M – STAR – Irresolute Topological Vector Spaces

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Abstract—**In 2016 A. Devika and A. Thilagavathi introduced a new class of sets called** $M^* - $open **sets and investigated some properties of these sets in topological spaces. In this paper, we introduce and study a new class of spaces, namely** $M^* - $irresolute **topological vector spaces via** $M^* - $open **sets. We explore and investigate several properties and characterizations of this new notion of** $M^* - $irresolute **topological vector space. We give several characterizations of** $M^* - $Hausdorff **space. Moreover, we show that the extreme point of the convex subset of** $M^* - $irresolute **topological vector space** X **lies on the boundary.**

Keywords—Topological space, $M^* - $open *set,* $M^* - $closed *set,* $M^* - $irresolute *topological vector space,* $M^* - $irresolute *function.*

I. INTRODUCTION

The concept of topological vector spaces was first given by Kolmogroff in 1934. Since then, different researchers have explored many interesting and useful properties of topological vector spaces. Due to nice properties, topological vector spaces earn a great importance and remain a topological notion in various advanced branches of mathematics like fixed point theory, operator theory, rational inequalities, etc. Now a days the researchers not only make use of topological vector spaces in other fields for developing new notions but also extend the concept of topological vector spaces with every possible way, making this field of study a more convenient and understandable. Recently, M. Khan et al. (2015) introduced the $s-topological$ vector spaces, a generalization of topological vector spaces. In 2016, Khan and Iqbal introduced the irresolute topological vector spaces which are particular brands of $s-topological$ vector spaces, but irresolute topological vector spaces are independent of topological vector spaces. Ibrahim (2017) initiated the study of $\alpha-topological$ vector spaces. Ram et al. (2018) introduced the notion of almost $s\text{-}topological$ vector spaces. Recently in 2016 A. Devika and A. Thilagavathi introduced a new class of open sets called $M^* - $open sets in topological spaces and showed that the family of all $M^* - $open subsets of a topological space (X, τ) forms a topology on X. We introduce $M^* - $irresolute topological vector spaces by using $M^* - $open sets and investigate several properties and characterizations of this notion of $M^* - $irresolute topological vector space.

2. M^*-OPEN SETS IN TOPOLOGICAL SPACES

The (X, τ) and (Y, σ) $(or\ simply,\ X\ and\ Y)$ denote topological spaces on which no separation axioms are assumed unless explicitly stated. A subset A of a topological space (X, τ) is said to be open if $A \in \tau$. A subset A of a topological space X is said to be closed if the set $X \mathbf{B} A$ is open. The interior of a subset A of a topological space X is defined as the union of all open sets contained in A. It is denoted by $Int(A)$. The closure of a subset A of a topological space X is defined as the intersection of all closed sets containing A.

We start recalling following definitions and results from $[\text{Devika \& Thilavagathi}:2016]$, which are necessary for this study in sequel.

Definition 2.1. A subset A of a topological space (X,τ) is said to be *semi-open* set if $A \subseteq Cl\lfloor Int(A) \rfloor$.

Definition 2.2. A subset A of a topological space (X,τ) is said to be regular open set if $A = Int\lfloor Cl(A) \rfloor$ `and a regular closed set if $A = Cl\lfloor Int(A) \rfloor$.

Definition 2.3. Let (X,τ) be a topological space and $A \subseteq X$. A point $x \in X$ is said to be $\theta-interior$ point of A if there exists an open set U containg x such that $U \subseteq Cl(U) \subseteq A$. The set of all $\theta-interior$ points of A is said to be the $\theta-interior$ of A and is denoted by $Int_\theta(A)$. A subset A of X is said to be $\theta-open$ if $A = Int_\theta(A)$. The set of all $\theta-open$ sets in a topological space (X,τ) is denoted by τ_θ. We note that τ_θ is a topology on X and $\tau_\theta \subseteq \tau$. For a subset A of X, $\theta-closure$ of A denoted by $Cl_\theta(A)$ is defined by $Cl_\theta(A) = \bigcap\{B \subseteq X : A \subseteq B \text{ and } B \text{ is } \theta-closed \text{ in } X\}$.

Definition 2.4. A point $x \in X$ is called the $\delta-cluster$ point of $A \subseteq X$ if $A \bigcap Int\lfloor Cl(U) \rfloor \neq \phi$ for every open set U of X containing x. The set of all $\delta-cluster$ points of A is called the $\delta-closure$ of A, denoted by $Cl_\delta(A)$. A subset $A \subseteq X$ is called $\delta-closed$ if $A = Cl_\delta(A)$. The complement of a $\delta-closed$ set is called $\delta-open$. The collection of all $\delta-open$ sets in a topological space (X,τ) forms a topology τ_δ on X, called the semigeneralization topology of τ, weaker than τ and the class of all regular open sets in τ forms an open basis for τ_δ.

We observe that for any topological space (X,τ) the relation $\tau_\theta \subseteq \tau_\delta \subseteq \tau$ always holds.

Definition 2.5. A subset A of a topological space (X,τ) is said to be θ-*semi-open* set if $A \subseteq Cl\lfloor Int_\theta(A) \rfloor$.

Definition 2.6. A subset A of a topological space (X,τ) is said to be M-*open* set if $A \subseteq Cl\lfloor Int_\theta(A) \rfloor \bigcup Int\lfloor Cl_\delta(A) \rfloor$.

Definition 2.7. The complement of a $semi-open$ $(resp.\ \theta-semi-open,\ M\text{-}open)$ set is called $semi-closed$ $(resp.\ \theta-semi-closed,\ M\text{-}closed)$.

Definition 2.8. The intersection of all $semi-closed$ $(resp.\ \theta-semi-closed,\ M-closed)$ sets containing A is called the $semi-closure$ $(resp.\ \theta-semi-closure,\ M-closure)$ of A and is denoted by $sCl(A)$ $(resp.\ sCl_\theta(A),\ M-Cl(A))$.

Definition 2.9. The union of all $semi-open$ $(resp.\ \theta-semi-open,\ M-open)$ sets containing A is called the $semi-interior$ $(resp.\ \theta-semi-interior,\ M-interior)$ of A and is denoted by $sInt(A)$ $(resp.\ sInt_\theta(A),\ M-Int(A))$.

Lemma 2.10. Let (X,τ) be a topological space and $A,B \subseteq X$. Then the following statements are true :

(1) A is $\theta-open$ open if and only if $A = Int_\theta(A)$.

(2) $Int_\theta(A)$ is the union of all $\theta-open$ sets of X whose closures are contained in A.

(3) For any subset A of X, $A \subseteq Cl(A) \subseteq Cl_\delta(A) \subseteq Cl_\theta(A)$ $(resp.\ Int_\theta(A) \subseteq Int_\delta(A) \subseteq Int(A) \subseteq A)$. (4)

$Int_\theta(A \bigcap B) = Int_\theta(A) \bigcap Int_\theta(B)$.

$Int_\theta(A) \bigcup Int_\theta(B) \subseteq Int_\theta(A \bigcup B)$.

(5) $Cl_\theta(A \bigcap B) = Cl_\theta(A) \bigcap Cl_\theta(B)$.

$Cl_\theta(A \bigcup B) = Cl_\theta(A) \bigcup Cl_\theta(B)$.

Definition 2.11. Let (X,τ) be a topological space. Then a subset A of a space (X,τ) is said to be,

(1) an M^*-*open* set if $A \subseteq Int\left\lfloor Cl\left(Int_\theta(A)\right)\right\rfloor$.

(2) an M^*-*closed* set if $Cl\left\lfloor Int\left(Cl_\theta(A)\right)\right\rfloor \subseteq A$.

Lemma 2.12. Let (X,τ) be a topological space. Then the following statements are true :

(1). Every $\theta-open$ set is an M^*- *open* set.

(2) Every M^*-*open* set is a $\theta-semi-open$ set.

(3) Every M^*-*open* set is an M-*open* set.

Lemma 2.13. Let (X,τ) be a topological space. Then the following statements are true:

(1) The arbitrary union of M^*-*open* sets is M^*-*open*.

(2) The arbitrary intersection of M^*-*closed* sets is M^*-*closed*.

Lemma 2.14. For a topological space (X,τ) the family of all M^*-open sets of X forms a topology denoted by $\tau_{M^*} = M^*\text{-}O(X,\tau) = M^*\text{-}O(X)$ for X.

Definition 2.15. Let A be a subset of a space (X,τ). Then,

(1) The intersection of all $M^*-closed$ sets containing A is called the M^*-*closure* of A and is denoted by $M^* - Cl(A)$.

(2) The union of all M^*-*open* sets contained in A is called the M^*-*Interior* of A and is denoted by $M^*\text{-}Int(A)$.

Theorem 2.16. The following statements hold for a subset A of a space (X,τ).

(1) A is M^*-*open* if and only if $A = A \cap Int\left\lfloor Cl\left(Int_\theta(A)\right)\right\rfloor$.

(2) A is M^*-*closed* if and only if $A = A \cup Cl\left\lfloor Int\left(Cl_\theta(A)\right)\right\rfloor$.

Theorem 2.17. The following statements hold for a subset A of a space (X,τ).

(1) $M^* - Int(A) = A \cap Int\left\lfloor Cl\left(Int_\theta(A)\right)\right\rfloor$.

(2) $M^*\text{-}Cl(A) = A \cup Cl\left[Int\left(Cl_\theta(A)\right)\right]$.

Theorem 2.18. Let A be a subset of a topological space (X,τ). Then the following statements hold :

(1) A is an M^*-*open* set if and only if $A = M^*\text{-}Int(A)$.

(2) A is an M^*-*closed* set if and only if $A = M^*\text{-}Cl(A)$.

Prpoosition 2.19. A subset A of a topological space (X,τ) is an $M^*-closed$ set if and only if $X \mathbf{B} A$ is an M^*-open set. The set of all $M^*-closed$ sets in (X,τ) is denoted by $M^* - C(X) = M^* - C(X,\tau)$.

Theorem 2.20 Let A be a subset of a topological space (X,τ). Then the following statements hold :

(1) $M^*\text{-}Cl(X \mathbf{B} A) = X \mathbf{B} M^*\text{-}Int(A)$.

(2) $M^*\text{-}Int(X \mathbf{B} A) = X \mathbf{B} M^*\text{-}Cl(A)$.

(3) If $A \subseteq B$ then $M^* - Cl(A) \subseteq M^*\text{-}Cl(B)$ and $M^*\text{-}Int(A) \subseteq M^*\text{-}Int(B)$.

(4) $M^*\text{-}Cl\left[M^*\text{-}Cl(A)\right] = M^*\text{-}Cl(A)$ and $M^* - Int\left\lfloor M^* - Int(A)\right\rfloor = M^* - Int(A)$.

(5) $M^*\text{-}Cl(A) \cup M^*\text{-}Cl(B) \subseteq M^*\text{-}Cl(A \cup B)$ and $M^*\text{-}Int(A) \cup M^*\text{-}Int(B) \subseteq M^*\text{-}Int(A \cup B)$.

(6) $M^*\text{-}Cl(A \cap B) \subseteq M^*\text{-}Cl(A) \cap M^*\text{-}Cl(B)$ and $M^*\text{-}Int(A \cap B) \subseteq M^*\text{-}Int(A) \cap M^*\text{-}Int(B)$.

(7) $M^*\text{-}Cl(A) = A \cup sInt_\theta\left[Cl_\theta(A)\right]$.

(8) $M^*\text{-}Int(A) = A \cap sCl_\theta\left[Int_\theta(A)\right]$.

Theorem 2.21. The following are equivalent for a subset A of (X,τ).

(1) A is an M^*-open set.

(2) $A \subseteq sCl_\theta \lfloor Int_\theta(A) \rfloor$.

(3) $sCl_\theta(A) = sCl_\theta \lfloor Int_\theta(A) \rfloor$.

Theorem 2.22. Let A be a subset of a space (X,τ). Then the following are equivalent :

(1) A is an $M^*-closed$ set.

(2) $sInt_\theta \lfloor Cl_\theta(A) \rfloor \subseteq A$.

(3) $sInt_\theta(A) = sInt_\theta \lfloor Cl_\theta(A) \rfloor$.

Definition 2.23. A subset A of a topological space (X,τ) is said to be an $M^*-neighborhood$ $\left(Briefly\ M^*-nbd\right)$ of a point $p \in X$ if there exists an M^*-open set W such that $p \in W \subseteq A.$ The class of $M^*-neighborhoods$ of $p \in X$ is called the $M^*-neighborhood$ system of p and denoted by $M^* - N_p(X) = M^* - N_p(X,\tau) = M^* - N(p).$

Theorem 2.24. A subset G of a space X is M^*-open if and only if it is M^*-nbd for every point $p \in G.$

Definition 2.25 A mapping $f:(X,\tau) \rightarrow (Y,\sigma)$ is called $M^*-irresolute$ at a point $x \in X$ if for all M^*-open subsets V in Y containing $f(x),$ there is an M^*-open subset U of X such that $f(U)$ is a subset of $V.$ The function f will be called $M^*-irresolute$ if f is $M^*-irresolute$ at each point $x \in X.$

Theorem 2.26 Let $f:(X,\tau) \rightarrow (Y,\sigma)$ be a function. Then the following statements are equivalent.

(1) f is $M^*-irresolute.$

(2) For each $x \in X$ and each $M^*-neighborhood$ V of $f(x)$ in Y, there is an $M^*-neighborhood$ U of x in X such that $f(U) \subseteq V.$

(3) The inverse image of every $M^*-closed$ subset of Y is an $M^*-closed$ subset of X.

(4) The inverse image of every M^*-open subset of Y is an M^*-open subset of X.

Definition 2.27. A function $f:(X,\tau) \rightarrow (Y,\sigma)$ is called $M^*-continuous$ if $f^{-1}(V)$ is M^*-open set in X for every open set V in $Y.$

Proposition 2.28. A function $f:(X,\tau) \rightarrow (Y,\sigma)$ is called $M^*-continuous$ if and only if $f^{-1}(V)$ is $M^*-closed$ set in X for every closed set V in $Y.$

Definition 2.29. A function $f:(X,\tau) \rightarrow (Y,\sigma)$ is called $Pre-M^*-open$ if and only if the image set $f(U)$ is M^*-open set in Y for every M^*-open set U in $X.$

Proposition 2.30. A bijection function $f:(X,\tau) \rightarrow (Y,\sigma)$ is called $M^*-homeomorphism$ if f is $pre-M^*-open$ and $M^*-irreseolute.$

3. PROPERTIES OF M$^\bullet$-IRRESOLUTE TOPOLOGICAL VECTOR SPACES

In this section, we define and investigate some basic properties of $M^*-irresolute$ topological vector spaces.

Definition 3.1. A topological space $\left(X_{(K)},\tau\right)$ is called an $M^*-irresolute$ topological vector space $\left(M^*ITVS\right)$ whenever the following conditions are satisfied.

(1) for each $x, y \in X$ and for each M^*-open neighborhood W of $x+y$ in $X,$ there exist M^*-open neighborhoods U and V in X of x and y respectively, such that $U+V \subseteq W.$

(2) for each $x \in X,$ $\lambda \in K$ and for each M^*-open neighborhood W of λx in $X,$ there exist M^*-open neighborhoods U of λ in K and V of x in $X,$ such that $U.V \subseteq W.$

Theorem 3.2. Let $\left(X_{(K)},\tau\right)$ be an $M^*-irresolute$ topological vector spaces. Then the following

assertions are true:

(1) The (left) right translation mapping $T_x : X \to X$ defined by $T_x(y) = y + x$; for all $x, y \in X$, is M^*-irresolute.

(2) The multiplication mapping $M_\lambda : X \to X$, defined by $M_\lambda(x) = \lambda x$, for all $x \in X$, is M^*-irresolute.

Proof. (1) Let W be an M^*-open neighborhood of $T_x(y) = x + y$. Then by definition, there exist M^*-open neighborhoods U and V in X containing y and x respectively, such that $U + V \subseteq W$. This gives that $T_x(U) = U + x \subseteq U + V \subseteq W$. This proves that, $T_x : X \to X$ is an M^*-irresolute mapping.

(2) Let $x \in X$, $\lambda \in K$. Then $M_\lambda(x) = \lambda x$. Let W be any M^*-open neighborhood W of λx. Then by definition of $M^* ITVS$, there exist M^*-open neighborhoods U in K of λ and V in X of x, such that $U.V \subseteq W$. This gives that $M_\lambda(V) = \lambda V \subseteq U.V \subseteq W$. This proves that $M_\lambda : X \to X$ is M^*-irresolute mapping.

Theorem 3.3. Let $\left(X_{(K)}, \tau\right)$ be an M^*-irresolute topological vector space. Let $A \in M^*$-$O(X)$. Then the following statements are true.

(1) $x + A \in M^*$-$O(X)$, for every $x \in X$.

(2) $\lambda A \in M^*$-$O(X)$, for every $non-zero$ scalar $\lambda \in K$.

Proof. (1) $y \in x + A$. Then $y = x + a$ for some $a \in A$. By definition of M^*-irresolute topological vector spaces, there exist M^*-open sets U and V in X containing $-x$ and y respectively such that $U + V \subseteq A$. This gives $-x + V \subseteq U + V \subseteq A$. This implies $y \in V \subseteq x + A$. Therefore $y \in M^*$-$Int(x + A)$. Hence, $x + A = M^*$-$Int(x + A)$. This proves that $x + A$ is M^*-open in X.

(2) Let $x \in \lambda A$. Then $x = \lambda a$ for some $a \in A$. Thus $a = \frac{1}{\lambda} x \in A$. By definition of M^*-irresolute topological vector spaces, there exist M^*-open sets

U in K containing $\frac{1}{\lambda}$ and V in X containing x such that $U.V \subseteq A$. This implies that $a = \frac{1}{\lambda} x \in \frac{1}{\lambda} V \subseteq U.V \subseteq A$. Hence $x \in V \subseteq \lambda A$. Thus we obtain $x \in M^*$-$Int(\lambda A)$. Therefore, it follows that $\lambda A \subseteq M^*$-$Int(\lambda A)$. Hence, $\lambda A = M^*$-$Int(\lambda A)$. This shows that $\lambda A \in M^*$-$O(X)$.

Corollary 3.4. Let $\left(X_{(K)}, \tau\right)$ be an M^*-irresolute topological vector space. Let A be an M^*-open subset of X. Then the following statements are true.

(1) $x + A \subseteq M^*$-$Cl\left[M^*$-$Int(x + A)\right]$ for each $x \in X$.

(2) $\lambda A \subseteq M^*$-$Cl\left[M^*$-$Int(\lambda A)\right]$ for any non-zero scalar λ.

Theorem 3.5. Let $\left(X_{(K)}, \tau\right)$ be an M^*-irresolute topological vector space. Let $A \in M^*$-$O(X)$ and B be any subset of X. Then prove that $A + B \in M^*$-$O(X)$.

Proof. Suppose $A \in M^*$-$O(X)$. and $B \subseteq X$. Then, for each $b \in B$ and by Theorem 3.3 (1), we have $A + b \in M^*$-$O(X)$. Then $A + B = \bigcup\{A + b : b \in B\}$ is a union of M^*-open sets. Since arbitrary union of M^*-open sets is M^*-open, therefore $A + B$ is M^*-open in X.

Corollary 3.6. Suppose $\left(X_{(K)}, \tau\right)$ is M^*-irresolute topological vector space and let $A \in M^*$-$O(X)$. Then the set $U = \bigcup_{n=1}^{\infty}(nA)$ is M^*-open in X.

Theorem 3.7. Let $\left(X_{(K)}, \tau\right)$ be an M^*-irresolute topological vector space. Let $A \subseteq X$. Then the following statements are true

(1) M^*-$Int(x + A) = x + M^*$-$Int(A)$ for any $x \in X$.

(2) M^*-$Int(\lambda A) = \lambda\left[M^*$-$Int(A)\right]$ for non-zero scalar $\lambda \in K$.

Proof. (1) By Theorem 3.3 (1), $x + M^*$-$Int(A)$ is M^*-open. Therefore $x + M^*$-$Int(A) \subseteq x + A$ implies $x + M^*$-$Int(A) \subseteq M^*$-$Int(x + A)$. Now let $z \in M^*$-$Int(x + A)$. Then $z = x + y$ for some

$y \in A$. By definition of M^*ITVS, there exist M^*-*open* sets U and V in X containing x and y respectively, such that $U+V \subseteq x + M^*$-$Int(A)$. This gives that $z = x+y \in x+V \subseteq U+V \subseteq x+M^*$-$Int(A)$. Therefore it follows that M^*-$Int(x+A) \subseteq x+M^*$-$Int(A)$.

(2) By Theorem 3.3(2), $\lambda \left[M^*$-$Int(A) \right]$ is M^*-*open*. Thus $\lambda \left[M^*$-$Int(A) \right] \subseteq \lambda A$ implies that $\lambda \left[M^*$-$Int(A) \right] \subseteq M^*$-$Int(\lambda A)$. Next, if $y \in M^*$-$Int(\lambda A)$, then $y = \lambda x$ for some $x \in A$. By definition of M^*ITVS, there exist M^*-*open* sets U of λ in K and V of x in X such that $U.V \subseteq M^*$-$Int(\lambda A)$. Therefore, $y = \lambda x \in \lambda V \subseteq U.V \subseteq M^*$-$Int(\lambda A) \subseteq \lambda A$. This implies that $x \in V \subseteq A$. Since V is M^*-*open*. Thus $x \in M^*$-$Int(A)$. Consequently, $y = \lambda x \in \lambda \left[M^*$-$Int(A) \right]$. Hence we conclude that M^*-$Int(\lambda A) = \lambda \left[M^*$-$Int(A) \right]$.

Theorem 3.8. Let $\left(X_{(K)}, \tau \right)$ be an M^*-*irresolute* topological vector space. Let $A, B \subseteq X$. Then M^*-$Int(A) + M^*$-$Int(B) \subseteq M^*$-$Int(A+B)$.

Proof. We know that M^*-$Int(A) \subseteq A$ and M^*-$Int(B) \subseteq B$. Thus M^*-$Int(A) + M^*$-$Int(B) \subseteq A+B$. By Theorem 3.5, M^*-$Int(A) + M^*$-$Int(B)$ is M^*-*open*. Therefore we have M^*-$Int(A) + M^*$-$Int(B) = M^*$-$Int \left[M^*$-$Int(A) + M^*$-$Int(B) \right] \subseteq M^*$-$Int(A+B)$. Hence M^*-$Int(A) + M^*$-$Int(B) \subseteq M^*$-$Int(A+B)$.

Theorem 3.9. Let F be any M^*-*closed* subset of an M^*-*irresolute* topological vector space X. Then the following statements are true:

(1) $x + F \in M^*$-$C(X)$, for every $x \in X$.

(2) $\lambda F \in M^*$-$C(X)$, for each non – zero scalar $\lambda \in K$.

Proof. (1) Suppose that $y \in M^*$-$Cl(x+F)$. Consider $z = -x+y$ and let W be any M^*-*open* set in X containing z. Then there exist M^*-*open*

sets U and V in X such that $-x \in U$, $y \in V$ and $U + V \subseteq W$. Since $y \in M^*$-$Cl(x+F)$, $(x+F) \bigcap V \neq \phi$. So, there is an element $a \in (x+F) \bigcap V$. Thus $a \in x+F$ and $a \in V$. Hence $-x+a \in F$ and $-x+a \in U+V$. Therefore $-x+a \in F \bigcap (U+V) \subseteq F \bigcap W$. Thus $F \bigcap W \neq \phi$. Therefore $z \in M^*$-$Cl(F) = F$. Hence $y \in x+F$. Thus $x+F = M^*$-$Cl(x+F)$. This proves that $x+F$ is M^*-*closed* set in X.

(2) Assume that $x \in M^*$-$Cl(\lambda F)$. Let W be any M^*-*open* neighborhood of $y = \dfrac{1}{\lambda} x$ in X. Since X is M^*ITVS, there exist M^*-*open* sets U in K containing $\dfrac{1}{\lambda}$ and V in X containing x such that $U.V \subseteq W$. By hypothesis, $(\lambda F) \bigcap V \neq \phi$. Therefore, there is an element $a \in (\lambda F) \bigcap V$. Thus $a \in \lambda F$ and $a \in V$. Hence $\dfrac{1}{\lambda} a \in F$ and $\dfrac{1}{\lambda} a \in \dfrac{1}{\lambda} V \subseteq U.V \subseteq W$. Therefore $F \bigcap W \neq \phi$. Hence $y \in M^*$-$Cl(F) = F$. Thus $x \in \lambda F$ and thereby, $\lambda F = M^*$-$Cl(\lambda F)$. Hence $\lambda F \in M^*$-$C(X)$.

Corollary 3.10. Let $\left(X_{(K)}, \tau \right)$ be an M^*-*irresolute* topological vector space and let $A \subseteq X$. Then M^*-$Cl \left[x + M^*$-$Cl(A) \right] = x + M^*$-$Cl(A)$ for each $x \in X$.

Theorem 3.11. Let $\left(X_{(K)}, \tau \right)$ be an M^*-*irresolute* topological vector space and S be a subspace of X. If S contains a $non-empty$ M^*-*open* subset of X, then S is M^*-*open* in X.

Proof. Suppose U is a $non-empty$ M^*-*open* subset of X such that $U \subseteq S$. By Theorem 3.3(1), for any $y \in S$, $U + y$ is an M^*-*open* subset of X. Since S is a subspace of X, so also we have $U + y \subseteq S$ for any $y \in S$, Thus $S = \bigcup \{ U + y : y \in S \}$ is M^*-*open* in X being a union of M^*-*open* sets.

Theorem 3.12. Let A be any subset of an M^*-*irresolute* topological vector space X. Then the following statements are true.

(1) $x + M^*\text{-}Cl(A) = M^*\text{-}Cl(x+A)$ for any $x \in X$.

(2) $M^*\text{-}Cl(\lambda A) = \lambda\left[M^*\text{-}Cl(A)\right]$ for any non-zero scalar λ.

Proof. (1) By applying Theorem 3.9(1), $x + M^*\text{-}Cl(A)$ is $M^*\text{-}closed$. Hence $x + A \subseteq x + M^*\text{-}Cl(A)$ implies $M^*\text{-}Cl(x+A) \subseteq x + M^*\text{-}Cl(A)$. For the reverse inclusion, let $z \in x + M^*\text{-}Cl(A)$. Then $z = x + y$, for some $y \in M^*\text{-}Cl(A)$. Let W be any $M^*\text{-}open$ neighborhood of z in X. Then, there exist $M^*\text{-}open$ neighborhoods U and V of x and y respectively in X such that $U + V \subseteq W$. Since $y \in M^*\text{-}Cl(A)$, $A \cap V \neq \phi$. Consider $a \in A \cap V$. Then $x + a \in (x+A) \cap (U+V) \subseteq (x+A) \cap W$. Therefore we have $(x+A) \cap W \neq \phi$. Consequently, $z \in M^*\text{-}Cl(x+A)$. Thus $x + M^*\text{-}Cl(A) \subseteq M^*\text{-}Cl(x+A)$. Hence, $x + M^*\text{-}Cl(A) = M^*\text{-}Cl(x+A)$

(2) By Theorem 3.9 (2), $\lambda\left[M^*\text{-}Cl(A)\right]$ is $M^*\text{-}closed$. Therefore $\lambda A \subseteq \lambda\left[M^*\text{-}Cl(A)\right]$ implies that $M^*\text{-}Cl(\lambda A) \subset \lambda\left[M^*\text{-}Cl(A)\right]$. Next, let $x \in M^*\text{-}Cl(A)$ and let W be any $M^*\text{-}open$ neighborhood of $z = \lambda x$ in X. Then we get $M^*\text{-}open$ sets U in K containing λ and V in X containing x such that $U.V \subseteq W$. Since $x \in M^*\text{-}Cl(A)$, there is an element $a \in A \cap V$ and thus
$y = \lambda a \in (\lambda A) \cap (\lambda V) \subseteq (\lambda A) \cap (UV) \subseteq (\lambda A) \cap W$.
Hence $(\lambda A) \cap W \neq \phi$. Therefore it follows that $z = \lambda x \in M^*\text{-}Cl(\lambda A)$. Thus $\lambda\left[M^*\text{-}Cl(A)\right] \subseteq M^*\text{-}Cl(\lambda A)$. Hence the assertion follows.

Theorem 3.13. Let $\left(X_{(K)}, \tau\right)$ be an $M^*\text{-}irresolute$ topological vector space. Let A and B be subsets of X. Then prove that $M^*\text{-}Cl(A) + M^*\text{-}Cl(B) \subseteq M^*\text{-}Cl(A+B)$.

Proof. Let $x \in M^*\text{-}Cl(A)$ and $y \in M^*\text{-}Cl(B)$. Let W be an $M^*\text{-}open$ neighborhood of $x + y$. Then there exist $M^*\text{-}open$ neighborhoods U and V of x and y respectively, such that $U + V \subseteq W$. Since, $x \in M^*\text{-}Cl(A)$, $y \in M^*\text{-}Cl(B)$, there are $a \in A \cap U$ and $b \in B \cap V$. Then, $a + b \in (A+B) \cap (U+V) \subseteq (A+B) \cap W$. Thus we have $(A+B) \cap W \neq \phi$. This implies that $x + y \in M^*\text{-}Cl(A+B)$. Hence eventually we obtain $M^*\text{-}Cl(A) + M^*\text{-}Cl(B) \subseteq M^*\text{-}Cl(A+B)$.

Theorem 3.14. Let $\left(X_{(K)}, \tau\right)$ be an $M^*\text{-}irresolute$ topological vector space. For given $y \in X$ and $\lambda \in K$ with $\lambda \neq 0$, each translation mapping $T_y : X \to X$ defined by $T_y(x) = x + y$ and multiplication mapping $M_\lambda : X \to X$ defined by $M_\lambda(x) = \lambda x$, where $x \in X$, is $M^*\text{-}homeomorphism$ onto itself.

Proof. First, we show that $T_y : X \to X$ is $M^*\text{-}homeomorphism$. It is obviously bijective. By Theorem 3.2 (1), T_y is $M^*\text{-}irresolute$. Moreover, T_y is $pre-M^*\text{-}open$ because for any $M^*\text{-}open$ U, by Theorem 3.3 (1), $T_y(U) = U + y$ is $M^*\text{-}open$. Similarly, we can prove that M_λ is $M^*\text{-}homeomorphism$.

Theorem 3.15. Let $\left(X_{(K)}, \tau\right)$ be an $M^*\text{-}irresolute$ topological vector space. Then any $M^*\text{-}open$ subspace of X is $M^*\text{-}closed$ in X.

Proof. Let G be an $M^*\text{-}open$ subspace of X. Then by Theorem 3.3 (1), for any $x \in X \mathbf{B} G$, $G + x$ is $M^*\text{-}open$. We also clearly have $x \in G + x \subseteq X \mathbf{B} G$. Then, $Z = \bigcup\{G + x : x \in X \mathbf{B} G\} = X \mathbf{B} G$ being a union of $M^*\text{-}open$ sets is $M^*\text{-}open$. Therefore, $G = X \mathbf{B} Z$ is $M^*\text{-}closed$.

Theorem 3.16. Let $\left(X_{(K)}, \tau\right)$ be an $M^*\text{-}irresolute$ topological vector space and B be an $M^*\text{-}open$ set in X, then for any subset A of X, we have $A + B = M^*\text{-}Cl(A) + B$.

Proof. Since we know that $A \subseteq M^*\text{-}Cl(A)$, so $A + B \subseteq s^*g\text{-}Cl(A) + B$. Conversely, let $y \in M^*\text{-}Cl(A) + B$ and write $y = x + b$, where

$x \in M^*\text{-}Cl(A)$ and $b \in B$. There exists an $M^*\text{-}open$ neighborhood V of zero such that $T_b(V) = V + b \subseteq B$. Now, V is $M^*\text{-}open$ neighborhood of 0 in X, this gives that $-V$ is also $M^*\text{-}open$ neighborhood of 0 in X. Then $x - V$ is an $M^*\text{-}open$ neighborhood of x. Since $x \in M^*\text{-}Cl(A)$, so, there exists an element $a \in A \bigcap (x - V)$. We know that $y = x + b = a - a + x + b \in a + V + b \subseteq A + B.$ Therefore, $M^*\text{-}Cl(A) + B \subseteq A + B.$ Hence, consequently we obtain $A + B = M^*\text{-}Cl(A) + B.$

Theorem 3.17. Let $\left(X_{(K)}, \tau\right)$ be an $M^*\text{-}irresolute$ topological vector space. Then the scalar multiple of $M^*\text{-}closed$ set is $M^*\text{-}closed$.

Proof. Let B be an $M^*\text{-}closed$ set in X and let $\lambda \in K - \{0\}$. Then $X - B$ is $M^*\text{-}open$ set in X. Now $M_\lambda(X - B) = \lambda(X - B) = \lambda X - \lambda B = X - \lambda B \in M^*\text{-}O(X)$. Therefore, $\lambda B \in M^*\text{-}C(X)$.

Definition 3.18. A topological space (X, τ) is said to be $M^*\text{-}compact$ if every cover of X by $M^*\text{-}open$ sets of X has a finite sub cover. A subset A of X is said to be $M^*\text{-}compact$ relative to X if every cover of A by $M^*\text{-}open$ sets of X has a finite sub cover.

Theorem 3.19. Let $\left(X_{(K)}, \tau\right)$ be an $M^*\text{-}irresolute$ topological vector space and let A be any $M^*\text{-}compact$ set in X. Then prove that $x + A$ is $M^*\text{-}compact$ for each $x \in X$.

Proof. Let $\Omega = \{U_\alpha : \alpha \in \Lambda\}$ be an $M^*\text{-}open$ cover of $x + A$. Then $A \subseteq \bigcup \{-x + U_\alpha : \alpha \in \Lambda\}$ and $\{-x + U_\alpha : \alpha \in \Lambda\} \subseteq M^*\text{-}O(X)$. By hypothesis, $A \subseteq \bigcup \{-x + U_\alpha : \alpha \in \Lambda_0\}$ for some finite $\Lambda_0 \subseteq \Lambda$. Whence we find that $x + A \subseteq \bigcup \{U_\alpha : \alpha \in \Lambda_0\}$. This shows that $x + A$ is $M^*\text{-}compact$. Hence, the proof is complete.

Theorem 3.20. Let $\left(X_{(K)}, \tau\right)$ be an $M^*\text{-}irresolute$ topological vector space. The scalar multiple of $M^*\text{-}compact$ set is $M^*\text{-}compact$.

Proof. Let A be an $M^*\text{-}compact$ subset of X. If $\lambda = 0$ we are nothing to prove. Assume $\lambda \in K - \{0\}$. Let $\Psi = \{U_\alpha : \alpha \in \Lambda\}$ be an $M^*\text{-}open$ cover of λA. Then

$$A \subseteq \left(\frac{1}{\lambda}\right)(\bigcup \Psi) = \left(\frac{1}{\lambda}\right)(\bigcup \{U_\alpha : \alpha \in \Lambda\}) =$$
$$\bigcup \left\{\left(\frac{1}{\lambda}\right)U_\alpha : \alpha \in \Lambda\right\}. \qquad \text{Since}$$

$\{U_\alpha : \alpha \in \Lambda\} \subseteq M^*\text{-}O(X).$ and $\left(X_{(K)}, \tau\right)$ is M^*ITVS, so we obtain $\left\{\left(\frac{1}{\lambda}\right)U_\alpha : \alpha \in \Lambda\right\} \subseteq M^*\text{-}O(X),$ By hypothesis A is $M^*\text{-}compact,$ therefore there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $A \subseteq \bigcup \left\{\left(\frac{1}{\lambda}\right)U_\alpha : \alpha \in \Lambda_0\right\}.$ This implies that $\lambda A \subseteq \bigcup \{U_\alpha : \alpha \in \Lambda_0\}.$ Hence λA is $M^*\text{-}compact$.

Definition 3.21. A mapping $f : \left(X_{(K)}, \tau_X\right) \to \left(Y_{(K)}, \tau_Y\right)$ is said to be linear if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y),$ for all $x, y \in X$ and $\alpha, \beta \in K.$

Definition 3.22. A mapping $f : X \to K$ is called linear functional if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y),$ for all $x, y \in X$ and $\alpha, \beta \in K.$ The kernel of f is defined by $Ker(f) = \{x \in X : f(x) = 0\}.$

Theorem 3.23. Let $f : \left(X_{(K)}, \tau_X\right) \to \left(Y_{(K)}, \tau_Y\right)$ be a linear mapping such that f is $M^*\text{-}irresolute$ at $0.$ Then f is $M^*\text{-}irresolute$ on $X.$

Proof. Let x be any non-zero element of X and V be any $M^*\text{-}open$ set in Y containing $f(x).$ Since translation of a $M^*\text{-}open$ set in an $M^*\text{-}irresolute$ topological vector spaces is $M^*\text{-}open,$ $V - f(x)$ is $M^*\text{-}open$ set in Y containing $0.$ Since f is $M^*\text{-}irresolute$ at $0,$ there exists an $M^*\text{-}open$ set U in X containing 0 such that $f(U) \subseteq V - f(x).$ Furthermore, linearity of f implies that $f(x + U) \subseteq V.$ By Theorem 3.3 (1), $x + U$ is $M^*\text{-}open$ and hence f is $M^*\text{-}irresolute$ at $x.$ By hypothesis, f is $M^*\text{-}irresolute$ at $0.$ This reflects that f is $M^*\text{-}irresolute.$

Corollary 3.24. Let $\left(X_{(K)}, \tau\right)$ be an M^*-*irresolute* topological vector space. Let $f : X \to K$ be a linear functional which is M^*-*irresolute* at $0.$ Then the set $F = \{x \in X : f(x) = 0\}$ is M^*-*closed*.

4. CHARACTERIZATIONS OF M$^{\bullet}$-IRRESOLUTE

TOPOLOGICAL VECTOR SPACES

In this section we give some characterizations of M^*-*irresolute* topological vector space.

Theorem 4.1. Let (X, τ) be an M^*-*irresolute* topological vector space. For $x \in X,$ the following assertions are true:

(1) If $U \in M^*$-$N(x),$ then $x \in U.$

(2) If $U \in M^*$-$N(x)$ and V is a neighborhood of $x,$ then $U \bigcap V \in M^*$-$N(x).$

(3) If $U \in M^*$-$N(x),$ then there exists $V \in M^*$-$N(x)$ such that $U \in M^*$-$N(y),$ for all $y \in V.$

(4) If $U \in M^*$-$N(x)$ and $U \subseteq V,$ then $V \in M^*$-$N(x).$

(5) If $U \in M^*$-$N(0),$ then $\lambda U \in M^*$-$N(0)$ for every non-zero element $\lambda \in R.$

(6) If $U \in M^*$-$N(x)$ and V is an M^*-*neighborhood* of $x,$ then $U \bigcap V \in M^*$-$N(x).$

(7) $U \in M^*$-$N(0)$ if and only if $x + U \in M^*$-$N(x).$

Proof. We will prove $(2),$ (5) and (7) while the proofs of others follow easily.

(2) If U is an M^*-*neighborhood* of $x,$ and V is a neighborhood of $x,$ then there is an M^*-*open* subset A and an open set B such that $x \in A \subseteq U$ and $x \in B \subseteq V.$ Then $x \in A \bigcap B \subseteq U \bigcap V$ and $\tau \subseteq M^*$-$O(X).$ Thus $A \bigcap B \in M^*$-$O(X).$ Therefore $U \bigcap V$ is an M^*-*neighborhood* of $x.$

(5) Let U be an M^*-*neighborhood* for zero. Then there exists an M^*-*open* neighborhood V of zero such that $V \subseteq U.$ Since the map $M_\lambda : X \to X,$ defined by $M_\lambda(x) = \lambda x,$ is M^*-*irresolute*. The

inverse map $M_\lambda; X \to X,$ defined by $M_\lambda(x) = \dfrac{1}{\lambda} x,$ is also an M^*-*irresolute*. Thus M_λ is M^*-*homeomorphism*, for each $\lambda \in R - \{0\}.$ Hence $M_\lambda(V) = \lambda V$ is an M^*-*open* neighborhood of zero. Furthermore, clearly $\lambda V \subseteq \lambda U.$ Thus consequently $\lambda U \in M^*$-$N(0).$

(7) Suppose U is an M^*-*neighborhood* for zero. Then there exists an M^*-*open* neighborhood V of zero such that $V \subseteq U.$ Since the map $T_x : X \to X,$ defined by $T_x(y) = y + x,$ is M^*-*irresolute*. The inverse map $T_x; X \to X,$ defined by $T_x(y) = x - y,$ is also an M^*-*irresolute*. Thus T_x is M^*-*homeomorphism*, for each $x \in X.$ Hence $T_x(V) = x + V$ is an M^*-*open* neighborhood for a point $x.$ Clearly $x + V \subseteq x + U.$ Thus $x + U \in N_x.$ The converse can be proved similarly.

Definition 4.2. A subset A of a topological vector space X is called balanced if and only if $\lambda A \subseteq A$ for each $\lambda \in R$ such that $|\lambda| \le 1.$

Definition 4.3. A subset A of a topological vector space X is called absorbing if for all $x \in X$ there exists a number $\varepsilon > 0$ such that $\lambda x \in A$ for $|\lambda| \le \varepsilon.$

Definition 4.4. A set C of a topological vector space X is said to be convex, if and only if it contains all segments between its points: $x \in C,$ $y \in C,$ for $t \in [0,1]$ implies $tx + (1-t) y \in C,$ or equivalently $tC + (1-t) C \subseteq C,$ for all $t \in [0,1].$ A set C of a topological vector space X is said to be absolutely convex if it is both convex and balanced.

Theorem 4.5. Let $\left(X_{(K)}, \tau\right)$ be an M^*-*irresolute* topological vector space. If a subset C of X is convex, then M^*-$Cl(C)$ is also convex.

Proof. The convexity of C implies $tC + (1-t) C \subseteq C.$ By Theorem 3.12 (2), and Theorem 3.13, it follows immediately that

$$t\left[M^*\text{-}Cl(C)\right] + (1-t)\left[M^*\text{-}Cl(C)\right] = M^*\text{-}Cl(tC) + M^*\text{-}Cl\left[(1-t)C\right] \subseteq M^*\text{-}Cl\left[tC + (1-t)C\right]$$
$$= M^*\text{-}Cl(C). \qquad \text{Thus}$$
$$t\left[M^*\text{-}Cl(C)\right] + (1-t)\left[M^*\text{-}Cl(C)\right] \subseteq M^*\text{-}Cl(C).$$

Hence we conclude that M^*-$Cl(C)$ is convex.

Theorem 4.6. Let $(X_{(K)},\tau)$ be an M^*-*irresolute* topological vector space. If a subset C of X is convex, then M^*-$Int(C)$ is also convex.

Proof. By Theorem 3.7 (2), and Theorem 3.8,

$t[M^*\text{-}Int(C)]+(1-t)[M^*\text{-}Int(C)]=$

$M^*\text{-}Int(tC)+ \qquad\qquad M^*\text{-}Int[(1-t)C]\subseteq$

$M^*\text{-}Int[tC+(1-t)C]=M^*\text{-}Int(C).$ Therefore

M^*-$Int(C)$ is convex.

Theorem 4.7. Let $(X_{(R)},\tau)$ be an M^*-*irresolute* topological vector space. Then the following statements are equivalent:

(a) Every M^*-*neighborhood* U of zero is absorbing.

(b) For every M^*-*neighborhood* U for zero, there exists a balanced set $V \in M^*\text{-}N(0)$ such that $V \subseteq U$.

Proof. (a) Suppose U is an M^*-*neighborhood* for zero. Then there exists an M^*-*open* subset $V \in M^*\text{-}N(0)$ such that $V \subseteq U$. By hypothesis X is an M^*ITVS. So there exist M^*-*open* sets V_1 of R containing *zero* and V_2 of X containing zero such that $V_1.V_2 \subseteq V$. The set V_1 contains an open interval of the form $(-\varepsilon,\varepsilon)$ for some $\varepsilon > 0.$ Therefore $tx \in V \subseteq U$ for all $t \in (-\varepsilon,\varepsilon)$ and for all $x \in V_2$. This implies U is absorbing.

(b) Let U be an M^*-*neighborhood* of zero. By hypothesis X is an M^*ITVS. So there exist M^*-*open* sets V_1 of R containing zero and V_2 of X containing zero such that $V_1.V_2 \subseteq U.$ Then there exists $\varepsilon > 0$ such that $(-\varepsilon,\varepsilon) \subseteq V_1.$ Define $W = \bigcup\{tV_2 : t \in R, |t| < \varepsilon\}.$ Since tV_2 is an M^*-*neighborhood* of zero, for $t \neq 0$ and $tV_2 \subseteq U$ for $t \in (-\varepsilon,\varepsilon).$ Thus W is an M^*-*neighborhood* for zero and $W \subseteq U.$ Now we have to show that W is balanced. Let $r \in R$ such that $|r| \leq 1.$ Let $t \in (-\varepsilon,\varepsilon)$ and $x \in V_2.$ Since $|rt| = |r||t| \leq t < \varepsilon.$ Thus $r(tx) = (rt)x \in (-\varepsilon,\varepsilon).V_2 \subseteq W.$ This shows that $rW \subseteq W.$ Therefore W is balanced.

Theorem 4.8. Let X be an M^*-*irresolute* topological vector space. Then $M^*\text{-}Cl(A)=\cap\{A+U : U \in M^*\text{-}N(0)\}.$

Proof. Assume $x \in M^*\text{-}Cl(A),$ and let U be an M^*-*neighborhood* of zero. Then by Theorem 4.7(b), there exits a balanced neighborhood V for zero such that $V \subseteq U.$ Thus $x+V$ is an M^*-*neoghborhood* for x and $x \in M^*\text{-}Cl(A),$ so $(x+V)\cap A \neq \phi.$ Take $a \in (x+V)\cap A.$ Then $a \in x+V$ and $a \in A.$ Let $a = x+v$ for some $v \in V.$ Since V is balanced, so $A-V = A+V.$ Take $x = a+(-v) \in A-V$ implies $x \in A+V \subseteq A+U.$ Thus $x \in A+U,$ for any M^*-*neighborhood* U of zero. Therefore, we obtain $M^*\text{-}Cl(A)\subseteq\cap\{A+U : U \in M^*\text{-}N(0)\}.$

Conversely if $x \notin M^*\text{-}Cl(A),$ then there exists a balanced neighborhood U for zero such that $(x+U)\cap A = \phi.$ Thus $x \notin A-U = A+U.$ It follows that $\cap\{A+U : U \in M^*\text{-}N(0)\}\subseteq M^*\text{-}Cl(A).$ Thus we get $M^*\text{-}Cl(A)=\cap\{A+U : U \in M^*\text{-}N(0)\}.$

Theorem 4.9. Let X be an $M^*ITVS.$ Then the following assertions are true:

(a) For every $U \in M^*\text{-}N(0),$ there exists symmetric set $V \in M^*\text{-}N(0)$ such that $V + V \subseteq U.$

(b) For every $U \in M^*\text{-}N(0),$ there exits an M^*-*closed* balanced set $V \in M^*\text{-}N(0)$ such that $V \subseteq U.$

(c) For every $U \in M^*\text{-}N(0),$ there exists symmetric set $V \in M^*\text{-}N(0)$ such that $V + V + V \subseteq U.$

Proof. (a) Assume $U \in M^*\text{-}N(0).$ By hypothesis X is an $M^*ITVS.$ There exist M^*-*open* neighborhoods V_1 and V_2 for zero in X such that $V_1 + V_2 \subseteq U.$ Let $V = V_1 \cap (-V_1) \cap V_2 \cap (-V_2).$ Then V is symmetric M^*-*open* neighborhood of zero and $V + V \subseteq V_1 + V_2 \subseteq U.$

(b) Let U be an M^*-*neihborhood* of zero in $X.$ By part (a) there is M^*-*neihborhood* V for zero with $V + V \subseteq U.$ By Theorem 4.7 (b), there exits M^*-*neihborhood* W for zero which is balanced and $W \subseteq V.$ By Theorem 4.8,

M^*-$Cl(W) \subseteq W + V \subseteq V + V \subseteq U$. This shows that U contains M^*-*closed* neighborhood of zero.

(c) Follows easily from (a).

Definition 4.10. A topological space (X, τ) is called M^*-*Hausdorff*, if each two distinct points x and y in X, there exist disjoint M^*-*open* sets U, V such that $x \in U$ and $y \in V$.

Now we give some properties of M^*-*Hausdorff* space.

Theorem 4.11. Let X be an M^*ITVS. Then the following statements are equivalent.

(a) X is M^*-*Hausdorff*.

(b) If $x \in X$, $x \neq 0$, then there exists $U \in M^*$-$N(0)$ such that $x \notin U$.

(c) If $x, y \in X$, $x \neq y$, there exists $V \in M^*$-$N(x)$ such that $y \notin V$.

Proof. By continuity of translation, it is sufficient to prove the equivalence between (a) and (b) only.

$(a) \Rightarrow (b)$: Assume x be a non-zero vector belongs to X. Therefore there are disjoint M^*-*open* sets $U, V \subseteq X$ such that $0 \in U$ and $x \in V$. Thus $U \in M^*$-$N(0)$, $V \in M^*$-$N(x)$ and $x \notin U$.

$(b) \Rightarrow (a)$: Let $x, y \in X$ be such that $x - y \neq 0$. Then there exists $U \in M^*$-$N(0)$ such that $x - y \notin U$. By Theorem 4.9 (a), there exists M^*-*neihborhood* W of zero such that $W + W \subseteq U$. By Theorem 4.7 (b), W can be assumed to be balanced. Let $V_1 = x + W$ and $V_2 = y + W$. We note that $V_1 \in N_x$, $V_2 \in N_y$ and $V_1 \cap V_2 = \phi$, since if $z \in V_1 \cap V_2$ then $z - x \in W$ and $z - y \in W$. Since W is balanced, so $-(z - x) \in W$. It follows that $x - y = (z - y) + \lfloor -(z - x) \rfloor \in W + W \subseteq U$, which is a contradiction. So we must have $V_1 \cap V_2 = \phi$. Finally, the space X is M^*-*Hausdorff*.

The following result follows from Theorem 4.11.

Corollary 4.12. Let X be an M^*ITVS. Then the following statements are equivalent.

(a) X is M^*-*Hausdorff*.

(b) $\cap \{U : U \in M^*$-$N(0)\} = \{0\}$.

(c) $\cap \{U : U \in M^*$-$N(x)\} = \{x\}$.

Theorem 4.12. An M^*ITVS X is M^*-*Hausdorff* if and only if one-point set in X is M^*-*closed* in X.

Proof. Let $x \in X$ and $y \in X$ **B** $\{x\}$. Then $y - x \neq 0$, and by assumption, there exists $U \in M^*$-$N(0)$ such that $y - x \notin U$. By Theorem 4.9 (b), there exists an M^*-*closed* and balanced set $V \in N_0$ such that $V \subseteq U$. It follows that $y - x \notin V$ that is $y - x \in X$ **B** V. Thus $y \in (X$ **B** $V) + \{x\}$. But $(X$ **B** $V) + \{x\}$ is M^*-*open*, since V is M^*-*closed*, and $(X$ **B** $V) + \{x\} \subseteq X$ **B** $\{x\}$. This shows that X **B** $\{x\}$ is M^*-*open*. For the converse, let $x \in X$ and assume that $\{x\}$ is M^*-*closed*. Then by Theorem 4.8, $\{x\} = M^*$-$Cl(\{x\}) = \cap \{U + \{x\} : U \in M^*$-$N(0)\}$ $= \cap \{V : V \in M^*$-$N(x)\}$, where $V = U + \{x\} \in M^*$-$N(x)$. Then by Corollary 4.12, X is M^*-*Hausdorff*. We know that translation is an M^*-*homeomorphism* and as a consequence of Theorem 4.12, we have the following result.

Corollary 4.13. An M^*ITVS X is M^*-*Hausdorff* if and only if $\{0\}$ is M^*-*closed* in X.

Theorem 4.14. Let C, K be disjoint sets in an M^*ITVS X with C M^*-*closed*, K M^*-*compact*. Then there exists $U \in M^*$-$N(0)$ with $(K + U) \cap (C + U) = \phi$.

Proof. If $K = \phi$, there is nothing to prove. Otherwise, let $x \in K$ by the invariance with translation; we can assume $x = 0$. Then X **B** C is an M^*-*open* set of zero. Since addition is M^*-*irresolute* and M^*-*continuous*, by $0 + 0 + 0 = 0$, there is an M^*-*open* neighborhood $U \in M^*$-$N(0)$ such that $3U = U + U + U \subseteq X$ **B** C. By defining $U = U \cap (-U) \subseteq U$ we have that U M^*-*open* symmetric and $3U = U + U + U \subseteq X$ **B** C. This means that $\phi = \{3x : x \in U\} \cap C = \{2x : x \in U\} \cap \{y - x : y \in C, x \in U\} \supseteq U \cap (C + U)$. This concludes the proof for a single point. Sine K is

M^*-*compact*, then repeating the above argument for all $x \in K$, we obtain symmetric M^*-*open* sets V_x such that $(x+2V_x) \cap (C+V_x) = \phi$. The sets $\{V_x : x \in K\}$ are an M^*-*open* covering of K, but K is M^*-*compact.* Hence there is a finite number of points $x_i \in K$, $i = 1,2,...,n$, such that $K \subseteq \bigcup_{i=1}^{n}(x_i + V_{x_i})$. Define the M^*-*open* neighborhood V of zero by $V = \bigcap_{i=1}^{n} V_{x_i}$. Then we get

$$(K+V) \cap (C+V) \subseteq \bigcup_{i=1}^{n}(x_i + V_{x_i} + V) \cap (C+V) \subseteq$$

$\bigcup_{i=1}^{n}(x_i + 2V_{x_i})(C+V_{x_i}) = \phi.$ This completes the proof.

Lemma 4.15. If U is a M^*-*open* set and $U \cap A = \phi$, then $U \cap M^*$-$Cl(A) = \phi.$

Proof. Suppose that there exists an $x \in U \cap M^*$-$Cl(A)$. Then $x \in M^*$-$Cl(A)$ and U is an M^*-*open* neighborhood of x and $X \mathbf{B} U$ is M^*-*closed* set containing A, hence M^*-$Cl(A) \subseteq X \mathbf{B} U$ and $x \notin M^*$-$Cl(A)$ which is contradiction, hence $U \cap M^*$-$Cl(A) = \phi.$

Corollary 4.16. Let C,K be disjoint sets in an M^*ITVS X with C M^*-*closed*, K M^*-*compact*. Then there exists $U \in M^*$-$N(0)$ with $\left[M^*$-$Cl(K+U)\right] \cap (C+U) = \phi.$

Proof. By Theorem 4.14, there exists $U \in M^*$-$N(0)$ such that $(K+U) \cap (C+U) = \phi.$ $C+U = \bigcup\{y+U : y \in C\}$ is M^*-*open* set being a union of M^*-*open* sets. Now by Lemma 4.15, we obtain $\left[M^*$-$Cl(K+U)\right] \cap (C+U) = \phi.$

Theorem 4.17. Let X be an M^*ITVS. Let $f : X \to K$ be a non-zero linear map. Then $f(G)$ is M^*-*open* in K whenever G is M^*-*open* in X.

Proof. Let G be a nonempty M^*-*open* set. Then once can assume that there is $x_0 \in X - \{0\}$ such that $f(x_0) = 1$. For any $a \in G$, it is required to show that $f(a) \in M^*$-$Int[f(G)]$. Since $G \in M^*$-$N(a)$ by Theorem 4.1 we have $G-a \in M^*$-$N(0)$. By

Theorem 4.7 (a) $G-a$ is absorbing, that is, absorbs x_0, namely there exists an $\varepsilon > 0$ such that $\lambda x_0 \in G-a$ whenever $\lambda \in R$ with $|\lambda| \le \varepsilon$. Now for any $\beta \in R$ with $|\beta - f(a)| < \varepsilon$ we have $(\beta - f(a))x_0 \in G-a$, hence $f\left[(\beta - f(a))x_0\right] \in f(G-a)$. Since f is linear. This implies that $(\beta - f(a))f(x_0) \in f(G-a)$. So we get $(\beta - f(a))(1) \in f(G-a) = f(G) - f(a)$. This implies that $\beta \in f(G)$. Thus shows that $(f(a) - \varepsilon, f(a) + \varepsilon) \subseteq f(G)$. Since $(f(a) - \varepsilon, f(a) + \varepsilon)$ is θ-*open* in K. Thus $f(a) \in Int_\theta [f(G)] \subseteq M^*$-$Int[f(G)]$; hence consequently $f(G) = M^*$-$Int[f(G)]$.

Lemma 4.18. $\left[\text{Horvath;}1966\right]$. Let X be vector space and $\phi \neq K \subseteq X$. For $a \in K$, the following statements are equivalent:

(1) a is an extreme point of K.

(2) If $x,y \in K$ are such that $a = \frac{1}{2}(x+y)$, then $a = x = y$.

(3) Let $x,y \in K$ be such that $x \neq y$, let $\lambda \in (0,1)$ and $a = \lambda x + (1-\lambda)y$. Then we have either $\lambda = 0$ or $\lambda = 1$.

Theorem 4.19. Let X be an M^*ITVS and C be a convex subset of X. Then $\left[M^*$-$Int(C)\right] \cap (\partial C) = \phi.$

Proof. If M^*-$Int(C) = \phi,$ then the result is trivial. Suppose that M^*-$Int(C) \neq \phi$ and let $x \in M^*$-$Int(C)$. Then there exists $V \in M^*$-$N(0)$ such that $x+V \subseteq C$. As the map $\Phi : \Re \to X,$ where $\Phi(\mu) = \mu x$ is continuous at $\mu = 1,$ for this M^*-*Ineighborhood* $x+V$ there is an $r > 0$ such that $\mu x \in x+V$ whenever $|\mu - 1| \le r$. In particular, we have $(1+r)x \in x+V \subseteq C$ and $(1-r)x \in x+V \subseteq C$. Now consider $x = \lambda(1+r)x + (1-\lambda)(1-r)x$ and set $\lambda = \frac{1}{2}.$ Consequently, we have $x = \frac{1}{2}(1+r)x + \left(1-\frac{1}{2}\right)(1-r)x,$ which implies that x is not an extreme point of K.

Theorem 4.20. Let X be an M^*ITVS and W an M^*-*neighborhood* of 0. Then there is an M^*-*neighborhood* U of 0 such that M^*-$Cl(U) \subseteq W$. Equivalently, if C is a M^*-*closed* subset of X and x a point of X outside C then there are disjoint M^*-*open* sets U_1 and U_2 with $x \in U_1$ and $C \subseteq U_2$.

Proof. Let x be a point outside an M^*-*closed* set $C \subseteq X$. We will produce an M^*-*open* set U containing x with M^*-$Cl(U) \bigcap C = \phi;$ then $U_1 = U$ and $U_2 = X \blacksquare M^*$-$Cl(U)$, the complement of the M^*-*closure* of U, are disjoint M^*-*open* sets with $x \in U_1$ and $C \subseteq U_2$, as desired. We know that X looks the same everywhere, so we may work with $x = 0$. Let W be the complement of C. Then W is an M^*-*open* set with $0 \in W$. By hypothesis X is M^*ITVS. So by Theorem 4.9 (a), there exists an M^*-*open* subset U of 0 such that $U + U \subseteq W$. This means that $U + U$ is disjoint from C. Equivalently, U is disjoint from $C \blacksquare U$. For otherwise there would be an $x \in U$ which could be expressed as $c - y$ with $c \in C$ and $y \in U$, which would imply that $c = x + y \in U + U \subseteq W$ is in W. Now the set $-U$ is M^*-*open* because the map
$$X \to X: \quad x \to (-1)x = -x \quad \text{is an}$$
M^*-*homeomorphism,* and hence so are all its translates $x - U$. So the set $U_2 = C \blacksquare U = \bigcup \{c - U : c \in C\}$ is M^*-*open,* being the union of M^*-*open* sets. Thus we have found an M^*-*open* set $U_1 = U$ containing 0 and an M^*-*open* set U_2, disjoint from U_1, with $C \subseteq U_2$.

Theorem 4.21. Let $\left(X_{(K)}, \tau \right)$ be an M^*ITVS. Then every M^*-*open* subspace S of X is also an M^*ITVS.

Proof. Let W be an M^*-*open* neighborhood of $x + y$ in S where x, y are two distinct points in S. Since S is an M^*-*open* subspace of X, then W is an M^*-*open* neighborhood of $x + y$ in X, and by definition of M^*ITVS, there exist M^*-*open* neighborhoods U of x in X and V of y in X such that $U + V \subseteq W$. Then the sets $G = U \bigcap S$ and

$H = V \bigcap S$ are M^*-*open* neighborhoods of x and y in S such that $G + H \subseteq U + V \subseteq W$. Now suppose $\lambda \in K$, $x \in S$ and let W be an M^*-*open* neighborhood of λx in S. Since S is an M^*-*open* subspace of X, then W is an M^*-*open* neighborhood of λx in X. Then there exist M^*-*open* neighborhoods U of λ in K and V of y in X such that $U.V \subseteq W$. Then the set $G = U \bigcap S$ is an M^*-*open* neighborhood of λ in K and the set $H = V \bigcap S$ is an M^*-*open* neighborhood of x in S. Also $G.H \subseteq U.V \subseteq W$. Hence S is an M^*ITVS.

Theorem 4.22. Let $\left(X_{(K)}, \tau \right)$ be an M^*ITVS. Then every M^*-*open* subspace S of X is M^*-*closed* in X.

Proof. By Theorem 3.3 (1), $S + x$ is M^*-*open* in X. Then $Y = \bigcup \{S + x : x \in X \blacksquare S\}$ being union of M^*-*open* sets is also M^*-*open*. Thus $S = X \blacksquare Y$ is M^*-*closed*.

Theorem 4.23. Suppose that $\left(X_{(K)}, \tau \right)$ is an M^*ITVS. If $S \subseteq X$ is a linear subspace, then so is M^*-$Cl(S)$.

Proof. Let S be a linear subspace of X. Thus $S + S \subseteq S$ and for all $\lambda \in K$, $\lambda.S \subseteq S$. By Theorem 3.13,
$$M^*\text{-}Cl(S) + M^*\text{-}Cl(S) \subseteq M^*\text{-}Cl(S + S) \subseteq M^*\text{-}Cl(S).$$
By Theorem 3.12, for every $\lambda \in K$,
$$\lambda \left[M^*\text{-}Cl(S) \right] = M^*\text{-}Cl(\lambda S) \subseteq M^*\text{-}Cl(S).$$
Therefore, M^*-$Cl(S)$ is linear subspace of X.

Definition 4.24. Suppose that $\left(X_{(K)}, \tau \right)$ is an M^*ITVS. A subset $E \subseteq X$ is said to be bounded if for all M^*-*open* set V containing 0, there exists $s \in R$ such that for all $t > s$, $E \subseteq tV$.

Theorem 4.25. Suppose that $\left(X_{(K)}, \tau \right)$ is an M^*ITVS. If E is a bounded subset of X, then M^*-$Cl(E)$ is bounded.

Proof. Let W be an M^*-*open* set containing 0, then by Theorem 4.20, there exists $U \in M^*$-$N(0)$ such that M^*-$Cl(U) \subseteq W$. Since E is bounded, so $E \subseteq tU \subseteq t \left[M^*\text{-}Cl(U) \right] \subseteq tW$, for sufficiently large

t. It follows that for large enough t, $M^*\text{-}Cl(E) \subseteq M^*\text{-}Cl(tU) \subseteq t\left[M^*\text{-}Cl(U)\right] \subseteq tW$.

Thus, $M^*\text{-}Cl(E)$ is bounded.

Theorem 4.26. Let (X,τ) be an M^*ITVS. Let V be an $M^*\text{-}open$ neighborhood of zero in X. Then for every sequence $\{r_n : n \in N\}$ of positive real numbers such that $r_n \to \infty$, $\bigcup_{n=1}^{\infty} r_n V = X$.

Proof. Let $x \in X$ and consider the sequence $\left\{\dfrac{x}{r^n} : n \in N\right\}$. This sequence converges to 0 by the $M^*\text{-}irresoluteness$ of the scalar multiplication $F \times X \to X$. Thus, for sufficiently large n, $\dfrac{x}{r^n} \in V$ i.e., $x \in r_n V$.

Theorem 4.27. Let (X,τ) be an M^*ITVS. Then every $M^*\text{-}compact$ set is bounded.

Proof. Let C be an $M^*\text{-}compact$ subset of X. We need to prove that it is bounded, namely, that for every $M^*\text{-}open$ neighborhood V of 0, $K \subseteq tV$ for sufficiently large t. Let V be an $M^*\text{-}open$ neighborhood of 0, then by Theorem 4.7(b), there exists a balanced $M^*\text{-}open$ neighborhood W of 0 such that $W \subseteq V$. By Theorem 4.26, $C \subseteq \bigcup_{j=1}^{\infty} nW$. Since, C is $M^*\text{-}compact$, therefore there exists a positive integer m such that $C \subseteq \bigcup_{j=1}^{m} n_j W = n_m \bigcup_{j=1}^{m} (n_j / n_m) W \subseteq n_m W$. Thus, for all $t > n$, $C \subseteq n_m W = \lfloor t(n_m / t) \rfloor W \subseteq tW \subseteq tV$, which implies that C is bounded.

Theorem 4.28. Let (X,τ) be an M^*ITVS. Then every Cauchy sequence in X is bounded.

Proof. Let $\{x_n : n \in N\}$ be a Cauchy sequence in X. Let W be an $M^*\text{-}open$ neighborhood of zero, then by Theorem 4.9 (a), there exists an $M^*\text{-}open$ neighborhood V of 0 such that $V + V \subseteq W$. By definition of a Cauchy sequence, there exists N such that for all $m, n \geq N$, $x_n - x_m \in V$ and in particular for all $n > N$, $x_n \in x_N + V$. Set $s > 1$ such that $x_n \in sV$, then for all $n > N$, $x_n \in sV + V \subseteq sV + sV \subseteq sW$. Since for balanced sets $sW \subseteq tW$ for $s < t$, and since every $M^*\text{-}open$ neighborhood of

zero contains a balanced neighborhood, this proves that the sequence is indeed bounded.

Definition 4.29. Let (X,τ) be a vector space over $\square$. A non-negative real-valued function p defined on X is a *pseudonorm* if it satisfies the following two conditions:

(i) $p(\lambda x) = |\lambda| p(x)$, for all $\lambda \in \square$;

(ii) $p(x+y) \leq p(x) + p(y)$, for all $x, y \in X$.

Now, we introduce the notion of locally convex M^*ITVS. Moreover, we give a necessary and sufficient condition, in terms of convex $M^*\text{-}neighborhoods$ of 0, for an M^*ITVS to be locally convex.

Definition 4.30. An M^*ITVS (X,τ) is locally convex if for all $x \in X$, every $S \in M^*\text{-}N(x)$ contains a convex set $U \in M^*\text{-}N(x)$.

Theorem 4.31. An M^*ITVS (X,τ) is locally convex if and only if every $S \in M^*\text{-}N(0)$ contains a convex set $U \in M^*\text{-}N(0)$.

Proof. The sufficiency part is trivial. Let $S \in M^*\text{-}N(x)$. Then by Theorem 4.1 (7), $S - x \in M^*\text{-}N(0)$ and by assumption, there exists a convex set $U \in M^*\text{-}N(0)$ such that $U \subseteq S - x$. Hence by Theorem 4.1 (7) again, $U + x \in M^*\text{-}N(x)$. As $U + x \subseteq S$ and as $U + x$ is convex, (X,τ) is a locally convex M^*ITVS.

Corollary 4.32. In a locally convex M^*ITVS (X,τ), a *pseudonorm* p is $M^*\text{-}irresolute$ if and only if p is $M^*\text{-}irresolute$ at zero.

Proof. If p is $M^*\text{-}irresolute$, then p is $M^*\text{-}irresolute$ at zero. Conversely, suppose p is $M^*\text{-}irresolute$ at 0, and let $x \in X$ and $V \in N_{p(x)}(\square)$. Then by Theorem 4.1 (7), $V - p(x) \in N_0(\square) = N_{p(0)}(\square)$ and thus $(-\varepsilon, \varepsilon) \subseteq V - p(x)$ for some $\varepsilon > 0$. Clearly $(-\varepsilon, \varepsilon)$ being an open set in $\square$ is $M^*\text{-}open$ set in $\square$. By assumption, there exists $U \in M^*\text{-}N(0)$ such that $p(U) \subseteq (-\varepsilon, \varepsilon)$ and as $p(y) \geq 0$ for all $y \in U$, $p(U) \subseteq [0, \varepsilon)$. Then by Theorem 4.1 (7), $U + x \in M^*\text{-}N(x)$. For all $y \in U$, $0 \leq p(x+y) \leq$

$p(x) + p(y) \leq p(x) + \varepsilon, \ p(x+y) \in [0, p(x) + \varepsilon].$

Therefore it follows that $p(U + x) \subseteq V$.

Definition 4.33. Let A be an absolutely convex subset of a vector space X. Then the functional defined by $p(x) = inf\{\lambda : \lambda > 0, x \in \lambda A\}$ is called the *gauge* of A.

Lemma 4.34. $[Horvath; 1966]$. In a vector space X, the *gauge* of an absolutely convex and absorbent subset is a *pseudonorm.*

Now, we prove a main result in which we characterize absolutely convex and absorbent M^*-*neoghborhoods* of zero in terms of their M^*-*irresolute gauges.*

Theorem 4.35. Let p be a *gauge* of an absolutely convex and absorbent subset U of an M^*ITVS (X, τ). Then p is M^*-*irresolute* if and only if U is an M^*-*neighborhood* of zero.

Proof. If p is M^*-*irresolute,* then as $(-1,1)$ is an M^*-*open* set in $\Box$. $V = \{x : p(x) < 1\} = p^{-1}[(-1,1)]$ is an M^*-*open* subset of X. Thus as $V \subseteq U$, $U \in M^*$-$N(0)$. Conversely, if $U \in M^*$-$N(0)$ and $\varepsilon > 0$, then by Theorem 4.1 (5), $V = \varepsilon U \in M^*$-$N(0)$ and $p(x) < \varepsilon$ for all $x \in V$. Thus $p(V) \subseteq (-\varepsilon, \varepsilon)$. Hence, p is M^*-*irresolute* at zero. By Lemma 4.34, p is a *pseudonorm* and by Corollary 4.33, p is M^*-*irresolute* at each $x \in X$. Therefore p is M^*-*irresolute.*

5. CONCLUSION

Topological vector spaces are a fundamental notion and play an important role in various advanced branches of mathematics like fixed point theory, operator theory, etc. This paper expounds the M^*-irresolute topological vector spaces. The author makes familiarity with the origin of M^*-irresolute topological vector spaces and presents several basic features of M^*-irresolute topological vector spaces. The author also presents the result that in an M^*-irresolute topological vector space any M^*-neighborhood for 0 contains an M^*-neighborhood for 0 which is balanced. M^*-irresolute topological vector space satisfies the separation axiom $(M^*$-Hausdorff$)$ and in an M^*-irresolute topological vector space the M^*-interior of a convex set does not intersect with the boundary of the set. The author also presents a result to characterize absolutely convex and absorbent M^*-*neoghborhoods* of zero in terms of their M^*-*irresolute gauges.*

Conflict of Interest
The author declares that this publication is not subject to conflict of interests.

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