \int f^*(x)\,d\mu(x)$ can be defined accordingly.

For a fuzzy-valued function $f^*: M \to \mathcal{F}_I(\mathbb{R})$, defined on a measure space $(M, \mathcal{A}, \mu)$ for which all δ-level functions $\underline{f}_\delta(\cdot)$ and $\overline{f}_\delta(\cdot)$ are integrable with finite integral $\int \overline{f}_\delta\,d\mu(x) < \infty$ and $\int \underline{f}_\delta\,d\mu(x) < \infty$, the generating family of compact intervals $A_\delta = [a_\delta, b_\delta]$, for I^* are defined in the following way:

$$a_\delta = \int \underline{f}_\delta\,d\mu(x) \text{ and } b_\delta = \int \overline{f}_\delta\,d\mu(x)$$

The characterizing function $\eta(\cdot)$ of the fuzzy interval I^* is given by its values

$$\eta(x) = sup\,\{\delta.\mathbb{1}_{A_\delta}(x) : \delta\in[0,1]\} \quad \forall x\in\mathbb{R}.$$

Definition 2.5:
Based on a measure space $(M, \mathcal{A}, \mu)$, a *fuzzy probability density* is a fuzzy valued function $f^*: M \to \mathcal{F}_I(\mathbb{R})$ whose δ-level functions are all integrable, with finite integral, and for which a classical probability density $g(\cdot)$ on $(M, \mathcal{A}, \mu)$ exists, as well as obeying

$$\underline{f}_1(\cdot) \le g(x) \le \overline{f}_1(\cdot) \quad \forall x\in M.$$

Remark 5: Probabilities of events $B\in\mathcal{A}$ based on fuzzy probability densities $f^*(\cdot)$ are fuzzy intervals, which are defined in the following way:

Definition 2.6:
Let $\mathcal{D}_\delta$ be the set of classical probability densities $h(\cdot)$, which obeys $\underline{f}_\delta(\cdot) \le h(x) \le \overline{f}_\delta(\cdot) \quad \forall x\in M$, the fuzzy probability

$P^*(B)$ is the fuzzy interval whose generating family A_δ of compact intervals $A_\delta = [a_\delta, b_\delta]$ is defined by

$$\begin{aligned} a_\delta &:= \inf\{\int h(x)\,d\mu(x): h \in \mathscr{D}_\delta\} \\ b_\delta &:= \sup\{\int h(x)\,d\mu(x): h \in \mathscr{D}_\delta\} \end{aligned} \qquad \forall\delta\in(0;1].$$

The characterizing function $\psi(\cdot)$ of the fuzzy interval $P^*(B)$ is given by $\psi(\cdot) = \sup\{\delta.\mathbb{1}_{A_\delta}(x): \delta\in[0,1]\} \quad \forall x \in \mathbb{R}$..

III. Example of fuzzy probability densities

As a simplified example, life times are sometimes modeled with an exponential distribution with parameter $\lambda\in(0,\infty)$, i.e.

$$f(x) = \lambda e^{-\lambda x} \quad \forall x \geq 0$$

The following figure 3.1 shows different δ-level functions (each with corresponding upper and lower limits) of the corresponding fuzzy density, generated using the method described in section II:

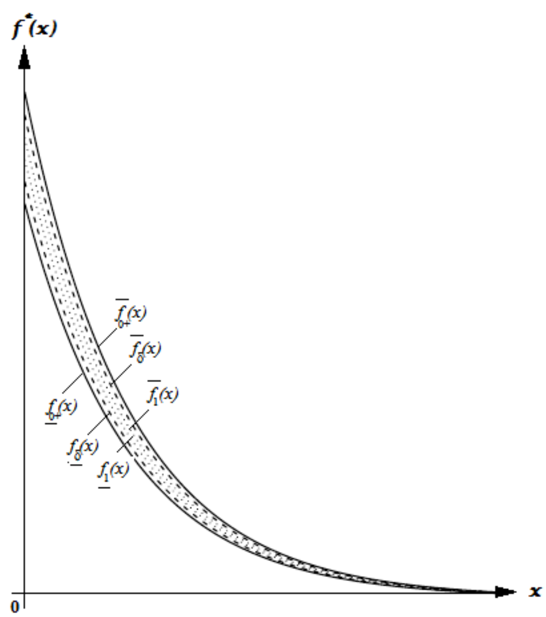

Fig. 3.1 Some δ-level functions of a fuzzy density function from an exponential distribution

Remark 6: The concept of fuzzy probability density differs from that of the lower/upper probabilities.

IV. Fuzzy reliability functions

Recalling classical reliability functions $R(\cdot)$ for life time X, given by their values $R(t) := Pr\{X > t\}\ \forall t \geq 0$, this means for probability densities $g(\cdot)$ on the Lebesgue measure space $(\mathbb{R}, \boldsymbol{\mathcal{B}}, \lambda)$

$$R(t) := \int_t^\infty g(x)\,d\lambda(x) \quad \forall t \geq 0.$$

Based on fuzzy probability densities $f^*(\cdot)$ a fuzzy extension $R^*(\cdot)$ of $R(\cdot)$ is obtained, based on the family $\mathscr{D}_\delta$ from Definition 2.6. The generating family (A_δ; $\delta\in(0,1]$) for the fuzzy intervals $R^*(t)$ are given by $[a_\delta(t), b_\delta(t)]$, where

$$\begin{aligned} a_\delta(t) &:= \inf\{\int_t^\infty h(x)\,d\lambda(x): h \in \mathscr{D}_\delta\} \\ b_\delta(t) &:= \sup\{\int_t^\infty h(x)\,d\lambda(x): h \in \mathscr{D}_\delta\} \end{aligned} \qquad \forall\delta\in(0;1].$$

In figure 4.1, an example of a fuzzy reliability function is depicted.

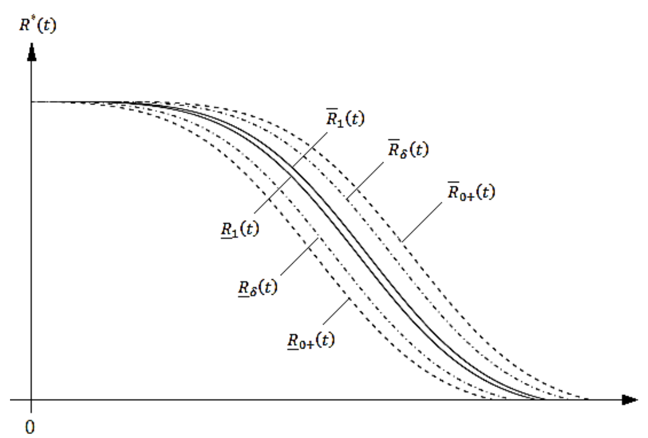

Fig. 4.1 Some δ-level functions of a fuzzy reliability function $R^*(\cdot)$.

Note: $\underline{R}_{0+}(t)$ and $\overline{R}_{0+}(t)$ denote the endpoints of the support of the characterizing function of $R^*(t)$.

As an extension to the example in Section III, the δ-level functions of a fuzzy reliability function $R^*(x)$, where t is exponentially distributed, are shown in Figure 4.2.

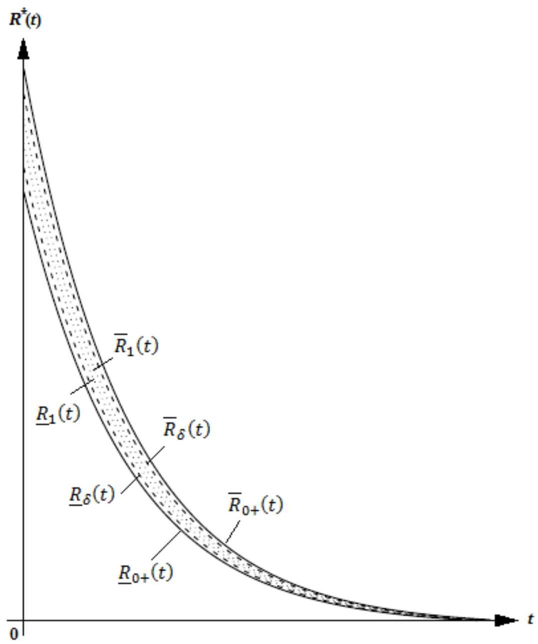

Fig. 4.2 Some δ-level functions of a fuzzy reliability function of an exponential distribution

Note: The standard reliability function of an exponential distribution is in the form of $R(t) := e^{-\lambda t}\ \forall\, t \geq 0$. Applying the method described in Section IV, The generating family (A_δ; $\delta \in (0,1]$) for the fuzzy intervals $R^*(t)$, shown in figure 4.2, are obtained. Also, as opposed to the densities in figure 3.1, the scale on the *y*-axis differs due to lacking of λ multiplication in this case, i.e. the exponential distribution.

V. Estimation of Reliability Functions Based on Fuzzy Life Time Data

The standard empirical reliability function $\hat{R}_n(\cdot)$ based on classical life time data $t_1, \ldots, t_n$ is given by

$$\hat{R}_n(t) := \frac{\#\{t_i > t\}}{n} \quad \forall t \geq 0.$$

For fuzzy data $t_1^*, \ldots, t_n^*$, there are different possibilities to generalize the estimator $\hat{R}_n(\cdot)$. See [5] for details.

Remark 7: For details on how to obtain the characterizing function of a fuzzy life time t^*, also compare with [3] and [7].

VI. Fuzzy Data in Databases

Data and information are obtained, analyzed, and stored in databases. Critical information generally is obtained from different sources and often cannot be replicated, e.g. life time data that are results from measurements. There are different types of uncertain data, i.e. imprecise, vague, ambiguous, inconsistent, and incomplete data [2]. Fuzzy theory allows us to develop models for imprecise or vague data, i.e. to integrate the vague knowledge into databases. To store this type of information, fuzzy databases are necessary for storing the fuzzy data.

The fuzzy relation and fuzzy set theory provide a requisite mathematical framework for dealing with such fuzzy data [1]. Applied databases have to be able to store fuzzy numbers and fuzzy vectors in order to provide realistic information concerning real data. Fuzzy numbers and fuzzy vectors can be represented in databases by storing δ-cuts (levels). Also, fuzzy multivariate data can be represented in databases by storing a suitable family of δ-cuts of the corresponding vector-characterizing function. Fuzzy meta-model keeps all relevant fuzzy data and manages links to relation of real entities (see also [5] for details).

VII. Final Remarks

For observations and measurements of continuous quantities, fuzziness is unavoidable. Therefore, suitable mathematical models are necessary to describe real data. This is possible and many research topics related to this are still waiting to be solved. The fuzziness of individual measurement results can be described by so-called fuzzy numbers, whereas the variability and errors are described by stochastic models.

Generalized estimation procedures for reliability characteristics based on fuzzy life time data are necessary. In this contribution, a generalized reliability function, which is a fuzzy valued function describing the adapted life time distribution is introduced.

Fuzziness is almost everywhere in the physical world, including lifetime data. In order to describe different facets of reality, the analysis methods have to capture this type of uncertainty. The related methods are available through mathematical models for fuzzy data. Application of such methods results in more realistic models for data analysis and, subsequently, better understanding of the collected data for further use of such information.

References

[1] Guglani, S., Katti, C.P., Saxena, P.C. , "Fuzzy Statistical Database and Its Physical Organization," *International Journal of Database Management Systems*, vol. 5, no. 4, 2013, pp. 27-47.

[2] Hudec, M. , *Fuzziness in Information Systems*, Springer, Switzerland, 2016.

[3] Kovářová, L. and Viertl, R., "The Generation of Fuzzy Sets and the Construction of Characterizing Functions of Fuzzy Data," *Iranian Journal of Fuzzy Systems*, vol. 12, no. 6, 2015, pp. 1-16.

[4] Popat, D., Sharda, H., Taniar, D., Classification of Fuzzy Data in Database Management System, in *Negoita, M. Gh. (Ed.) Proceedings of Knowledge-Based Intelligent Information and Engineering Systems 8th International Conference*, New Zealand, 2004, pp. 691-697.

[5] Shafiq, M. and Viertl, R., "Empirical Reliability Functions based on Fuzzy Life Time Data," *Journal of Intelligent & Fuzzy Systems*, vol. 28, 2015, pp. 707-711.

[6] Sunanta, O. and Viertl, R., "Fuzziness and Statistics - Mathematical Models for Uncertainty," *Theoretical and Practical Research in Economic Fields*, vol. 8, no. 1(15), 2017, pp. 31-46.

[7] Viertl, R., *Statistical Methods for Fuzzy Data*, Chichester: Wiley, 2011, pp. 53-66.
[8] Viertl, R., “Measurement of Continuous Quantities and their Statistical Evaluation,” *Austrian Journal of Statistics*, vol. 44, 2015, pp. 25-32
[9] Zadeh, L. A., “Fuzzy Sets,” *Information and Control*, vol. 8, no. 3, 1965, pp. 338-353.
[10] Zadeh, L. A., “The Concept of Linguistic Variable and Its Application to Approximate Reasoning,” *Information Science*, vol. 8, 1975, pp. 43-80.