Coupled FCT-HP for Analytical solutions of the Generalized time-fractional Newell-Whitehead-Segel Equation

S. O. Edeki, J. I. Ejiogu, S. A. Ejoh, and G. A. Adeyemi

***Abstract*—**This paper considers the generalized form of the time-fractional Newell-Whitehead-Segel model (TFNWSM) with regard to exact solutions via the application of Fractional Complex Transform (FCT) coupled with He's polynomials method of solution. This is applied to two forms of the TFNWSM viz: linear and nonlinear versions of the time-fractional Newell-Whitehead-Segel equation whose derivatives are based on Jumarie's sense. The results guarantee the reliability and efficiency of the proposed method with less computation time while still maintaining high level of accuracy.

***Keywords*—**Fractional complex transform; Decomposition method; fractional calculus; Analytical solutions; NWSM.
Mathematics Subject Classification— 83C15, 65H20, 35Q30

I. Introduction

FRACTIONAL calculus (FC) as a branch of applied mathematics deals with different possibilities of how the powers of operators being it differential or integral are defined in terms of real number or complex number. It generalizes the classical corresponding differential calculus of integer orders. Mathematical modelling in most cases, involves partial differential equations (PDEs) of nonlinear and/or linear forms. Meanwhile, obtaining solutions to these model-equations has been a great concern. Hence, the adoption and construction of numerical, semi-analytical methods, and modified semi-analytical techniques and methods [1-11]. Therefore, this work will be on an effective coupled method applied to the time-fractional Newell-Whitehead-Segel equation of the form:

$$\begin{cases} \dfrac{\partial^\alpha w(x,t)}{\partial t^\alpha} = k\dfrac{\partial^2 w(x,t)}{\partial x^2} + aw(x,t) - bw^j(x,t), \\ w(x,0) = f(x),\ \alpha \in (0,1] \end{cases} \tag{1.1}$$

where $a, b \in \mathbb{R}$, and $k, j \in \mathbb{Z}^+$.

Many researchers have considered solving (1.1) at $\alpha = 1$ via some solution techniques [12-15]. The aim of this paper is to obtain the solution of the *time-fractional* NWSE via the application of FCT coupled with He's polynomial solution method [16-20]. This coupled method involves less computational time and work.

II. Method Part I: Jumarie's Fractional Derivative (JFD) [21, 22]

JFD is a modified form of the Riemann-Liouville derivatives. Hence, the definition of JFD and its basic properties as follows:

Let $h(v)$ be a continuous real function of v (not necessarily differentiable), and $D_v^\alpha h = \dfrac{\partial^\alpha h}{\partial v^\alpha}$ denoting JFD of h, of order α w.r.t. v. Then,

$$D_v^\alpha h = \begin{cases} \dfrac{1}{\Gamma(-\alpha)}\dfrac{d}{dv}\displaystyle\int_0^v (v-\lambda)^{-\alpha-1}\left(h(\lambda)-h(0)\right)d\lambda,\ \alpha \in (-\infty, 0) \\ \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dv}\displaystyle\int_0^v (v-\lambda)^{-\alpha}\left(h(\lambda)-h(0)\right)d\lambda,\ \alpha \in (0,1) \\ \left(h^{(\alpha-m)}(v)\right)^{(m)},\ \alpha \in [m, m+1),\ m \geq 1 \end{cases} \tag{2.1}$$

where $\Gamma(\cdot)$ denotes a gamma function. As summarized in [21], the basic properties of JFD are stated as *P1-P5*:

P1: $D_v^\alpha k = 0,\ \alpha > 0$,

P2: $D_v^\alpha \left(kh(v)\right) = kD_v^\alpha h(v),\ \alpha > 0$,

P3: $D_v^\alpha v^\beta = \dfrac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} v^{\beta-\alpha},\ \beta \geq \alpha > 0,$

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P4: $D_v^\alpha\left(h_1(v)h_2(v)\right)=D_v^\alpha h_1(v)\left(h_2(v)\right)+h_1(v)D_v^\alpha h_2(v)$,

P5: $D_v^\alpha\left(h\left(v\left(g\right)\right)\right)=D_v^1 h\cdot D_g^\alpha v$,

where k is a constant.

Note: *P1, P2, P3, P4,* and *P5* are referred to as fractional derivative of: constant function, constant multiple function, power function, product function, and function of function respectively. P5 can be linked to Jumarie's chain rule of fractional derivative.

A. The He's Polynomials and the Generalized non-integer NWSE

Here, we make reference to [15] for details of the He's Polynomials method. Hence, presented below is the structure of the He's Polynomials solution method on the generalized version of in terms of NWSE in terms of integer order.

Consider (1.1) for $\alpha=1$ where $I_0^t(\cdot)$ denotes an integral operator, then:

$$\begin{cases} w=w(0)+I_0^t\left(kw_{xx}+aw-bw^j\right), \\ w(0)=f(x),\ w=w(x,t), w(0)=w(x,0). \end{cases} \quad (2.2)$$

The series solution form can be expressed as:

$$w(x,t)=\sum_{n=0}^{\infty}p^n w_n, \quad (2.3)$$

which is calculated as $p\to\infty$. Hence, by the application of convex homotopy method, (2.2) becomes:

$$\sum_{\eta=0}^{\infty}p^\eta w_\eta=g(x)+I_0^t\left(\begin{array}{l} k\sum_{\eta=0}^{\infty}p^{\eta+1}w_{xx,\eta} \\ +a\sum_{\eta=0}^{\infty}p^{\eta+1}w_\eta-bH_\eta \end{array}\right), \quad (2.4)$$

where H_n, $\eta\in\{0\}\cup\mathbb{N}$ denotes He's polynomials for the concerned nonlinear term, $w^j(x,t)$.

So, by comparison of the $p's$ powers in (2.4), we have:

$$p^{(0)}: w_0=g(x)$$
$$p^{(1)}: w_1=I_0^t\left(kw_{xx,0}+aw_0-bH_0\right)$$
$$p^{(2)}: w_2=I_0^t\left(kw_{xx,1}+aw_1-bH_1\right)$$
$$p^{(3)}: w_3=I_0^t\left(kw_{xx,2}+aw_2-bH_2\right)$$
$$\vdots$$
$$p^{(i)}: w_i=I_0^t\left(kw_{xx,i-1}+aw_{i-1}-bH_{i-1}\right), i\geq 1.$$

Hence, the solution: $w(x,t)=\sum_{n=0}^{\infty}p^n w_n\to\sum_{n=0}^{\infty}w_n$ as $p\to\infty$.

B. The Fractional Complex Transform

In general form, the fractional differential equation of the following form is considered:

$$\begin{cases} f\left(\varpi, D_t^\alpha\varpi, D_x^\beta\varpi, D_y^\lambda\varpi, D_z^\gamma\varpi\right)=0, \\ \varpi=\varpi(t,x,y,z) \end{cases}. \quad (2.5)$$

Then, the Fractional Complex Transform [22] is defined as follows:

$$\begin{cases} T=\dfrac{at^\alpha}{\Gamma(1+\alpha)},\ \alpha\in(0,1], \\ X=\dfrac{bx^\beta}{\Gamma(1+\beta)},\ \beta\in(0,1], \\ Y=\dfrac{cy^\lambda}{\Gamma(1+\lambda)},\ \lambda\in(0,1], \\ Z=\dfrac{dz^\gamma}{\Gamma(1+\gamma)},\ \gamma\in(0,1], \end{cases} \quad (2.6)$$

with a, b, c, and d as unknown constants.

From *P3*,

$$D_v^\alpha v^\beta=\frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}v^{\beta-\alpha},\ \beta\geq\alpha>0,$$

$$\therefore\ D_t^\alpha T=D_t^\alpha\left[\frac{at^\alpha}{\Gamma(1+\alpha)}\right]=\frac{a}{\Gamma(1+\alpha)}D_t^\alpha t^\alpha=a. \quad (2.7)$$

Obviously in a similar manner, using properties *P1-P5*, and the FCT in (2.6), the following are easily obtained:

$$\begin{cases} D_t^\alpha T=\dfrac{\partial^\alpha T}{\partial t^\alpha}=a,\ D_x^\beta X=\dfrac{\partial^\beta X}{\partial x^\beta}=b, \\ D_y^\lambda Y=\dfrac{\partial^\lambda Y}{\partial y^\lambda}=c,\ D_z^\gamma Z=\dfrac{\partial^\gamma Z}{\partial z^\gamma}=d. \end{cases} \quad (2.8)$$

Hence,

$$\begin{cases} D_t^\alpha p(t,x,y,z)=D_t^\alpha p(T(t))=D_T^1 p\cdot D_t^\alpha T=a\dfrac{\partial p}{\partial T}, \\ D_x^\beta p(t,x,y,z)=D_x^\beta p(X(x))=D_X^1 p\cdot D_x^\beta X=b\dfrac{\partial p}{\partial X}, \\ D_y^\lambda p(t,x,y,z)=D_y^\lambda p(Y(y))=D_Y^1 p\cdot D_y^\lambda Y=c\dfrac{\partial p}{\partial Y}, \\ D_z^\gamma p(t,x,y,z)=D_z^\gamma p(Z(z))=D_Z^1 p\cdot D_z^\gamma Z=d\dfrac{\partial p}{\partial Z}. \end{cases} \quad (2.9)$$

Thus, for $p = p(t,x,y,z)$, we have:

$$D_t^\alpha p = a\frac{\partial p}{\partial T},\ D_x^\beta p = b\frac{\partial p}{\partial X},\ D_y^\lambda p = c\frac{\partial p}{\partial Y},\ D_z^\gamma p = d\frac{\partial p}{\partial Z}. \tag{2.10}$$

III. Applications/Examples

A. Consider the linear NWSE as follows [13, 14]:

$$\begin{cases} w_t^\alpha - w_{xx} + 3w = 0, \\ w(x,0) = e^{2x}, \end{cases} \tag{3.1}$$

with an exact solution:

$$w(x,t) = e^{2x+t}. \tag{3.2}$$

Procedure:

By FCT,

$T = \dfrac{at^\alpha}{\Gamma(1+\alpha)}$, which according to section 3 gives $D_t^\alpha w = \dfrac{\partial w}{\partial T} = w_T$ for $a = 1$. Hence, (3.1) becomes:

$$\begin{cases} w_T - w_{xx} + 3w = 0, \\ w(x,0) = e^{2x}, \end{cases} \tag{3.3}$$

The analytical solution of (3.3) according to He's polynomial with reference to (1.1) where $k = 1,\ a = -3,\ b = 0$, and $g(x) = e^{2x}$ gives:

$$\begin{cases} w(x,T) = \left(1 + T + \dfrac{T^2}{2} + \dfrac{T^3}{6} + \dfrac{T^4}{24} + \cdots\right)e^{2x} \\ \qquad = e^{2x+T} \end{cases} \tag{3.4}$$

Whence, the exact solution of (3.1) is:

$$w(x,t) = e^{2x+T} = \exp(2x)\exp\left(\frac{t^\alpha}{\Gamma(1+\alpha)}\right) = e^{2x+\left(\frac{t^\alpha}{\Gamma(1+\alpha)}\right)} \tag{3.5}$$

Note**:** When $\alpha = 1$, we have $w(x,t) = e^{2x+t}$ which corresponds to the exact solution of the classical NWSE of integer order [13, 14].

B. Consider the following linear NWSE

$$\begin{cases} w_t^\alpha = w_{xx} + 2w - 3w^2, \\ w(x,0) = 1, \end{cases} \tag{3.6}$$

Procedure w.r.t Case2:

When (3.6) and (1.1) are compared, we have: $a = 2,\ k = 5,\ j = 2\ b = -1$, and $g(x) = \eta$. Hence, by the FCT,

$T = \dfrac{at^\alpha}{\Gamma(1+\alpha)}$, which according to section 3 gives $D_t^\alpha w = \dfrac{\partial w}{\partial T} = w_T$ for $a = 1$. Hence, (3.6) becomes:

$$\begin{cases} w_T = w_{xx} + 2w - 3w^2, \\ w(x,0) = 1, \end{cases} \tag{3.7}$$

The analytical solution of (3.7) according to He's polynomial with reference to (1.1) where $k = 1,\ a = -3,\ b = 0$, and $g(x) = e^{2x}$ gives:

$$w(x,T) = 1 - T + 2T^2 - \frac{11T^3}{3} + \frac{20T^4}{3} + \cdots$$

$$= \frac{2\eta e^{2T}}{-1 + e^{2T}} \tag{3.8}$$

Whence, the exact solution of (3.6) is:

$$w(x,t) = \frac{2\exp\left(\dfrac{2t^\alpha}{\Gamma(1+\alpha)}\right)}{-1 + \exp\left(\dfrac{2t^\alpha}{\Gamma(1+\alpha)}\right)} \tag{3.9}$$

Note**:** When $\alpha = 1$, we have $w(x,t) = e^{2x+t}$ which corresponds to the exact solution of the classical NWSE of integer order [13, 14].

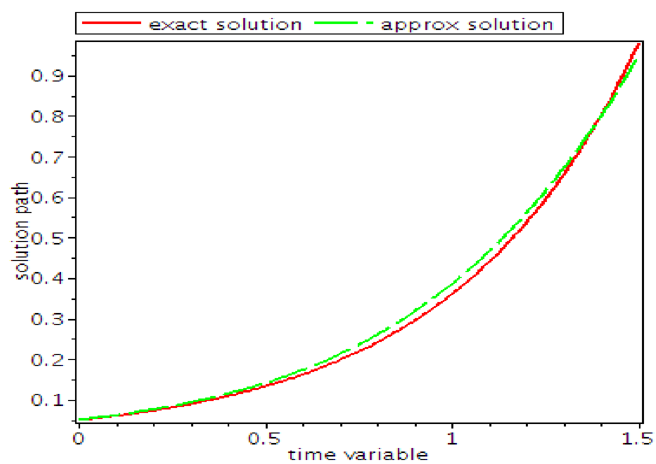

Figure 2: Exact and FCT-HP solutions for case 2

IV. CONCLUDING REMARKS

Fractional Complex Transform (FCT) coupled with He's polynomials method were successfully obtained. The associated derivatives were defined in terms of Jumarie's sense. Based on the solved linear and nonlinear problems, reliability and efficiency of the proposed solution method are assured by the obtained results as less computational time is involved with high basic knowledge of fractional calculus not necessarily required. The method is therefore, recommended for space-fractional derivatives of higher-ordered problems arising from other areas of pure and applied sciences. This can also be benchmarked with the Restarted Adomian decomposition Method and other approaches [23, 24].

ACKNOWLEDGMENT

The authors are grateful to Covenant University for enabling working environment and financial support. Sincere thanks to the anonymous referee(s) for their productive comments.

Conflict of Interests

The authors declare that they have no conflict of interest regarding the publication of this paper.

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