

Introduction to Hyperspaces

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Abstract— The development of mathematics brought mathematicians to infinite structures. This process started with transcendent real numbers and infinite sequences going through infinite series to transfinite numbers to nonstandard numbers to hypernumbers. From mathematics, infinity came to physics where physicists have been trying to get rid of infinity inventing a variety of techniques for doing this. In contrast to this, mathematicians as well as some physicists suggested ways to work with infinity introducing new mathematical structures such distributions and extrafunctions. The goal of this paper is to extend mathematical tools for treating infinity by considering hyperspaces and developing their theory.

Keywords— vector space, hypervector, hyperspace, series, sequence, summation

I. INTRODUCTION

The basic structures of the standard calculus and functional analysis are constructed using infinite operations with numbers and functions such as convergence of sequences and summation of series [1, 2]. The calculus and functional analysis have many applications beyond mathematics. For instance, summation of series is widely applied in physics (cf., for example, [3 – 5]). However, the classical summation methods provide sums not for all series of real or complex numbers. That is why mathematicians developed various techniques for extending the domain of summable series [6 – 8]. Examples are Cesàro, Abel or Borel summation, each of which extends the classical technique of series summability.

Similar problems of divergence emerged in applications of integration in theoretical physics [9, 3]. Although renormalization techniques allowed elimination of some divergencies of integrals and series, these methods did not give adequate solutions of these problems (cf., for example, [10]). Nonstandard analysis also was not very helpful in this respect [11].

Nevertheless, neither of the known generalized techniques allows summability of arbitrary series of real (complex) numbers. Only the extension of real (complex) numbers to real (complex) hypernumbers establishes universal summability of real (complex) numbers [12]. A similar construction of extrafunctions extended the scope and applications of distributions [13]. These constructions found utilization in different areas including probability theory and its applications [14] as well as integration theory in general and the theory of the Feynman integral in particular [15].

At the same time, the extension of numbers to hypernumbers does not provide summability of arbitrary series and convergence of arbitrary sequences of real (complex) hypernumbers. To achieve this, we need to make one more extension to the numerical hyperspaces over the space of real (complex) hypernumbers [16, 17].

Nevertheless, even this extension does provide sufficient tools for infinite operations such series summation for sequences of functions. That is why in this paper, we develop a theory of general hyperspaces. The structure and properties of general hyperspaces used for rigorous determination of irregular operations with hypernumbers and extrafunctions, as well as their application to the problems of summation of hypernumbers is studied and integration of extrafunctions based on hypermeasures [18] is considered.

II. CONSTRUCTING HYPERSPACES

Let us consider a real normed vector space V with a norm $\|\cdot\|$ and take the set $V^\omega = \{ (u_n)_{n \in \omega} ; u_n \in V \}$ of all V -sequences, i.e., sequences of elements from V , where a sequence $u = (u_n)_{n \in \omega}$ of elements from V is represented by a mapping $f_a : \omega \rightarrow V$ and ω is the ordered set $\{1, 2, 3, \dots, m, \dots\}$ of all natural numbers. In what follows, we use two notations for sequences $u = (u_n)_{n \in \omega}$ and $u = \{u_n ; n = 1, 2, 3, \dots, m, \dots\}$.

Definition 1. For arbitrary sequences $u = (u_n)_{n \in \omega}$ and $v = (v_n)_{n \in \omega}$ from V^ω , the relation

$$u \approx v \text{ means that } \lim_{n \rightarrow \infty} \|u_n - v_n\| = 0$$

Lemma 1. The relation \approx is an equivalence relation in V^ω .

Lemma 2. If in the sequences $(u_n)_{n \in \omega}$ and $(v_n)_{n \in \omega}$, almost all elements are the same, then these sequences are equivalent, i.e., $(u_n)_{n \in \omega} \approx (v_n)_{n \in \omega}$.

Definition 2. Classes of equivalent with respect to \approx sequences from V^ω are called *hypervectors* and the set of all hypervectors (equivalence classes) is denoted by V_ω is called a *hyperspace* over the vector space V .

From this perspective, each hypervector has the form

$$HvF(u_n)_{n \in \omega} = \{ (v_n)_{n \in \omega} \in V^\omega ; (b_n)_{n \in \omega} \approx (u_n)_{n \in \omega} \}$$

It is also possible to treat hypervectors not as classes but as names of these classes. In this case, $HvF(u_n)_{n \in \omega}$ is a name of the class $\{ (v_n)_{n \in \omega} \in V^\omega ; (v_n)_{n \in \omega} \approx (u_n)_{n \in \omega} \}$ of sequences.

Definition 2 defines the *natural projection* $p: V^\omega \rightarrow V_\omega$ where $p((u_n)_{n \in \omega}) = \text{Hv}_F(u_n)_{n \in \omega}$.

Example 1. When V is the one-dimensional space \mathbf{R} of all real numbers, then V_ω is the space \mathbf{R}_ω of all real hypernumbers [12].

Example 2. When V is the space \mathbf{C} of all complex numbers as vectors over \mathbf{R} , then V_ω is the space \mathbf{C}_ω of all complex hypernumbers [17].

Example 3. When V is the space \mathbf{F} of real functions, then V_ω is the space \mathbf{F}_ω of all real extrafunctions [12]. It is necessary to note that the theory of extrafunctions is a far-reaching development of the theory of distributions [13].

According to Definition 2, any sequence $u = (u_n)_{n \in \omega}$ with elements from V determines (and represents) a hypervector $\chi = \text{Hv}_F(u_n)_{n \in \omega}$ where $u \in \chi$.

Definition 3. The sequence u is called a *defining sequence* or *representing sequence* or *representation* of the hypervector $\chi = \text{Hv}_F(u_n)_{n \in \omega}$.

Definition 2 implies that any two hypernumbers $\text{Hv}_F(u_n)_{n \in \omega}$ and $\text{Hv}_F(v_n)_{n \in \omega}$ are equal if and only if their defining sequences $(u_n)_{n \in \omega}$ and $(v_n)_{n \in \omega}$ are equivalent.

Note that by Lemma 2, any hypervector χ in V_ω does not depend on a finite number of elements from its representation, i.e., we have the following result.

Lemma 3. If $\chi = \text{Hv}_F(u_n)_{n \in \omega}$ and almost all elements in sequences $(u_n)_{n \in \omega}$ and $(v_n)_{n \in \omega}$ are the same (equal), then $\chi = \text{Hv}_F(v_n)_{n \in \omega}$.

Definition 4. A sequence $u = (u_n)_{n \in \omega}$ is called *bounded* if there is a positive real number c such that $\|u_n\| < c$ for all $n \in \omega$.

Bounded vector sequences define bounded hypervectors.

Definition 5. A hypervector χ is called *bounded* if it has a bounded defining sequence.

Lemma 4. If a hypervector χ is bounded, then each of its defining sequences is bounded.

III. PROPERTIES OF HYPERSPACES

Let us obtain some properties of hyperspaces.

Taking a subset P of a vector space V , it is possible to define its ω -extension P_ω as follows

$$P_\omega = \{ \chi \in V_\omega ; \text{there is a sequence } (u_n)_{n \in \omega} \text{ with all elements from } P \text{ and } \chi = \text{Hv}_F(u_n)_{n \in \omega} \}$$

Let us consider some properties of ω -extensions.

Proposition 1. If $P \subseteq V$, then $p(P^\omega) = P_\omega$.

ω -extensions preserve inclusion of sets, i.e., we have the following result.

Proposition 2. If $Q \subseteq P \subseteq V$, then $Q_\omega \subseteq P_\omega \subseteq V_\omega$.

Let us consider set-theoretical operations with ω -extensions.

Proposition 3. If $Q, P \subseteq V$, then $Q_\omega \cup P_\omega \subseteq (Q \cup P)_\omega$.

The following example shows that in a general case, the inclusion in Proposition 3 is proper.

Example 4. Let us take $Q = \{0\}$ and $P = \{1\} \subseteq \mathbf{R}$, which is considered as a one-dimensional vector space, then Q_ω contains only one element and P_ω contains only one element while $(Q \cup P)_\omega$ contains infinitely many elements. For instance, the sequence $u = (0, 1, 0, 1, 0, 1, 0, 1, \dots)$ defines the hypervector that does not belong either to Q_ω or to P_ω . Consequently, we have $Q_\omega \cup P_\omega \neq (Q \cup P)_\omega$.

Proposition 4. If $Q, P \subseteq V$, then $(Q \cap P)_\omega \subseteq Q_\omega \cap P_\omega$.

The following example shows that in a general case, the inclusion in Proposition 4 is proper.

Example 5. Let us take $Q = \{1/2^n ; n = 2, 3, 4, 5, \dots\}$ and $P = \{1/3^n ; n = 2, 3, 4, 5, \dots\} \subseteq \mathbf{C}$, then $Q \cap P$ is empty and consequently, $(Q \cap P)_\omega$ is empty. At the same time, Q_ω contains the hypervector $\text{Hv}\{1/2^n ; n = 2, 3, 4, 5, \dots\} = 0$ and P_ω contains the hypervector $\text{Hv}\{1/3^n ; n = 2, 3, 4, 5, \dots\} = 0$. We see that $Q_\omega \cap P_\omega$ is not empty containing the hypervector $0 = \text{Hv}(0, 0, \dots, 0)$ and consequently, $(Q \cap P)_\omega \neq Q_\omega \cap P_\omega$.

In some cases, sets P_ω and V_ω coincide demonstrating that ω -extensions do not always preserve the strict inclusion relations of sets.

Theorem 1. If P is a dense subset of V , then there is a one-to-one correspondence between the set P_ω and the set V_ω of all hypervectors, i.e., these sets are equipotent.

Example 6. The set Q of all rational numbers is a proper subset of the one-dimensional vector space \mathbf{R} of all real numbers. However, by Theorem 1, $Q_\omega = \mathbf{R}_\omega$.

Taking different norms in the space V , it is possible to build different hyperspaces V_ω , which coincide only in some cases. It is possible to ask a question whether there is the largest hyperspace over V . It exists and to build it, we need further development of the theory.

Definition 6. Two sequences $u = (u_n)_{n \in \omega}$ and $v = (v_n)_{n \in \omega}$ from V^ω are *cofinitely equivalent* if almost all elements in u and v are the same (equal).

It is possible to consider the factoring V_{cofin} of the set V^ω by this equivalence. At the same time, any norm in a vector space defines a metric [2, 19]. So, we can take the space V with some metric \mathbf{d} and build the hyperspace $V_{\omega \mathbf{d}}$. This hyperspace has the following property.

Proposition 5. If the metric \mathbf{d} in the vector space V defines the discrete topology in V , then $V_{\omega \mathbf{d}} = V_{\text{cofin}}$.

It means that V_{cofin} is a hyperspace over V .

Lemma 3 implies the following result.

Proposition 6. For any metric \mathbf{d} in the vector space V , there is a natural projection $q: V_{\text{cofin}} \rightarrow V_{\omega \mathbf{d}}$.

Corollary 1. For any normed vector space V , there is a natural projection $q: V_{\text{cofin}} \rightarrow V_\omega$.

This shows that V_{cofin} is the largest hyperspace over V .

IV. OPERATIONS IN HYPERSPACES

Operations in F induce corresponding operations in F^ω .

Let $u = (u_i)_{i \in \omega}$ and $v = (v_i)_{i \in \omega}$ be elements from V^ω and assume a is a real number. Then we have the following operations in V^ω .

Definition 7. a) Operation of *addition* in V^ω is defined as $u + v = (z_i)_{i \in \omega}$ where $z_i = u_i + v_i$ for all $i \in \omega$;

b) Operation of *subtraction* in V^ω is defined as $u - v = (w_i)_{i \in \omega}$ where $w_i = u_i - v_i$ for all $i \in \omega$;

c) Operation of *multiplication by real numbers* is defined in V^ω by the formula $a \cdot v = (z_i)_{i \in \omega}$ where $z_i = av_i$ where for all $i \in \omega$;

By the definition of addition, multiplication by real numbers and subtraction in the set V^ω , all laws of operations with real numbers, such as commutativity of addition, associativity of addition, and associativity of multiplication, are valid for corresponding operations with sequences from V^ω .

In particular, we have the following result.

Proposition 7. V^ω is a real vector space.

Let us consider bounded sequences in this space.

Proposition 8. a) The sum and difference of two bounded vector sequences are bounded.

b) Multiplication of a bounded vector sequence by a real number gives a bounded vector sequence.

Proposition 8 implies the following result.

Corollary 2. Bounded vector sequences form a subspace B^ω of the vector space V^ω .

Relations between the spaces V^ω and V_ω imply the following result.

Proposition 9. Operations of addition and subtraction in V^ω induce similar operations in V_ω with all identities preserved.

Namely, if $\chi = \text{Hv}_F(u_n)_{n \in \omega}$ and $\zeta = \text{Hv}(v_n)_{n \in \omega}$ are hypervectors, then we have $\chi + \zeta = \text{Hv}(u_n + v_n)_{n \in \omega}$ and $\chi - \zeta = \text{Hv}(u_n - v_n)_{n \in \omega}$.

Proposition 10. Multiplication by real numbers in V^ω induces a similar operation in V_ω with all identities preserved.

Namely, if $\chi = \text{Hv}_F(u_n)_{n \in \omega}$ is a hypervector and a is a real number, then $a\chi = \text{Hv}(au_n)_{n \in \omega}$.

Propositions 9 and 10 imply the following result.

Theorem 2. V_ω is a real vector space.

A subset P of a vector space V is called *balanced* [2] if for any $0 \leq t \leq 1$, we have

$$tP \subseteq P$$

Theorem 3. If P is a balanced subset of a vector space V , then the set P_ω is a balanced subset of the vector space V_ω .

Corollary 3. If Q and P are balanced subsets of a vector space V , then $(Q \cap P)_\omega$, $Q_\omega \cap P_\omega$, $Q_\omega \cup P_\omega$ and $(Q \cup P)_\omega$ are balanced subsets of the vector space V_ω .

A subset P of a vector space V is called *convex* [2] if for any $0 \leq t \leq 1$, we have

$$tP + (1 - t)P \subseteq P$$

Theorem 4. If P is a convex subset of a vector space V , then the set P_ω is a convex subset of the vector space V_ω .

Corollary 4. If Q and P are convex subsets of a vector space V , then $(Q \cap P)_\omega$ and $Q_\omega \cap P_\omega$, are convex subsets of the vector space V_ω .

Now let us look how arithmetical operations transform bounded hypervectors.

Proposition 11. a) The sum and difference of two bounded hypervectors are bounded.

b) Multiplication of a bounded hypervector by a real number gives a bounded hypervector.

Proposition 11 implies the following result.

Corollary 5. Bounded hypervectors form a subspace B_ω of the vector space V_ω .

Let us consider arithmetical operations with ω -extensions in real vector spaces. There are two such operations – addition and multiplication by real numbers [2].

Proposition 12. If $Q, P \subseteq V$ and both Q and P contain the zero vector 0 , then $Q_\omega + P_\omega \subseteq (Q + P)_\omega$.

The following example shows that in a general case, this inclusion is proper.

Example 7. Let us take subsets $Q = \{0, 2\}$ and $P = \{0, 1\}$ of the one-dimensional vector space \mathbf{R} , then neither Q_ω nor P_ω contain the element $3 = 1 + 2$ from $(Q + P)_\omega$. Consequently, we have $Q_\omega + P_\omega \neq (Q + P)_\omega$.

Note that having the zero vectors in both subsets is an essential condition in Proposition 12 because otherwise the inclusion of set can be invalid.

Proposition 13. If $P \subseteq V$ and a is a real number, then $a(P)_\omega = (aP)_\omega$.

An interesting question is whether it is possible to extend the norm in V to the norm in V_ω . However, we can show that in a general case even if V is a Euclidean space, V_ω is not a normed space. At the same time, it is possible to prove the following result utilizing the concept of a hypernorm [17].

We remind that a *hypernorm* in a vector space X is a function $v: X \rightarrow \mathbf{R}_\omega^+$, which satisfies the following conditions:

N1. For any x from L , $q(x) = 0$ if and only if $x = \mathbf{0}$.

N2. $q(ax) = \|a\| \cdot q(x)$ for any x from L and any element a from F .

N3 (the triangle inequality or subadditivity).

$$q(x + y) \leq q(x) + q(y) \text{ for any } x \text{ and } y \text{ from } L$$

A vector space L with a hypernorm is called a *hyponormed vector space* or simply, a *hyponormed space*.

The real hypernumber $q(x)$ is called the *hypernorm* of the element x from the hypernormed space L .

Note that *norms* in vector spaces coincide with those hypernorms that take values only in the set of real numbers. That is, we have the following result [17].

Proposition 14. A hypernorm is a norm in a vector space if and only if it takes values in the set of all non-negative real numbers.

Thus, any normed vector space is a hypernormed vector space. Besides, any hypernormed vector space is a hyperseminormed vector space.

Theorem 5. V_ω is a hypernormed real vector space.

Taking a hypervector $\chi = \text{HvF}(u_n)_{n \in \omega}$, we define its hypernorm as the hypernumber $\alpha = \text{HvF}(\|u_n\|)_{n \in \omega}$.

In normed vector spaces, the norm defines a metric [2, 19]. In a similar way, the hypernorm defines a hypermetric in hypernormed vector spaces [17].

We remind that a mapping $\mathbf{d}: X \times X \rightarrow \mathbf{R}_\omega^+$ is called a *hypermetric* (or a *hyperdistance function*) in a set X if it satisfies the following axioms:

M1. For any x and y from X , $\mathbf{d}(x, y) = 0$ if and only if $x = y$.

M2 (Symmetry). $\mathbf{d}(x, y) = \mathbf{d}(y, x)$ for all $x, y \in X$.

M3 (the triangle inequality or subadditivity).

$$\mathbf{d}(x, y) \leq \mathbf{d}(x, z) + \mathbf{d}(z, y) \text{ for all}$$

$x, y, z \in X$.

A set X with a hypermetric \mathbf{d} is called a *hypermetric space*.

The real hypernumber $\mathbf{d}(x, y)$ is called the *distance* between x and y in the hypermetric space X .

Thus, Theorem 5 implies the following result.

Theorem 6. V_ω is a hypermetric space.

A metric defines a topology in the metric space [2, 19]. In a similar way, hypermetric also defines a topology in the hypermetric space [17]. Consequently, we have the following result.

Theorem 7. V_ω is a topological space.

Topology in V_ω is defined by the system of neighborhoods of 0 having the form

$$U_a = \{ \chi = \text{HvF}(u_n)_{n \in \omega}; \|\chi\| < a \}$$

where a is a positive real number.

A subset P of a vector space V is called *bounded* [2] if for any neighborhood U_a of 0, there is a positive real number t such that $P \subseteq tU_a$.

Theorem 8. If P is a bounded subset of a vector space V , then the set P_ω is a bounded subset of the vector space V_ω .

Corollary 6. If Q and P are bounded subsets of a vector space V , then $(Q \cap P)_\omega$, $Q_\omega \cap P_\omega$, $Q_\omega \cup P_\omega$ and $(Q \cup P)_\omega$ are bounded subsets of the vector space V_ω .

Theorem 8 and Propositions 11 and 12 imply the following result

Corollary 7. If Q and P are bounded subsets of a vector space V , then $Q_\omega + P_\omega$, $(Q + P)_\omega$ and $(aP)_\omega$ are bounded subsets of the vector space V_ω .

In a topological vector space, any compact set is bounded [19]. In contrast to this, in the hyperspace V_ω , even one hypervector can be unbounded. For instance, the hypervector (hypernumber) $\chi = \text{HvF}(n)_{n \in \omega}$ is unbounded in the hyperspace \mathbf{R}_ω . Thus, for hyperspaces, we need a more general definition of boundedness.

Let us consider an element u in a real vector space U .

Definition 8. A subset P of the space U is called *u -bounded* if for every neighborhood U of u , there is a real number a such that $P \subseteq aU$.

This definition implies the following result.

Lemma 5. A subset P of the space U is bounded if and only if it is 0-bounded.

For normed vector spaces, the concepts of boundedness and u -boundedness coincide.

Lemma 6. If U is a normed vector space and $u \in U$, then any u -bounded in U subset P is bounded.

In hyperspaces, this is not true in a general case.

Proposition 15. A χ -bounded subset P of the space U is bounded if and only if the hypervector χ is bounded.

The following result generalizes Corollary 6.

Proposition 16. If Q and P are χ -bounded subsets of a vector space V , then $(Q \cap P)_\omega$, $Q_\omega \cap P_\omega$, $Q_\omega \cup P_\omega$ and $(Q \cup P)_\omega$ are χ -bounded subsets of the vector space V_ω .

Note that the union of a χ -bounded subset and ζ -bounded subset of the vector space V_ω can be neither χ -bounded nor ζ -bounded. However, Proposition 16 and Corollary 6 imply the following result.

Proposition 17. If the hypervectors χ and ζ are bounded, then the union of a χ -bounded subset and ζ -bounded subset of the vector space V_ω is bounded.

A similar result is true for addition of sets in hyperspaces.

Proposition 18. If Q and P are χ -bounded subsets of a vector space V , then $Q_\omega + P_\omega$ and $(Q + P)_\omega$ are 2χ -bounded subsets of the vector space V_ω .

Note that the sum of a χ -bounded subset and ζ -bounded subset of the vector space V_ω can be neither χ -bounded nor ζ -bounded. However, Proposition 16 and Corollary 6 imply the following result.

Proposition 19. If the hypervectors χ and ζ are bounded, then the sum of a χ -bounded subset and ζ -bounded subset of the vector space V_ω is bounded.

An important class of vector spaces is formed by topological vector spaces. As Rudin writes, the concept of topological vector spaces pervades all of functional analysis [2]. However, in a general case, V_ω is not a topological vector space although it is proved that any hypernormed real vector space is a semitopological vector space [17]. This gives us the following result.

Corollary 8. V_ω is a semitopological vector space.

Note that although V_ω is not a topological vector space in a general case, V_ω is a topological space because a hypernorm defines a topology in a vector space [17].

Many properties of the real vector spaces V and V_ω are closely related. In particular, the space V_ω of hypervectors is a natural extension of the vector space V as we have the following result.

Theorem 9. The vector space V is isomorphic to the subspace $\underline{V} = \{\chi = \text{Hv}_F(u_n)_{n \in \omega} ; u_n = u \in V \text{ for all } n \in \omega\}$ of the vector space V_ω .

Identifying elements from V with stable sequences, i.e., sequences in which all elements are equal to the same element from V , we can treat the vector space V as a subset of the set V_ω of all hypervectors.

The norm in V is a restriction of the hypernorm in V_ω . This implies the following result.

Corollary 9. The normed vector space V is (or more exactly, is isomorphic to) a normed subspace of the hypernormed real vector space V_ω .

Let us consider ω -extensions of subsets of the real normed vector space V .

Proposition 20. a) If a subset P of the vector space V is closed with respect to addition, then P_ω is also closed with respect to addition.

b) If a subset P of the vector space V is closed with respect to subtraction, then P_ω is also closed with respect to subtraction.

c) If a subset P of the vector space V is closed with respect to multiplication by real numbers, then P_ω is also closed with respect to multiplication by real numbers.

Proposition 20 implies the following result.

Theorem 10. If U is a vector subspace of a normed vector space V , then U_ω is a vector subspace of V_ω .

The inclusion relation is preserved by ω -extensions.

Proposition 21. For any subspace W of a subspace U of the space V , the vector space W_ω is a subspace of U_ω .

However, the strict inclusion relation is not always preserved by ω -extensions as the following result demonstrates.

Let U be a dense subspace of V .

Theorem 11. The vector spaces U_ω and V_ω are isomorphic.

There are natural relations between linear mappings of vector spaces and linear mapping of their hyperspaces.

Proposition 22. A linear mapping of a vector space W into a vector space U can be extended to a linear mapping of the vector space W_ω into the vector space U_ω .

There are natural relations between linear functionals in vector spaces and linear hyperfunctionals in their hyperspaces.

We remind that a real hyperfunctional in a real vector space L is a mapping $F: L \rightarrow \mathbf{R}_\omega$ [17].

Proposition 23. A linear functional in a vector space V can be extended to a linear hyperfunctional in the vector space V_ω .

Completeness is an important property of normed vector spaces [1, 2].

Theorem 12. For any real normed vector space V , the vector space V_ω contains a completion of the space V .

Corollary 10. If U is a Banach space that contains V , then there is a linear mapping of U into V_ω .

It would be interesting to know whether a proper counterpart of the Hahn-Banach theorem is true for hyperspaces.

Completeness properties of hyperspaces allow proving the following result.

Theorem 13. a) Any series s of vectors from the space V has the sum in the space V_ω .

b) If a series s of vectors from the space V has the classical sum u in V , then its sum in the space V_ω coincides with u .

This shows that summation of vector series in hyperspaces is a natural extension of the conventional summation of vector series.

Summation in hyperspaces provides the base for integration in these spaces. However, to define integrals, it is necessary to consider hypermeasures in the sense of [18].

Let us consider some properties of summation related to operations with vectors and hypervectors. Sequences and series of vectors from the space V belong to the vector space V^ω . So, it is possible to perform operations of addition and multiplication by real numbers with them.

Theorem 14. a) If a hypervector χ is the sum of a series s of vectors from the space V and ζ is the sum of a series r of vectors from the space V , then $\chi + \zeta$ is the sum of a series $s + r$ in the space V_ω .

b) If a hypervector χ is the sum of a series s of vectors from the space V and a is a real number, then $a\chi$ is the sum of a series as in the space V_ω .

This shows that summation commutes with operations in hyperspaces.

V. CONCLUSION

We see that hyperspaces make vector spaces complete and allowing to assign the sum to any vector series. This provides for effective application of these spaces in physics where series and their summation play a very important role. The next step is to study integration in hyperspaces. The results of integration with hypernumbers obtained in [12, 15, 16] show that integration in hyperspaces can essentially extend the scope and applications of the integration in the initial vector space.

A broader problem is the development of functional analysis in hyperspaces. Specifically, it is important to find general counterparts of the basic principles of linear functional analysis for hyperspaces [19]. The goal is to bring new tools to the theory of operator equations in general and differential equations in particular making possible to solve much more equations than it is possible by traditional methods including distributions.

One more interesting problem for the future study is exploration of the categories of hyperspaces as well as additional algebraic structures related to hyperspaces. For instance, it is possible to consider linear hyperalgebras, i.e., hyperspaces with multiplication, and linear Ω -hyperalgebras, i.e., hyperspaces with an additional system Ω of operations.

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