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On Fuzzy L-paracompact Topological Spaces

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***Abstract*— The aim of this paper is to study fuzzy extensions of some covering properties defined by L. Kalantan as a modification of some kinds of paracompactness-type properties due to A.V.Arhangels'skii and studied later by other authors. In fact, we obtain that: if (X,T) is a topological space and A is a subset of X, then A is Lindelöf in (X,T) if and only if its characteristic map χ_{A} is a Lindelöf subset in (X,ω(T)). If (X,τ) is a fuzzy topological space, then, (X,τ) is fuzzy L-paracompact if and only if (X,ι(τ)) is L-paracompact, i.e. fuzzy L-paracompactness is a good extension of L-paracompactness. Fuzzy L_2-paracompactness is a good extension of L_2-paracompactness. Every fuzzy Hausdorff topological space (in the Srivastava, Lal and Srivastava' or in the Wagner and McLean' sense) which is fuzzy locally compact (in the Kudri and Wagner' sense) is fuzzy L_2-paracompact.**

Keywords—Fuzzy sets, Topology, paracompactness, covering properties.

I. Introduction

It is well known that fuzzy sets are been introduced by the engineer L.A. Zadeh who worked on Electrical Engineering, Artificial Intelligence and Mathematics, for this reason most of the publications about fuzzy sets appear in journals and books devoted to Computer Science or Engineering. Many authors worked on fuzzy extensions of paracompactness [1-14] [16] [18-19] [21-31] [33-38] [40] [42-45]. On 2016, A.V.Arhangel'skii defined other paracompact-type properties: C-paracompactness and C_2- -paracompactness. Later, other authors [32] investigated these two properties, and L. Kalantan [15] defined and studied a modification of these properties. In this paper, we define fuzzy extensions of these notions and obtain some results about them.

First, we give some previous definitions:

Definition 1. Let (X,τ) be a fuzzy topological space. We will say that (X,τ) is fuzzy C-paracompact if there exits a fuzzy paracompact space (Y,ς) (in various senses, which we will specify) and a bijection map

f: X—> Y, such that restriction $f|_K : K \to f(K)$ is a fuzzy homeomorphism for each $K \subset X$ such that its characteristic map χ_K is a Lowen's fuzzy compact subset.

Definition 2. Let (X,τ) be a fuzzy topological space. We will say that (X,τ) is fuzzy C_2-paracompact if there exists a fuzzy paracompact (in various senses, which we will specify in next definitions) Hausdorff space (Y,ς) and a bijective map $f: X \to Y$, such that restriction $f|_K : K \to f(K)$ is a fuzzy homeomorphism for each $K \subset X$ such that its characteristic map χ_K is a Lowen's fuzzy compact subset.

Remark. The kinds of fuzzy paracompact, fuzzy compact and fuzzy Hausdorff spaces cited in above definitions should be good extensions of paracompactness, compactness and Hausdorff topological spaces. We list these definitions:

Definition 3. [24] Let $r \in (0,1]$, μ be a set in a fuzzy topological space (X,τ). We say that μ is r-paracompact (resp. r^*-paracompact) if for each r-open Q-cover of μ there exists an open refinement of it which is both locally finite (resp. ⋆-locally finite) in μ and a r - Q-cover of μ. And μ is called S-paracompact (resp. S^*-paracompact) if for every r ∈ (0,1], μ is r-paracompact (resp. r^*-paracompact). We say that (X,τ) is r-paracompact (resp. r^*-paracompact, S-paracompact, S^*-paracompact) if set X verifies this property.

Definition 4. [2] Let μ be a fuzzy set in a fuzzy topological space (X,τ) .We say that μ is fuzzy paracompact (resp. r^*-paracompact) if for each open L-cover $\mathscr{U}$ of μ and for each $r \in (0,1]$, there exists an open refinement $\mathscr{V}$ of $\mathscr{U}$ which is both locally finite (resp. ⋆-locally finite) in μ and L-cover of μ-r. We say that a fuzzy topological space (X,τ) is fuzzy paracompact (resp. ⋆-fuzzy paracompact) if each constant fuzzy set in X is fuzzy paracompact (resp. ⋆-fuzzy paracompact).

Definition 5. [2] A fuzzy topological space (X,τ) is called fuzzy paracompact if for each $\mathscr{U}\subset\tau$ and for each $r\in(0,1]$ such that $sup\ \{\mu \mid \mu\in \mathscr{U}\}\geqslant$r, and for all ε (0<ε ⩽ r), there exists a locally finite open refinement $\mathscr{V}$ of $\mathscr{U}$ such that

$sup\ \{\mu \mid \mu\in \mathscr{V}\}\geq r-\varepsilon.$

Definition 6. [2] A fuzzy set μ in a fuzzy topological space (X,τ) is called fuzzy Lindelöf if for all family $\mathscr{U}\subset\tau$ such that $sup\{\upsilon \mid \upsilon\in\mathscr{U}\}\geq\mu$ } and for all ε>0 there exists a countable subfamily $\mathscr{U}_0\subset\mathscr{U}$ such that $sup\{\upsilon \mid \upsilon\in\mathscr{U}_0\}\geq\mu-\varepsilon$. The fuzzy topological space (X,τ) is fuzzy Lindelöf if each constant fuzzy set in (X,τ) is fuzzy Lindelöf.

Definition 7. [39] A fuzzy topological space (X,τ) is said to be fuzzy Hausdorff if for any two distinct fuzzy points p,q ∈ X, there are disjoint $U,V\in\tau$ with $p\in U$ and $q\in V$.

II. Main results

Lemma. Let (X,T) be a topological space and A be a subset of X. Then A is Lindelöf in (X,T) if and only if its characteristic map χ_A is a Lindelöf subset in $(X,\omega(T))$.

Proof. (⇐) For each $\mathscr{U}\subset T$, such that $A\subset\bigcup_{U\in\mathscr{U}}U$ is $\sup\{\chi_U|\mathrm{U}\in\mathscr{U}\}\geq\chi_A$. For each ε ∈ (0,1), from the hypothesis there exists a countable subfamily $\mathscr{U}_0\subset\mathscr{U}$ such that $\sup\{\chi_U|\mathrm{U}\in\mathscr{U}\}\geq\chi_A-\varepsilon$. Then $\mathscr{U}_0$ is a countable subcovering of $\mathscr{U}$.

(⇒) Let $\mathscr{F}\subset\omega(T)$ such that $\sup\{\mu|\mu\in\mathscr{F}\}\geq\chi_A$. For each ε>0 and for each $\mu\in\mathscr{F}$ if $\mu^\varepsilon=\mu+\varepsilon$ we have that, $\mathscr{S}(\mu^\varepsilon)=\left\{(x,r)/\mu^\varepsilon(x)\geq r,\mu^\varepsilon(x)\neq r\right\}$ is open in $X\times\mathbf{R}$ And $\bigcup_{\mu\in\mathscr{F}}\mathscr{S}(\mu^\varepsilon)\supset A\times\mathbf{I}$ (which is Lindelöf), because for each $(x,r)\in A\times\mathbf{I}$ (where ε<r), is $\sup\{\mu(x)|\mu\in\mathscr{F}\}=1\geq r$ and r >ε, then, there exists $\mu_0\in\mathscr{F}$ such that $\varepsilon\leq r\leq\mu_0(x)$ (with $\varepsilon\neq r$). Thus $\varepsilon\leq r\leq\mu_0(x)+\varepsilon$ (with $\varepsilon\neq r$ and $\mu_0(x)+\varepsilon\neq r$, then $(x,r)\in\mathscr{S}(\mu^\varepsilon)$.

Finally, there exists a countable subfamily $\mathscr{F}_0\subset\mathscr{F}$ such that $\bigcup_{\mu\in\mathscr{F}}\mathscr{S}(\mu^\varepsilon)\supset A\times\mathbf{I}$ and $\sup\{\mu|\mu\in\mathscr{F}_0\}\geq\chi_A-\varepsilon$, because for each $(a,1)\in A\times\mathbf{I}$ there is $\mu_0\in\mathscr{F}_0$ such that $(a,1)\in\mathscr{S}({\mu_0}^\varepsilon)$, then $1\leq\mu_0(a)+\varepsilon$ (but $\neq 1$), and $\sup\{\mu|\mu\in\mathscr{F}_0\}\geq\chi_A-\varepsilon$.

Proposition 1. Let (X,τ) be a fuzzy topological space. Then, (X,τ) is fuzzy L-paracompact if and only if $(X,\iota(\tau))$ is L-paracompact, i.e. fuzzy L-paracompactness is a good extension of L-paracompactness.

Proof. (X,τ) is fuzzy L-paracompact, i.e. there exists a fuzzy paracompact space (Y,ς) (in the sense of some good extension of paracompactness [2], [5], [24], [25]) and a bijection map $f:X\to Y$ such that restriction $f|_A:A\to f(A)$ is a fuzzy homeomorphism for each $\mathrm{A}\subset\mathrm{X}$ such that its characteristic map χ_A is a fuzzy Lindelöf subset [15]. That is, there is a paracompact space $(Y,\iota(\varsigma))$ and a bijection map $f:X\to Y$ such that the restriction $f|_A:A\to f(A)$ f is a homeomorphism for each Lindelöf subspace $\mathrm{A}\subset\mathrm{X}$, i.e. $(X,\iota(\tau))$ is L-paracompact.

Corollary. Fuzzy L_2 -paracompactness is a good extension of L_2-paracompactness.

Proposition 2. Every fuzzy Hausdorff topological space (in the Srivastava, Lal and Srivastava' or in the Wagner and McLean' sense) which is fuzzy locally compact (in the Kudri and Wagner' sense) is fuzzy L_2-paracompact.

Proof. It follows from [39, Prop.3.2], [41, Prop.3.1], [17,Th.3.3] and above Corollary.

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