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Remarks on Frobenius groups

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Abstract- Let the finite group G act transitively and non-regularly on a finite set Ω whose cardinality $|\Omega|$ is greater than one. Use N to denote the full set of fixed-point-free elements of G acting on Ω along with the identity element. Write H to denote the stabilizer of some $\alpha \in \Omega$ in G. In the note, it is proved that the subset N is a subgroup of G if and only if G is a Frobenius group. It is also proved $G = \langle N \rangle H$, where $\langle N \rangle$ is the subgroup of G generated by N.

Keywords- finite group, Frobenius group, permutation group

I. Introduction

Finite group G is a transitive permutation group acting on a set Ω, where $|\Omega| > 1$. Recall that an element g of G is a *derangement* if g acts fixed-point-freely on Ω. Let N be the subset of G consisting of all derangements together with the identity, so N is clearly a normal subset of G, but it need not be a subgroup in general. We refer to N as the derangement kernel of G. Observe that G is the union of the derangement kernel N together with all of point stabilizers, which are conjugate in G, hence $|N| > 1$. Recall that a transitive action of G on Ω is said to be a Frobenius action if every point stabilizer is nontrivial but the intersection of any two point stabilizers is trivial. A group G is called a Frobenius group when it has a Frobenius action on some set Ω whose cardinality is greater than one. A celebrated theorem of Frobenius asserts that if G is a Frobenius group, then its derangement kernel N is a proper subgroup of G ([5, Theorem 7.2]), and in that case N is called the Frobenius kernel. In [6], it is proved that if all elements in N are involutions,then N is an elementary abelian 2-group such that either $G = N$ or G is a Frobenius group with kernel N. In this note, we show that if the derangement kernel N is a proper subgroup, then the action of G on Ω is of Frobenius. When N is a subgroup, it is easy to prove $G = NH$, where H is a point stabilizer in G. In fact, there are other conditions to guarantee $G = NH$. For example, we show that $G = NH$ when G is 2-transitive on Ω (Proposition 4). Also we show that it is always true that $G = \langle N \rangle H$, where $\langle N \rangle$ is the subgroup of G generated by N (Theorem 3). We even guess that $G = NH$ whenever G has a transitive action on Ω. However, we can neither prove the claim nor give a counterexample. Under only the hypothesis that G acts transitively and non-regularly on Ω, the subset N is not generally a subgroup of G. We prove that N is a group if and only if G is Frobenius group (Theorem 1).

We mention that Frobenius groups paly a prominent role in the theory of finite groups, they usually act as either a starting point or a reduced goal (by the minimal counterexample argument) when investigating some problems of group theory, for example, see [1, 2, 8].

Unless otherwise stated, the notation and terminology is standard, as presented in [5].

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II. Results

The following result indeed shows that the derangement kernel N is a group exactly when G is a Frobenius group or a regular group. It is clear that $|G| = |N| = |\Omega|$ when G is a regular group on Ω.

Theorem 1. *Let G be a transitive and non-regular group acting on Ω with the derangement kernel N, then $|\Omega| \leq |N|$. Furthermore, the following statements are equivalent.*

1) *The action of G on Ω is Frobenius.*
2) *The set N is a subgroup of G.*
3) *The equality $|\Omega| = |N|$ holds.*

Proof. Write H for $C_G(\alpha_1)$. Since G acts transitively on Ω, it follows that $|\Omega| = n = |G : H|$ and $G = N \cup C_G(\alpha_1) \cup C_G(\alpha_2) \cup \cdots \cup C_G(\alpha_n)$, then we may deduce that $|G| \leq |N| + n(|H| - 1)$. Thus we get that

$$|G| \leq |N| + n|H| - n = |N| + |G : H||H| - n,$$

hence $|\Omega| = n \leq |N|$, as desired.

Now assume part 1. Then the derangement kernel N is just the Frobenius kernel, and Frobenius' theorem ([4,

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Satz V.7.6] or [5, Theorem 7.2]) yields that N is a group, part 2 follows.

Assuming part 2 we deduce that $|\Omega| = |\alpha_1^G| = |\alpha_1^{C_G(\alpha_1)N}| = |\alpha_1^N| = |N : C_N(\alpha_1)| = |N|$, hence $|N| = |\Omega|$, part 3 follows.

Finally assume part 3. We have $|\Omega| = |G : H|$, so $|G| = |\Omega||H|$. We are assuming that $|N| = |\Omega| = n$, and thus $|G| = |N||H|$. By the definition of N, we have $G = N \cup \bigcup_{1\le i\le n} C_G(\alpha_i)$, so if we write $C_i = C_G(\alpha_i) - 1$, we have $G = N \cup \bigcup_{1\le i\le n} C_i$. Now, $|C_i| = |H| - 1$ and by assumption, $|N| = n$, and thus we have $|N||H| = |G| = |N \cup \bigcup_{1\le i\le n} C_i| \le |N| + \sum_{1\le i\le n} |C_i| = |N| + n(|H| - 1) = |N| + |N|(|H| - 1) = |N||H|$. Equality holds, and thus the union is disjoint. Then $C_G(\alpha_i) \cap C_G(\alpha_j) = 1$ when i and j are different, and thus by definition, the action of G on Ω is Frobenius. The proof is finished. □

The following consequence may be regarded as a slight improvement of Frattini argument (see [7, Theorem 2.1.4], for example).

Lemma 2. *Let G act transitively on the set Ω where $|\Omega| > 1$, $H = C_G(\alpha)$ for $\alpha \in \Omega$ and N a subset of G. Then $G = HN$ if and only if $\alpha^N = \Omega$, where $\alpha^N = \{\alpha^n \mid n \in N\}$.*

Proof. If $G = HN$, then $\Omega = \alpha^G = \alpha^{HN} = \alpha^N$, as wanted. For $g \in G$, if $\alpha^g = \beta$ for $\beta \in \Omega$, then since $\alpha^N = \Omega$, there exists some $n \in N$ such that $\alpha^n = \beta$, thus $\alpha^{gn^{-1}} = \alpha$, so $gn^{-1} \in H$, hence $g \in HN$, and so $G = HN$, as desired. □

Theorem 3. *Let G be a transitive group acting on Ω with the derangement kernel N and $H = C_G(\alpha_1)$. Then the subgroup $\langle N\rangle$ is transitive on Ω and $\langle N\rangle H = G$. Furthermore, if NH is subgroup, then $NH = G$.*

Proof. Since N is a normal subset, it follows that$\langle N\rangle$ is a normal subgroup, and thus $\langle N\rangle H$ is a subgroup that contains N. Now $\bigcup_{g\in G}(\langle N\rangle H)^g$ contains N and all conjugates of H, and since G is the union of N and the conjugates of H, it follows that $\bigcup_{g\in G}(\langle N\rangle H)^g = G$. But it is a fact that if the union of all conjugates of some subgroup of a group is the whole group, then the subgroup must be the whole group. We have $G = \langle N\rangle H = H\langle N\rangle$. By Lemma 2, all α_i are in the $\langle N\rangle$-orbit containing α_1, and thus $\langle N\rangle$ acts transitively. Finally, suppose NH is a subgroup. Then NH contains both $\langle N\rangle$ and H, so it contains $\langle N\rangle H = G$, and thus $NH = G$. □

Observe that G may be expressible as $G = NH$ even though N is not a subgroup, as shown in the following consequence.

Proposition 4. *Let G act on the set $\Omega = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ with the derangement kernel N and $H = C_G(\alpha_1)$, $n > 1$. If the action is 2-transitive, then $G = NH$.*

Proof. Let $g \in G - H$ and $\alpha_1^g = \alpha_i$. Pick $1 \ne z \in N$ and let $\alpha_1^z = \alpha_j$. By the 2-transitivity, we know that H acts transitively on the difference set $\Omega - \{\alpha_1\}$, and so there exists $h \in H$ such that $\alpha_j^h = \alpha_i$, then $\alpha_1^{zhg^{-1}} = \alpha_1$, which implies $zhg^{-1} \in H$ and so $g \in HNH$. Because $HNH = NHH = NH$, it follows $g \in NH$. We therefore conclude $G = NH$, as desired. □

It is known that Symmetric group S_n and Alternating group A_n are 2-transitive when $n \ge 4$. Thus they have the above product form.

For the alternating group A_5 of degree 5, we may get via GAP ([3]) that
$N = \{(), (1,5,4,3,2), (1,4,2,5,3), (1,3,5,2,4), (1,2,3,4,5), (1,4,5,3,2), (1,2,4,3,5), (1,5,3,2,4), (1,4,5,2,3), (1,5,4,2,3), (1,3,4,5,2), (1,5,3,4,2), (1,3,2,4,5), (1,3,2,5,4), (1,2,4,5,3), (1,5,2,3,4), (1,2,5,4,3), (1,4,3,2,5), (1,2,3,5,4), (1,4,3,5,2), (1,3,4,2,5), (1,5,2,4,3), (1,4,2,3,5), (1,3,5,4,2), (1,2,5,3,4)\}$, and its subset

$$\{(), (1,5,4,3,2), (1,4,2,5,3), (1,3,5,2,4), (1,2,3,4,5)\}$$

is actually a right transversal for A_4 in A_5, thus we achieve that $A_5 = A_4N = NA_4$ (as N is a normal subset). As $(1,4,5,3,2) * (1,5,4,3,2) = (1,3)(2,5) \notin N$, we see that N is not a group. (The nonabelian simple group A_5 has a proper normal subset N and a nontrivial factorization form $A_5 = NA_4$. This is really an interesting thing!) For A_6, we may also verify via GAP ([3]) that $A_6 = A_5N = NA_5$, where N is the derangement kernel of A_5.

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