Filters in Michálek's Fuzzy Topological Spaces

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Abstract—The aim of this paper is to study some properties of filters in Michálek's fuzzy topological spaces, which are quite different of the classic properties of fuzzy topology. That continues a previous paper of this author.

Keywords—fuzzy sets, topology, filters, convergence

I. Introduction

The first notion of fuzzy topological spaces has been defined by C.L. Chang, in 1968 [1]. This definition is the natural translation to fuzzy sets of the ordinary notion of topological spaces. J.Michálek defined and studied another concept of fuzzy topological space [2] which is quite different of the classic Chang's definition. We have studied in [3] some properties of these new spaces, as C.K. Wong proposed in his review [4] of Michálek's paper. Now, we will study filters in Michálek's fuzzy topological spaces.

First, we give some previous definitions:

Definition 1.[2] Let X be a non-empty set, let $\mathcal{P}$ (X) be the system of all subsets of the set X, and I^X is the system of all fuzzy sets in X. A pair $\langle$ X,u $\rangle$ is called fuzzy topological space supposing that u is a mapping from $\mathcal{P}$ (X) to I^X satisfying the following three axioms:

1. if A⊂X, then u A(x)=1 for all x∈A,

2. if A⊂X contains at most one element, then u A(x)= χ_A (x), where χ_A is the characteristic function of the set A,

3. if A_1⊂X, A_2⊂X, then u(A_1∪A_2)(x)=max{uA_1(x),uA_2(x)}

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Definition 2. ([1]) Let X and Y be two sets and let φ be a map from X to Y. Let μ be a fuzzy set in Y, then the inverse image of μ, written as φ^{-1}(μ), is defined by φ^{-1}(μ)(x)=μ(φ(x)) for all x in X. Conversely, if υ is a fuzzy set in X, the image of ν, written as φ(υ) is a fuzzy set in Y given by $\varphi(\nu)(y) =$

$$\{\sup \nu(x) / x \in \varphi^{-1}(y)\}, if\, \varphi^{-1}(y) \neq \emptyset$$

or 0, otherwise.

Definition 3. ([3]) Let $\langle$ X,u $\rangle$, $\langle$ Y,v $\rangle$ be two Michálek's fuzzy topological spaces and let φ be a map from X to Y. We say that φ is compatible with u and v if, for all B∈ $\mathcal{P}$ (Y), we have that u(φ^{-1}(B))= φ^{-1}(v(B)).

Definition 4. ([2]) Let A⊂X, A^c = X-A, then the fuzzy set μ_{A^o} where μ_{A^o} (x)=1-uA^c(x) is called the fuzzy interior set of the set A.

Definition 5. ([2]) A subset U⊂X is called to be a fuzzy neighborhood of an element a∈X if u(a)(x)≤ μ_{U^o} (x) for every x∈X.

Lemma 1. ([2]) A set U⊂X is a fuzzy neighborhood of an element a∈X if and only if uU^c(a)=0.

Definition 6. ([2]) Let $\langle$ X,u $\rangle$ be a fuzzy topological space, a∈X, we denote Σ(a)={U⊂X|uU^c(a)=0}.

II. Main results

Proposition 1. Let $\langle$ X,u $\rangle$ and $\langle$ Y,v $\rangle$ be two fuzzy topological spaces, and φ:X→Y be a compatible map with u and v, W⊂Y, a∈X. Then W is a fuzzy neighborhood of φ (a) in 〈Y,v〉 if and only if φ^{-1}(W) is a fuzzy neighborhood of a in 〈X,u〉.

Proof. u(φ^{-1} (W))^c(a)=uφ^{-1}(W^c))(a)=
φ^{-1}(v(W^c))(a)=v(W^c)(φ(a)), and W is a fuzzy neighborhood of φ (a), if and only if v(W^c)(φ(a))=0 (i.e. u(φ^{-1}(W))^c(a)=0), and φ^{-1}(W) is a fuzzy neighborhood of a.

Corollary. Let 〈X,u〉 and 〈Y,v〉 be two fuzzy topological spaces, and φ:X→Y be a compatible map with u and v, then φ^{-1}(Σ(φ(a))⊂Σ(a) for every a∈X.

Definition 7. Let X be a non-empty set. A filter on X is a non-empty family F ⊂ P (X) with all members of F non-empty, and which has the following properties:

i) Every finite intersection of sets in F belongs to it.

ii) ii) Every subset of X which contains a set of F belongs to F.

Remark 1. Let X and Y be two non-empty sets and φ be a map from X to Y. If F is a filter on X, then

{C⊂Y / C⊃ φ (F) for some F∈ F } is other filter on Y. We will denote this filter as φ (F).

We will define and study convergence of filters on Michálek's fuzzy topological spaces. This is interesting, because the results are quite different of respective for ordinary topological spaces.

Definition 8. Let 〈X, u〉 be a fuzzy topological space and be a filter F on X. A point a∈X will be a limit of F, if F contains the systems of neighborhoods Σ(a) of a.

Definition 9. Let 〈X,u〉 be a fuzzy topological space and F be a filter on X. A point a∈X will be a cluster point of F, if every fuzzy neighborhood of a meets every member of F.

Remark. Obviously, every limit of a filter is a cluster point of it.

Proposition 2. Let 〈X,u〉 and 〈Y,v〉 be two fuzzy topological spaces, and φ:X→Y be a compatible map with u and v. Then, for every point a∈X and every filter F on X convergent to a, we have that φ (F) converges to φ (a).

Proof. For every fuzzy neighborhood W of φ (a), we have (by Proposition 1) that φ^{-1}(W) is a fuzzy neighborhood of a. Then, φ^{-1}(W) ∈ F and W∈φ(F).

Proposition 3. Let 〈X,u〉 and 〈Y,v〉 be two fuzzy topological spaces, and φ:X→Y be a map such that for every point a∈X and every filter F on X such that F converges to a, is φ (F) convergent to φ (a).Then, φ is not necessarily compatible with u and v.

Proof. Let X be an infinite set, and u: P (X)→I^X defined by u A= χ_A if A is finite, and u A=c_1 (the constant map of value 1) if A is infinite.

Let Y be a set, and v: P (Y)→I^Y defined by v B= χ_B for all subset B of Y.

Thus, the respective systems of neighborhoods are:

If a∈X, Σ(a)={A⊂X | uA^c(a)=0}={A⊂X | uA (a)=1}=

{A⊂X | A is finite and a∈A}∪{A⊂X | A is infinite}.

If b∈Y, Σ (b)={B⊂Y | b∈B}.

Let a be an arbitrary point of X and F be an arbitrary filter on X which converges to a (i.e. Σ(a)⊂ F, then {a}∈ F, and {φ (a)}∈ φ (F)).Then Σ (φ (a))⊂ φ (F), that is, φ (F) converges to φ (a).

But, if B⊂Y, φ^{-1}(vB)= $\chi_{\varphi^{-1}(B)}$ and u(φ^{-1}(B)) is $\chi_{\varphi^{-1}(B)}$ only if φ^{-1}(B) is finite, and c_1 in other case. Then, φ is not compatible with u and v.

Remark. In General Topology, for topological spaces, a point is a cluster point of a filter on X if and only if it lies in the closure of all the members of the filter. For Michálek's fuzzy topological spaces, the situation is also quite different.

Proposition 4. Let 〈X,u〉 be a fuzzy topological space, a∈X, and F be a filter on X. Then, if a is cluster point of F, that is not equivalent to uF (a)=1 for all F∈ F.

Proof. Let X be an infinite set, and u: P (X)→I^X defined (as above) by u A=χ_A if A is finite, and u A=c_1 (the constant map of value 1) if A is infinite. Then 〈 X,u 〉is a fuzzy topological space and for every a∈X, Σ(a)={A⊂X| uA^c(a)=0}={A⊂X | uA (a)=1}= {A⊂X| A is finite and a∈A}∪{A⊂X| A is infinite}.

So, if F ={X-F| F⊂X, and F is finite}, for every a ∈X, {a}∈Σ(a) and X-{a}∈ F .

Then, the filter F has not cluster points in 〈 X,u〉. But, for every F∈ F , F is infinite, then F∈ Σ(a), or equivalently, uF (a)=1 for every F∈ F .

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