Fibers of Polynomial Mappings Over

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Abstract: We find sufficient conditions on a polynomial mapping $f = (p_1, \ldots, p_n) : \mathbb{R}^n \to \mathbb{R}^n$ to be surjective. One such a condition is the existence of a non-trivial solution of the induced homogeneous system of the equations $\sum_{j=1}^n (p_j)^{\alpha_j} g_{ij} = 0$, $i = 1, \ldots, n$. Here $\alpha_j \in \mathbb{Z}^+$, $g_{ij} \in \mathbb{R}[X_1, \ldots, X_n]$ and $\det(g_{ij})$ never vanishes on $\mathbb{R}^n$. A conclusion that follows is that if $\prod_{j=1}^n (\deg\, p_j)$ is an odd integer, then surjectivity $f(\mathbb{R}^n) = \mathbb{R}^n$ follows if the homogeneous system $\overline{p_1} = \ldots = \overline{p_n} = 0$ ($\overline{p}$ is the highest homogeneous component of p) has only the trivial solution. We also investigate mappings f for which the determinant of their Jacobian matrix, $\det J(f)$ never vanishes on $\mathbb{R}^n$. These polynomial mappings are in the core of the Real Jacobian Conjecture. One conclusion is that for such a local polynomial diffeomorphism the system $\overline{p_j \frac{\partial p_j}{\partial X_i}} = 0$, $i = 1, \ldots, n$ must have non-trivial solutions, and for any $j = 1, \ldots, n$. Also, such a local diffeomorphism is surjective if the induced homogeneous system of $\sum_{j=1}^n \alpha_j (p_j)^{\alpha_j - 1} \frac{\partial p_j}{\partial X_i} = 0$, $i = 1, \ldots, n$, has only the trivial (zero) solution. These last two theorems give a new point of view on S. Pinchuk's solution of the Real Jacobian Conjecture. Other obvious applications of our results are for the existence of solutions of the corresponding polynomial equations in n unknowns over the real field, $\mathbb{R}$.

Keywords: Pinchuk polynomial mapping, polynomial mappings, surjective polynomial mappings, the Jacobian conjecture

I. THE RESULTS

Definition 1.1: Let $p(X_1, \ldots, X_n) \in \mathbb{R}[X_1, \ldots, X_n]$. We denote by $\overline{p}(X_1, \ldots, X_n)$ the leading homogeneous component of $p(X_1, \ldots, X_n)$ with respect to the standard grading, $\deg X_j = 1$ for $1 \le j \le n$.

Theorem 1.2: Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(X_1, \ldots, X_n) = (p_1(X_1, \ldots, X_n), \ldots, p_n(X_1, \ldots, X_n))$ be a polynomial mapping (i.e. $(p_1, \ldots, p_n) \in \mathbb{R}[X_1, \ldots, X_n]^n$). Let $g_{ij}(X_1, \ldots, X_n) \in \mathbb{R}[X_1, \ldots, X_n]$, $i, j = 1, \ldots, n$. Let $(\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}^+)^n$. We assume that the following 2 conditions hold true:
(i) The determinant $\det(g_{ij}(X_1, \ldots, X_n))_{i,j=1,\ldots,n}$ never vanishes in $\mathbb{R}^n$.
(ii) The following system of n equations in n unknowns is such that the degree of each of the equations is an odd number:

$$\sum_{j=1}^n (p_j(X_1, \ldots, X_n))^{\alpha_j} g_{ij}(X_1, \ldots, X_n) = 0, \ \ i = 1, \ldots, n. \tag{1}$$

Then the following 2 assertions are true:

(a) If the induced homogeneous system of the system (1):

$$\overline{\sum_{j=1}^n (p_j(X_1, \ldots, X_n))^{\alpha_j} g_{ij}(X_1, \ldots, X_n)} = 0, \ \ i = 1, \ldots, n, \tag{2}$$

has only the zero solution $(X_1, \ldots, X_n) = (0, \ldots, 0)$ over $\mathbb{R}$, then $f(\mathbb{R}^n) = \mathbb{R}^n$.
(b) If the induced homogeneous system of the system in equation (1), i.e. the system (2) has only the zero solution over $\mathbb{C}$, then $\forall (a_1, \ldots, a_2) \in \mathbb{R}^n$, either $|f^{-1}(a_1, \ldots, a_n)| = \infty$ over $\mathbb{C}$ under the extra assumption that $\det(g_{ij}(Z_1, \ldots, Z_n))_{i,j=1,\ldots,n} \in \mathbb{R}^\times$, or there exists an integer $k = k(a_1, \ldots, a_n) \ge 0$ such that $|f^{-1}(a_1, \ldots, a_n)| = 2k + 1$ over $\mathbb{R}$.
Proof.
(a) Let $(a_1, \ldots, a_n) \in \mathbb{R}^n$. We will prove that $(a_1, \ldots, a_n) \in f(\mathbb{R}^n)$. we consider the following system of equations:

$$X_{n+1}^{d_i} \sum_{j=1}^n \left(p_j \left(\frac{X_1}{X_{n+1}}, \ldots, \frac{X_n}{X_{n+1}} \right) - a_j \right)^{\alpha_j} \times$$
$$\times g_{ij} \left(\frac{X_1}{X_{n+1}}, \ldots, \frac{X_n}{X_{n+1}} \right) = 0, \tag{3}$$

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$$where \quad d_i = \deg\left(\sum_{j=1}^{n} (p_j)^{\alpha_j} g_{ij}\right), \quad i = 1, \ldots, n.$$

This is a system of n homogeneous real polynomial equations in the $n+1$ unknowns $X_1, \ldots, X_n, X_{n+1}$, and by condition (ii) the degrees d_i, $i = 1, \ldots, n$ of all of these equations are odd integers. By well known facts on varieties over $\mathbb{R}$ (see pages 200-202 in [3]), it follows that the system (3) has a non-zero real solution $(X_1, \ldots, X_n, X_{n+1}) = (X_1^0, \ldots, X_n^0, X_{n+1}^0)$. We must have $X_{n+1}^0 \neq 0$, for otherwise $(X_1^0, \ldots, X_n^0) \neq (0, \ldots, 0)$ and $(X_1^0, \ldots, X_n^0)$ is a solution of (1),i.e. (2). This contradicts the assumption of the theorem in part (a). Thus we get from equation (3):

$$\sum_{j=1}^{n} \left(p_j\left(\frac{X_1^0}{X_{n+1}^0}, \ldots, \frac{X_n^0}{X_{n+1}^0}\right) - a_j\right)^{\alpha_j} \times$$

$$\times g_{ij}\left(\frac{X_1^0}{X_{n+1}^0}, \ldots, \frac{X_n^0}{X_{n+1}^0}\right) = 0,$$

$$for \quad i = 1, \ldots, n. \tag{4}$$

By condition (i) of our theorem, this implies that

$$f\left(\frac{X_1^0}{X_{n+1}^0}, \ldots, \frac{X_n^0}{X_{n+1}^0}\right) = (a_1, \ldots, a_n).$$

(b) Let us consider the system (3) over $\mathbb{C}$. By the Bezout Theorem (see pages 198-199 in [3]), either the system (3) has infinitely many solutions over $\mathbb{C}$, or it has exactly

$$\prod_{i=1}^{n} \deg\left(\sum_{j=1}^{n} (p_j)^{\alpha_j} g_{ij}\right)$$

solutions over $\mathbb{C}$, counting multiplicities and not counting the zero solution. In the case we have infinitely many solutions over $\mathbb{C}$, we must have for each such a solution $(Z_1^0, \ldots, Z_n^0, Z_{n+1}^0)$ that $Z_{n+1}^0 \neq 0$, for by the assumption in part (b) of our theorem, the induced homogeneous system (2), of the system (1) has only the zero solution over $\mathbb{C}$. Since we also assume in this case that $\det(g_{ij}(Z_1, \ldots, Z_n))_{i,j=1,\ldots,n} \in \mathbb{C}^{\times}$ it follows as before by equation (4) that the fiber over $\mathbb{C}$, $f^{-1}(a_1, \ldots, a_n)$ contains infinitely many points:

$$\left(\frac{Z_1^0}{Z_{n+1}^0}, \ldots, \frac{Z_n^0}{Z_{n+1}^0}\right).$$

In the second case, in which we have exactly

$$\prod_{i=1}^{n} \deg\left(\sum_{j=1}^{n} (p_j)^{\alpha_j} g_{ij}\right)$$

solutions over $\mathbb{C}$, noting that by condition (ii) this number is an odd integer and that non-real solutions $(Z_1^0, \ldots, Z_n^0, Z_{n+1}^0)$ come in conjugate pairs, we deduce that the fiber over $\mathbb{R}$, $f^{-1}(a_1, \ldots, a_n)$, contains an odd number of points. □

Corollary 1.3: Let the polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, be given by

$$f(X_1, \ldots, X_n) = (p_1(X_1, \ldots, X_n), \ldots, p_n(X_1, \ldots, X_n)).$$

Let

$$g_{ij}(X_1, \ldots, X_n) \in \mathbb{R}[X_1, \ldots, X_n], \quad i, j = 1, \ldots, n.$$

Let $(\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}^+)^n$. We assume that the following two conditions hold true:

(i) The determinant $\det(g_{ij}(X_1, \ldots, X_n))_{i,j=1,\ldots,n}$ never vanishes in $\mathbb{R}^n$.

(ii) For each $i = 1, \ldots, n$ the set $\{\alpha_j \deg p_j + \deg g_{ij} \mid j = 1, \ldots, n\}$ contains a unique maximal element $\alpha_{j(i)} \deg p_{j(i)} + \deg g_{ij(i)}$, which is an odd integer. We agree that $\deg 0 = -\infty$.

Let us consider the following homogeneous system:

$$\overline{p}_{j(i)} \overline{g}_{ij(i)} = 0, \quad i = 1, \ldots, n. \tag{5}$$

Then the following two assertions are true:

(a) If the system (5) has only the zero solution over $\mathbb{R}$, then $f(\mathbb{R}^n) = \mathbb{R}^n$.

(b) If the system (5) has only the zero solution over $\mathbb{C}$, then for any $(a_1, \ldots, a_n) \in \mathbb{R}^n$ either $|f^{-1}(a_1, \ldots, a_n)| = \infty$ over $\mathbb{C}$, provided that also the following assumption holds true, $\det g_{ij}(Z_1, \ldots, Z_n))_{i,j=1,\ldots,n} \in \mathbb{R}^{\times}$, or that there exists an integer $k = k(a_1, \ldots, a_n) \geq 0$ such that $|f^{-1}(a_1, \ldots, a_n)| = 2k + 1$ over $\mathbb{R}$.

Proof.

This is a special case of Theorem 0.2, where the system (5) is precisely the system (2) because of the maximality and the uniqueness of $\alpha_{j(i)} \deg p_{j(i)} + \deg g_{ij(i)}$ among the elements of the set $\{\alpha_j \deg p_j + \deg g_{ij} \mid j = 1, \ldots, n\}$. □

Corollary 1.4: Let the polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, be given by

$$f(X_1, \ldots, X_n) = (p_1(X_1, \ldots, X_n), \ldots, p_n(X_1, \ldots, X_n)).$$

Suppose that the product $(\deg p_1) \cdot \ldots \cdot (\deg p_n)$ is an odd integer. Then the following two assertions are true:

(a) If $|\overline{f}^{-1}(0, \ldots, 0)| = 1$ over $\mathbb{R}$, then $f(\mathbb{R}^n) = \mathbb{R}^n$.

(b) If $|\overline{f}^{-1}(0, \ldots, 0)| = 1$ over $\mathbb{C}$, then $\forall (a_1, \ldots, a_n) \in \mathbb{R}^n$ either the fiber size $|f^{-1}(a_1, \ldots, a_n)| = \infty$ over $\mathbb{C}$, or there exists an integer $k = k(a_1, \ldots, a_n) \geq 0$ such that $|f^{-1}(a_1, \ldots, a_n)| = 2k + 1$ over $\mathbb{R}$.

Proof.

This follows by Corollary 0.3, where $(\alpha_1, \ldots, \alpha_n) = (1, \ldots, 1)$ and where $g_{ij} = \delta_{ij}$, $i, j = 1, \ldots, n$ because the system (5) becomes $\overline{p}_j = 0$, $j = 1, \ldots, n$ which has the solution set $\overline{f}^{-1}(0, \ldots, 0)$. □

Remark 1.5: We note that if in Corollary 0.4 we have $\deg p_j = 1$, $j = 1, \ldots, n$, i.e. if all the $p_j = \overline{p}_j$ are linear forms then we get the well known fact from linear algebra.

Namely, if $A\overline{X} = \overline{0}$ is an $n \times n$ linear homogeneous system that has only the trivial solution, then $A\overline{X} = \overline{b}$ is consistent $\forall \overline{b} \in \mathbb{R}^n$.

Remark 1.6: If for $j = 1, \ldots, n$, $bj \geq 0$ is an integer and if we have

$$p_j(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_{ij} X_i^{2b_j+1} +$$

$$+ elements\, of\, degrees < 2b_j + 1.$$

Then the polynomial mapping $f(X_1, \ldots, X_n) = (p_1(X_1, \ldots, X_n), \ldots, p_n(X_1, \ldots, X_n))$ is a surjective mapping, i.e. $f(\mathbb{R}^n) = \mathbb{R}^n$, provided that the only solution of the following system:

$$\sum_{i=1}^{n} a_{ij} X_i^{2b_j+1} = 0, \quad j = 1, \ldots, n,$$

is the trivial solution: $X_1 = \ldots = X_n = 0$.

For in this case the above system is the system (5) of Corollary 0.4 ($g_{ij} = \delta_{ij}$). For example, this is the case for the equal-degree case $b_1 = \ldots = b_n = b$ provided that $\det(a_{ij})_{i,j=1,\ldots,n} \neq 0$. Another example is the following: we pick 4 non-zero real numbers, a, b, c and d such that $sgn(ad) = -sgn(bc)$. Then any mapping of the form:

$$f : \mathbb{R}^2 \to \mathbb{R}^2,$$

$$f(X, Y) = (aX^{2k+1} + bY^{2k+1} + \ldots, cX^{2j+1} + dY^{2j+1} + \ldots),$$

is a surjective mapping. For the system (5) is:

$$\left\{ \begin{array}{lcl} aX^{2k+1} + bY^{2k+1} & = & 0 \\ cX^{2j+1} + dY^{2j+1} & = & 0 \end{array} \right. .$$

If $k \leq j$ then the system can be written as follows:

$$\left\{ \begin{array}{lcl} aX^{2k+1} + bY^{2k+1} & = & 0 \\ (cX^{2(j-k)})X^{2k+1} + (dY^{2(j-k)})Y^{2k+1} & = & 0 \end{array} \right. .$$

We view this as a linear homogeneous system in the unknowns X^{2k+1} and Y^{2k+1}. Then the coefficients matrix is:

$$\left(\begin{array}{cc} a & b \\ cX^{2(j-k)} & dY^{2(j-k)} \end{array} \right).$$

The determinant of this matrix is $(ad)Y^{(2(j-k)} - (bc)X^{2(j-k)}$ and this can not be 0 because of the assumption $sgn(ad) = -sgn(bc)$, unless $j > k$ and $X = Y = 0$. In the other cases the only solution is, again, $X = Y = 0$.

Theorem 1.7: Let $g_{ij}(X_1, \ldots, X_n) \in \mathbb{R}[X_1, \ldots, X_n]$ for $i, j = 1, \ldots, n$ satisfy the condition that

$$\det(g_{ij}(X_1, \ldots, X_n))_{i,j=1,\ldots,n}$$

never vanishes in $\mathbb{R}^n$. Then for any j_0, $1 \leq j_0 \leq n$, such that the degrees $\deg g_{ij_0}$, $i = 1, \ldots, n$ are all odd integers the system:

$$\overline{g}_{ij_0}(X_1, \ldots, X_n) = 0, \quad i = 1, \ldots, n, \tag{6}$$

has non-zero real solutions.

Proof.

Let j_0 be such that the degrees $\deg g_{ij_0}$, $i = 1, \ldots, n$, are all odd integers. In Corollary 0.3 we take the following:

$$f : \mathbb{R}^n \to \mathbb{R}^n, \quad f(X_1, \ldots, X_n) = (\delta_{1j_0}, \ldots, \delta_{j_0j_0}, \ldots, \delta_{nj_0}).$$

$$and \quad (\alpha_1, \ldots, \alpha_n) = (1, \ldots, 1).$$

Then conditions (i) and (ii) of Corollary 0.3, with the choice $j(i) = j_0$ are satisfied. Since $f(\mathbb{R}^n) \neq \mathbb{R}^n$ it must be that the system (5) has non-zero solutions over $\mathbb{R}$. But in this case the system (5) coincides with the system above, (6). □

Theorem 1.8: Let the polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, be given by

$$f(X_1, \ldots, X_n) = (p_1(X_1, \ldots, X_n), \ldots, p_n(X_1, \ldots, X_n)).$$

Suppose that the determinant $\det J(f)(X_1, \ldots, X_n)$ never vanishes in $\mathbb{R}^n$. Then $\forall\, j$, $1 \leq j \leq n$ the system:

$$\overline{p}_j \overline{\frac{\partial p_j}{\partial X_i}} = 0, \quad i = 1, \ldots, n, \tag{7}$$

has non-trivial solutions over $\mathbb{R}$.

Proof.

Let $j = j_0$ be such that the system (7) has only the zero solution over $\mathbb{R}$. We will arrive at a contradiction by showing that this assumption implies on the one hand, $f(\mathbb{R}^n) = \mathbb{R}^n$, and it also implies, on the other hand, $f(\mathbb{R}^n) \neq \mathbb{R}^n$.

1) We first prove that $f(\mathbb{R}^n) = \mathbb{R}^n$. To see that, we use Corollary 0.3 with:

$$g_{ij}(X_1, \ldots, X_n) = \frac{\partial p_j}{\partial X_i} \quad for\ i, j = 1, \ldots, n.$$

We can assume without losing the generality that:

$$\deg g_{ij_0} = \deg p_{j_0} - 1, \quad i = 1, \ldots, n. \tag{8}$$

For the assumptions of our theorem as well as the conclusion $f(\mathbb{R}^n) = \mathbb{R}^n$, are invariant with respect to a real, non-singular change of the variables. More precisely, instead of working with the original mapping, $f(X_1, \ldots, X_n)$, we could have, first performed a change of the variables, as follows:

$$X_j = \sum_{i=1}^{n} a_{ij} U_i, \quad j = 1, \ldots, n, \tag{9}$$

where $(a_{ij})_{i,j=1,\ldots,n}$ is a real non-singular matrix. Then we could have proved that the mapping given by $F(U_1, \ldots, U_n) = f(X_1, \ldots, X_n)$ is epimorphic and that would have implied that the original mapping $f(X_1, \ldots, X_n)$ is epimorphic. The linear transformation we choose in equation (9) is such that $a_{ij} \neq 0$ for all $i, j = 1, \ldots, n$. With this choice of the linear transformation it is clear that generically (in the $a_{ij} \neq 0$), each of the components $\tilde{p}_j(U_1, \ldots, U_n) = p_j(X_1, \ldots, X_n)$, $j = 1, \ldots, n$, of the mapping $F(U_1, \ldots, U_n)$ has the property that for each $i = 1, \ldots, n$ it contains all the monomials of the form $aU_1^{m_1} \ldots U_n^{m_n}$ where $a \neq 0$, and where $\sum_{k=1}^{n} m_k = \deg p_j$, and $m_i \neq 0$. This justifies equation (8). Next we choose in Corollary 0.3 the following:

For $j \neq j_0$ we take $\alpha_j = 1$. We choose the positive integer α_{j_0} so large that $\alpha_{j_0} \deg p_{j_0} + (\deg p_{j_0} - 1)$ is strictly larger than $\deg p_j + \deg g_{ij}$ for $i = 1, \ldots, n$ and $j \neq j_0$. Also α_{j_0} is such that $\alpha_{j_0} \deg p_{j_0} + (\deg p_{j_0} - 1)$ is an odd integer. That is always possible to do: If $\deg p_{j_0}$ is an even integer, then there is no other restriction on α_{j_0} (except for being large enough). If $\deg p_{j_0}$ is an odd integer, then α_{j_0} must also be an odd integer. Now conditions (i) and (ii) of Corollary 0.3 are satisfied with $j(i) = j_0$. The system (5) Corollary 0.3 reduces to the system (7) with $j = j_0$ and so by part (a) of Corollary 0.3 it follows that $f(\mathbb{R}^n) = \mathbb{R}^n$.

2) In order to conclude the proof of Theorem 0.8, we now prove that the existence of such a j_0 implies that $f(\mathbb{R}^n) \neq \mathbb{R}^n$. We may assume that $\overline{p}_{j_0}(X_1, \ldots, X_n) \geq 0 \ \forall (X_1, \ldots, X_n) \in \mathbb{R}^n$, and there is an equality $\overline{p}_{j_0}(X_1^0, \ldots, X_n^0) = 0$ if and only if $(X_1^0, \ldots, X_n^0) = (0, \ldots, 0)$. Let us denote $d = \deg \overline{p}_{j_0}$. We claim that $\forall i$, $1 \leq i \leq n$ we have $\deg_{X_i} \overline{p}_{j_0} = d$:

For let $\overline{p}_{j_0}(X_1, \ldots, X_n) = \sum_{k=0}^{N} h_k(X_1, \ldots, \hat{X}_i, \ldots, X_n) X_i^k$ where h_k is an homogeneous polynomial in

$$(X_1, \ldots, \hat{X}_i, \ldots, X_n)$$

of degree $d - k$. Then $\overline{p}_{j_0}(0, \ldots, 0, X_i, 0, \ldots, 0) \equiv 0$ for any choice of X_i which is impossible. Hence we obtain:

$$\overline{p}_{j_0}(X_1, \ldots, X_n) = \sum_{i=1}^{n} \lambda_i X_i^d + h(X_1, \ldots, X_n), \quad (10)$$

where $\lambda_i > 0, \forall i, 1 \leq i \leq n$ and where h is homogeneous of degree d such that $\deg_{X_i} h < d$, $\forall i$, $1 \leq i \leq n$. Since $\overline{p}_{j_0} \geq 0$ it follows that d is an even integer and now equation (10) implies the existence of an $M > 0$ such that $\forall (X_1, \ldots, X_n) \in \mathbb{R}^n$ we have $p_{j_0}(X_1, \ldots, X_n) \geq -M$. Hence we conclude that $f(\mathbb{R}^n) \neq \mathbb{R}^n$. Now the proof of the theorem is completed. □

Theorem 1.9: Let the polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, be given by $f(X_1, \ldots, X_n) = (p_1(X_1, \ldots, X_n), \ldots, p_n(X_1, \ldots, X_n))$. Suppose that the determinant $\det J(f)(X_1, \ldots, X_n)$ never vanishes in $\mathbb{R}^n$. If there is an even integral vector $(\alpha_1, \ldots, \alpha_n) \in (2\mathbb{Z}^+)^n$ such that the induced homogeneous system of:

$$\sum_{j=1}^{n} \alpha_j \cdot (p_j(X_1, \ldots, X_n))^{\alpha_j - 1} \frac{\partial p_j}{\partial X_i} = 0, \quad i = 1, \ldots, n, \quad (11)$$

has only the zero solution over $\mathbb{R}$, then $f(\mathbb{R}^n) = \mathbb{R}^n$.

Proof.

Let us consider the following polynomial: $F(X_1, \ldots, X_n) = \sum_{j=1}^{n} (p_j(X1, \ldots, X_n))^{\alpha_j}$. Since $p_j(X_1, \ldots, X_n) \in \mathbb{R}[X_1, \ldots, X_n]$, $\forall j = 1, \ldots, n$ and since the vector $(\alpha_1, \ldots, \alpha_n)$ is an even integral vector, it follows that $\deg F$ is an even integer. Say $\deg F = 2N$ for some $N \in \mathbb{Z}^+$. Clearly, the assumptions as well as the conclusion of Theorem 0.9 are invariant with respect to a real non-singular linear change of the variables. Thus, as we explained in the proof of Theorem 0.8 we can assume that:

$$\deg \left(\frac{\partial F}{\partial X_i} \right) = \deg F - 1 = 2N - 1, \quad i = 1, \ldots, n.$$

Let us take in Theorem 0.2:

$$g_{ij}(X_1, \ldots, X_n) = \frac{\partial p_j}{\partial X_i}, \ \ for \ \ i, j = 1, \ldots, n.$$

The vector of integers in Theorem 0.2 will be $(\alpha_1 - 1, \ldots, \alpha_n - 1)$, and the j'th component of the mapping in Theorem 0.2 will be:

$$(\alpha_j)^{1 - \alpha_j^{-1}} p_j(X_1, \ldots, X_n).$$

These satisfy the conditions (i) and (ii) of Theorem 0.2 and now part (a) of Theorem 0.2 implies that $f(\mathbb{R}^n) = \mathbb{R}^n$. □

Pinchuk's example. See [1], [2].

Pinchuk defined the following:

$$t = xy - 1, \ \ s = 1 + xt, \ \ h = ts, \ \ f = s^2(t^2 + y),$$

and then set,

$$p = h + f, \ \ q = -t^2 - 6th(h+1) - u(f, h),$$

where

$$u = A(h)f + B(h),$$

$$A = h + \frac{1}{45}(13 + 15h)^3,$$

$$B = 4h^3 + 6h^2 + \frac{1}{2}h^2 + \frac{1}{2700}(13 + 15h)^4.$$

Thus we have:

$$\deg h = 5, \ \ \deg f = 10, \ \ \deg p = 10, \ \ \deg q = 25.$$

Pinchuk's example is the following mapping:

$$(p, q) = (x^6 y^4 - 2x^5 y^3 + \ldots, \frac{15^3}{45} x^{15} y^{10} + \ldots).$$

We are interested only in the leading homogeneous components. Thus:

$$p = x^6 y^4 + \ldots, \ \ \frac{\partial p}{\partial x} = 6x^5 y^4 + \ldots, \ \ \frac{\partial p}{\partial y} = 4x^6 y^3 + \ldots.$$

$$q = \frac{15^3}{45} x^{15} y^{10} + \ldots, \ \ \frac{\partial q}{\partial x} =$$

$$= \frac{15^4}{45} x^{14} y^{10} + \ldots, \ \ \frac{15^3 \cdot 10}{45} x^{15} y^9 + \ldots.$$

There are, in this case, two homogeneous systems of equations in (7) of Theorem 0.8:

$$\overline{p} \frac{\partial \overline{p}}{\partial x} = \overline{p} \frac{\partial \overline{p}}{\partial y} = 0,$$

and

$$\overline{q} \frac{\partial \overline{q}}{\partial x} = \overline{q} \frac{\partial \overline{q}}{\partial y} = 0.$$

These reduce to:

$$\begin{array}{ccccc} x^{11}y^8 & = & x^{12}y^7 & = & 0 \\ x^{29}y^{20} & = & x^{30}y^{19} & = & 0 \end{array}.$$

Thus both systems have non-zero solutions:

$$\{(0,y)\,|\,y\in\mathbb{R}\}=\{(x,0)\,|\,x\in\mathbb{R}\},$$

as should be the case according to Theorem 0.8. As for Theorem 0.9: we look at the system

$$\alpha_1 p(x,y)^{\alpha_1-1}\frac{\partial p}{\partial x}+\alpha_2 q(x,y)^{\alpha_2-1}\frac{\partial q}{\partial x}=0$$
$$\alpha_1 p(x,y)^{\alpha_1-1}\frac{\partial p}{\partial y}+\alpha_2 q(x,y)^{\alpha_2-1}\frac{\partial q}{\partial y}=0$$

then since it is well known that the image of Pinchuk's mapping (p,q) equals $\mathbb{R}^2-\{w_0,w_1\}$, the compliment of two points, the induced homogeneous system must have non-trivial solutions for any two even natural numbers α_1 and α_2.

Remark 1.10: The Pinchuk construction gives coordinates with a single element as their highest homogeneous component. This element has the form $\alpha x^m y^k$ where $\alpha\in\mathbb{R}^\times$, $m,k\geq 1$. Thus the equations in (7) of Theorem 0.8 are of the form:

$$x^m y^k\cdot x^{m-1}y^k=x^m y^k\cdot x^m y^{k-1}=0,$$

i.e.

$$x^{2m-1}y^{2k}=x^{2m}y^{2k-1}=0,$$

and so the solution set is the union of both axis:

$$\{(0,y)\,|\,y\in\mathbb{R}\}=\{(x,0)\,|\,x\in\mathbb{R}\},$$

which, of course, is non-trivial in agreement with Theorem 0.8.

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