The relationship profitability - risk for an optimal portfolio building with two risky assets and a risk-free asset

Florentin Serban, Maria Viorica Stefanescu, Silvia Dedu

***Abstract*—**This paper present a combination of classification theory with risk estimation theory to determine the best assets through those that we analyzed. We present also building to the efficient frontier for a portfolio with 2 or 3 assets We use a data analysis method to obtain two classes of assets and then we estimate the risk of each asset corresponding to each class. Thus, we get the best two assets among those considered risky for which we build the efficient frontier if we consider that the portfolio consists of these two risky assets and a risk-free asset. To illustrate the effectiveness of the method used, we present a case study that refers to the Bucharest Stock Exchange. We construct the efficient frontier, based on the correlation of the best stocks that we have obtained through data analysis (for classification), and by assessing the loss distribution (for risk assessment), taking into account that the portfolio contains an asset without risk.

***Keywords*—** efficient frontier, optimization, portfolio selection, principal component analysis, risk estimation

1. INTRODUCTION

Financial assets portfolio optimization is an important area, which developed the theory of Markowitz's mean-variation and the expected utility theory. Mean-variance theory has some limitations and can be applied successfully only if the expected returns of financial securities are normally distributed random variables. However, the literature contradicts the hypothesis of normality for the expected returns and it is a strong argument against the use of the mean-variation techniques, which is why they introduced new measures of risk. Value-at-Risk (VaR) is a measure of risk, which plays an important role in investment, risk management and regulatory control of financial institutions. Since in most cases the distribution of random variable risk is not known, a method of evaluation or estimation of VaR is required.

We propose to solve the problem of a portfolio optimization in two stages: assets selection and risk estimation.

The selection of assets is realized by applying principal components analysis in order to reduce the number of characteristics of the assets to be taken into account. Methodology based on clustering techniques is a useful tool for understanding the structure and the hierarchy in the financial data. The idea to obtain clusters that characterize a set of assets can be found also in Mantegna[8], Kaski et al.[6], Stefanescu et al. [12], Serban et al [10,11], Brida and Risso [2] who applied clustering techniques to classify the stocks of Milan and Frankfurt Stock Exchanges using the Pearson correlation coefficient. Since the stock market in Romania is not mature enough we can not afford to consider only the closing price to determine the "distance" between stocks.

We will use data analysis to consider several criteria which we think are important in determining proximity or remoteness of the stocks.

In the second stage, we will present an approach to estimating risk using historical simulation method. Then we will construct the efficient frontier of the profitability of a portfolio consisting of the best two risky assets that we obtain and a risk-free asset. At the end we solve a case study for the stocks listed on the Bucharest Stock Exchange. We will build a portfolio and we construct the efficient frontier.

2. BUILD OF THE PORTFOLIO

2.1. ASSETS CLUSTERING

In the context of nowadays financial markets it is a huge amount of available financial data. It is therefore very difficult to make use of such an amount of information and to find basic patterns, relationships or trends in data. We apply data analysis techniques in order to discover information relevant to financial data, which will be useful during the selection of assets and decision making. Consider that we have collected information on a number N of assets, each with P features, which represent various financial ratios, still called variables. Denote by x_i^j the j-th variable for action i. Multivariate data set will be represented by a matrix $X = \left(x_i^j\right)_{\substack{i=\overline{1,N} \\ j=\overline{1,P}}}$ and can be viewed as a set of N points in a P-dimensional space.

Principal components analysis (PCA) is a useful technique for analyzing data to find patterns of data in a large-scale data space. PCA involves a mathematical procedure that transforms P variables, usually correlated in a number of $p \le P$ uncorrelated variables called principal components. After applying the PCA, each asset i will be characterized by p variables, represented by a set of parameters $x_i^1, x_i^2, ..., x_i^p$ therefore, it is possible to form the arrays $X_i = \left(x_i^1, x_i^2, ..., x_i^p\right),\ i = \overline{1,N}$, which correspond to a set of S assets. Suppose now that we obtained a data set $X_i = \left(x_i^1, x_i^2, ..., x_i^p\right),\ i = \overline{1,N}$. We then use clustering techniques in order to find similarities and differences between the actions under consideration.

The idea of clustering is an assignment of the vectors $X_1, X_2, ..., X_N$ in T classes $C_1, C_2, ..., C_T$. Once completed the selection of activities, we construct the initial portfolio by selecting low-risk asset in each class

2.2. PHASE ESTIMATION RISK

We evaluate the performance of an asset using expected future income, an indicator widely used in financial analysis. Denote by $S_j(t)$ the closing price for an asset j at time t. Expected future income attached to the time horizon $[t, t+1]$ is : $R_j(t) = \ln P_j(t+1) - \ln P_j(t),\ j \in \overline{1, N}.$

Similary, we define the loss random variable, the variable L_j, for asset j for $[t, t+1]$ as : $L_j(t) = -R_j(t) = \ln P_j(t) - \ln P_j(t+1),\ j \in \overline{1, N}.$

Using Rockafellar et al..[9] define the risk measure VaR corresponding loss random variable L_j. Probability of L_j not to exceed a threshold $z \in \mathbf{R}$ is $G_{L_j}(z) = P(L_j \le z)$..

Value at risk of loss random variable L_j associated with the value of asset j income and corresponding probability level $\alpha \in (0,1)$ is: $VaR_\alpha(L_j) = \min\{z \in \mathbf{R} \mid G_{L_j}(z) \ge 1 - \alpha\}$ or P (X > VaR) = α. If G_{L_j} is strictly increasing and continuous, $VaR_\alpha(L_j)$ is the unique solution of equation $G_{L_j}(z) = 1 - \alpha$ then $VaR_\alpha(L_j) = G_{L_j}^{-1}(1-\alpha)$.

One of the most frequently used mehods for estimating the risk is the *historical simulation method.* This risk assessment method is useful if empirical evidence indicates that the random variables in question may not be well approximated by normal distribution or if we are not able to make assumptions on the distribution. Historical simulation method calculates the value of a hypothetical changes in the current portfolio, according to historical changes in risk factors. The great advantage of this method is that it makes no assumption of probability distribution, using the empirical distribution obtained from analysis of past data. Disadvantage of this method is that it predicts the future development based on historical data, which could lead to inaccurate estimates if the trend of the past no longer corresponds.

If L_j is the loss random variable and $\hat{G}_n$ is empirical distribution function of L_j and $\alpha \in (0,1)$ a fixed level of probability, then $\hat{G}_n(z) = \frac{1}{n}\sum_{i=1}^{n} F_{\{L_j \le z\}}$. We can prove that

$$VaR(L_j) = \min\left\{ z \in \mathbf{R} \mid \frac{1}{n}\sum_{i=1}^{n} F_{\{L_j^i \le z\}} \ge 1 - \alpha \right\}.$$

3. THE EFFICIENT FRONTIER OF THE RENTABILITY

Assume that on the market there are 2 risky actives. The active „*i*" has percentage of rentability r_i, where:

-The average $M(r_i) = \mu_i$

-Standard deviation of $r_i = \sigma_i$

Definition 1: A portofolio is called "profitable" if among all the portofolio with the same standard deviation of the rentability, has the best average

Definition 2 : A portofolio is called „efficient" if , starting from a set of shares and taking all the linear combination of the titles within the portofolio, we are looking for those titles which dominates other (profitable) titles

Definition 3: The set of profitable portofolios is called „the frontier of rentability"(Markowitz), .

Consider that the sell of the live actives is not permitted (this means that the weights $X_{1,2}$ of the actives within the portofolio are strictly positive) and that the risks $r_{1,2}$ of the actives have the averages $\mu_{1,2}$, with $\mu_1 < \mu_2$ and deviations $\sigma_{1,2}$, with $\sigma_1 < \sigma_2$

Therefore,

-covariance $\sigma_{12} = \rho \cdot \sigma_1 \cdot \sigma_2$, where „$\rho$" is the correlation coefficient.

- the weights $X_{1,2}$ of the assets of the portfolio are $X_1 = X$, respectively $X_2 = 1 - X$, $X \in (0,1)$.

- the profitability rate of the portofolio is a random variable r_p

$$r_p = X \cdot r_1 + (1 - X) \cdot r_2$$

-the average of the profitability rate is :

$$\mu_p = X \cdot \mu_1 + (1 - X) \cdot \mu_2 \quad (1)$$

- the variance of the profitability, assimilated with a risk index, is: $\sigma_p^2 = X^2 \cdot \sigma_1^2 + (1-X)^2 \cdot \sigma_2^2 + 2X \cdot (1-X) \cdot \rho \cdot \sigma_1 \sigma_2$ (2)

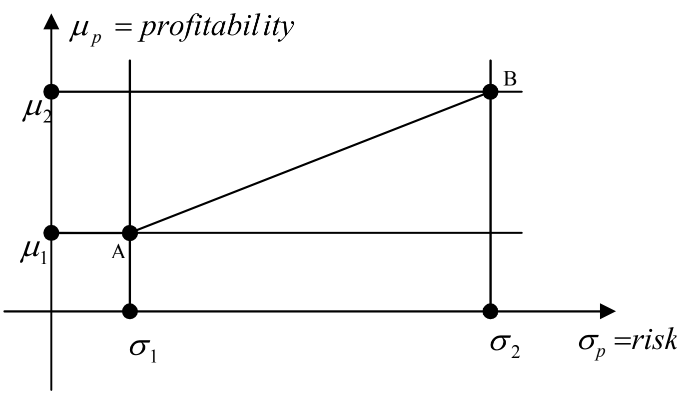
In what follows, we will highlight the contribution of the correlation coefficient „ ρ ":

A. CASE $\rho = 1$:

Relation (2) becomes: $\sigma_p^2 = \left[X \cdot \sigma_1 + (1 - X) \cdot \sigma_2\right]^2$, therefore: $\sigma_p = X \cdot \sigma_1 + (1 - X) \cdot \sigma_2$ (3)

$$\left.\begin{array}{l} \textit{From } (1) \Rightarrow X = \dfrac{\mu_p - \mu_2}{\mu_1 - \mu_2} \\ \textit{From } (3) \Rightarrow X = \dfrac{\sigma_p - \sigma_2}{\sigma_1 - \sigma_2} \end{array}\right\} \Rightarrow \mu_p = \frac{\mu_2\sigma_1 - \mu_1\sigma_2}{\sigma_1 - \sigma_2} + \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2} \cdot \sigma_p \quad (4)$$

Relation (4) represent the equation of a line in the (σ, μ)-plane, more precisely



Comments for the case $\rho = 1$:

-One cannot expect any gain from diversifying the portfolio: The frontier of efficiency AB coincide with the set of all portfolios possible to achieve with the given assets by linear combination;

-Any raise in the average of profitability (i.e. μ) will be accompanied by a corresponding growth of the risk (i.e. σ).

B. CASE $\rho = -1$:

From relation (2) we deduce:

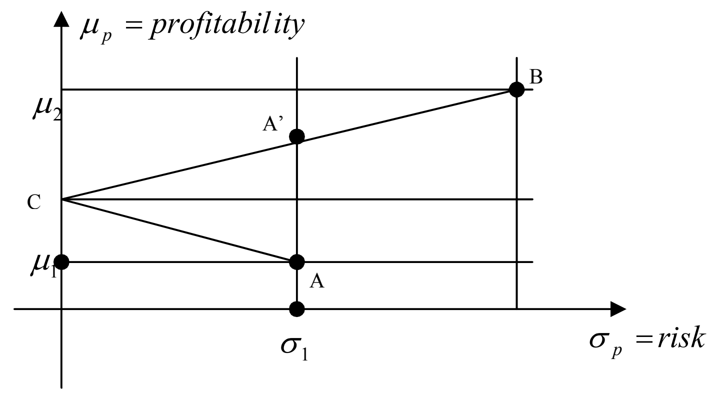
$\sigma_p^2 = \left[X \cdot \sigma_1 + (1 - X) \cdot \sigma_2\right]^2$; since $\sigma_p > 0$, we get

$$\sigma_p = \left|X \cdot \sigma_1 - (1 - X) \cdot \sigma_2\right| \quad (5)$$

From relations (1), (5) , we deduce:

$$\mu_p = \frac{\mu_2\sigma_1 + \mu_1\sigma_2}{\sigma_1 + \sigma_2} \pm \frac{\mu_1 - \mu_2}{\sigma_1 + \sigma_2} \cdot \sigma_p \quad (6)$$

Graph of the relation (6) is given by the line (ACB) in the following figure:



Comments for the case $\rho = -1$:

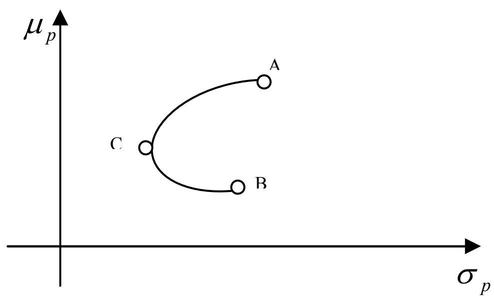
-Only the segment (CB) represents the frontier of eficiency: a cautious investor cannot accept the segment (AC), because any portfolio on the segment AC is strictly dominated by one on the segment CA', which has the same risk, but a better profitability.

-It is clear the gain obtained by the diversification of the portfolio: we start from A with $X = 1$; when „*X*" decreases, we move towards C, realising the following: the growth of the average of the profitability and the diminuation of the risk; On the segment AC, the marginal risk of the asset A_2 is negative, therefore, the growth of the weight of the asset 2 within a portfolio leads to a diminuation of the risk; the point „C" represents a portfolio without risk, composed by two risky assets.

- On the segment (CB) is displayed yet another risk for any growth of the average: therefore, the marginal risk of the asset 2 on the segment (CB) becomes positive.

C. CASE $-1<\rho<1$:

In this case, σ_p^2 is not a perfect square: the graph of the function implicitely expressed by the functions (1), (2) is presented below. Figure 3 contains the parabola (ACB) which is exactly the frontier of the eficiency of the portfolio.



Comments:

- The minimal risk achieved in the point (C) is not zero.
- Only the segment (CA) is a frontier of rentability: a prudent investitor cannot accept the segment (BC) because any portofolio on the segment (BC) is strictly dominated by a portofolio on the segment (CA) which has the same risk but a better rentability.

D. CASE FOR 3 ASSETS

Assume now that there are 3 assets on the market. The asset *i* has percentage of rentability r_i, where the average $M(r_i)=\mu_i$ and standard deviation of r_i is σ_i. For $i,j=\overline{1,3}$; $i\neq j$, assume that we knows the covariance σ_{ij}. Total correlation matrix is $V=\left\|\sigma_{ij}\right\|_{i,j=\overline{1,3}}$. We denote by $x_i\left(i=\overline{1,3}\right)$ - the weight of the active A_i within the portfolio.

The random variable which gives the profitability of the portfolio: $r_p=\sum x_i\cdot r_i$

Consider that the weights $x_{1,2,3}$ of the assets within the portofolio are positive and that the risks $r_{1,2,3}$ of the actives have the averages $\mu_{1,2,3}$ and deviations $\sigma_{1,2,3}$.

Therefore, the weights $\{x_i\}_{i=\overline{1,3}}$ satisfy the relations :

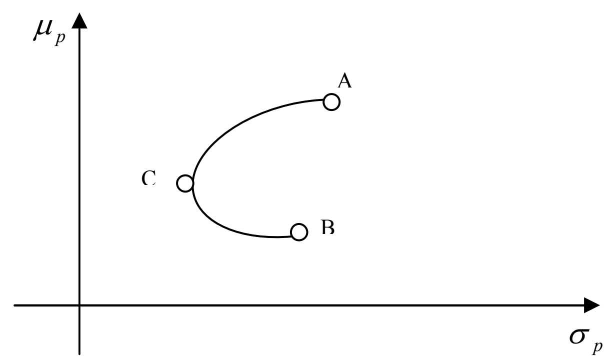
$x_i>0$; $i=\overline{1,3}$

$$\sum_i x_i=1;\ m_p=\sum_i x_i\cdot\mu_i;\ \sigma_p^2=\sum_{i=1}^{n}\sum_{j-1}^{n}x_i\cdot x_j\cdot\sigma_{ij}$$

In order to determine the optimal $\{x_i\}_{i=\overline{1,n}}$, we use the nonlinear model:

$$\begin{cases}(\min)\sigma_p^2=\sum_{i=1}^{3}\sum_{j-1}^{3}x_i\cdot x_j\cdot\sigma_{ij}\\ \left|\sum_i x_i=1\right.\\ \left|\sum_i x_i\cdot\mu_i=m_p\right.\end{cases},x_i>0;i=\overline{1,3}$$

The efficient frontier of the portfolio in the $\left(\sigma_p^2;m_p\right)$plane is a parabola:



Comments:

- Only the segment (CA) is a frontier of rentability: a prudent investitor cannot accept the segment (BC) because any portofolio on the segment (BC) is strictly dominated by a portofolio on the segment (CA) which has the same risk but a better profitability.

4. APPLICATION OF OPTIMIZATION OF A PORTFOLIO OF STOCKS ON BSE

4.1. FINANCIAL RATIOS USED IN THE STOCKS EVALUATION

In any process of making investment decisions, the fundamental aspect of information is accurate and current knowledge. The investment decision is therefore based on an investment analysis that is based on qualitative factors of a portfolio assessment and continues with determining the optimal timing of sale and purchase of financial assets in which it is established. This analysis of the investment portfolio consists of a fundamental analysis and technical analysis. This analysis takes into account the "health" the company's financial, macroeconomic and political conditions of the environment in which the company operates global development factor that is part of the industry and forecasts on the company's performance in future. The fundamental analyses search to find the stocks who are available to investors on the capital market, whether it is the primary or secondary. These indicators are obtained based on analysis of financial results contained in the company's financial reports (profit and loss, balance sheet) and the price formed on the supply and demand confrontation. While fundamental analysis seeks to determine a course of action theory, an intrinsic value and from this, if the securities in question are over-, under-or correctly evaluated. For this type of analysis necessary to obtain information and data from company balance sheets. Based on these data can produce forecasts of future income flows, dividends and the price.

We will present the financial indicators that we will use in our study:

- *The BV (Accounting value)* of a stock is calculated by dividing the total amount of equity in the company by the total number of stock issued by it and that are in circulation. The value of proper equity is determined by subtracting total liabilities from the company's total assets owned and it represents the "wealth of shareholders", that is what they remain with in the assumption that these assets would be valued and all debts would be paid
- *The PBR* (*The price-to-book ratio)* of a stock is calculated by dividing the company's current stock price by the book value per stock
- *PSR indicator* is calculated by dividing the current market value per stock of turnover in the last 4 consecutive quarters
- *Evolution of price*: to observe the price level at a given time we take into acount the maximum price and minimum price achieved in the last 9 months

We used information on a total of 60 stocks representing stocks of Class I and II, traded on the Bucharest Stock Exchange on 01.08.2011. Then we considered only the stocks very volatile and for which it is possible to calculate the indicators mentioned and resulting in 40 stocks. We take into account several characteristics for each stock, we use data analysis techniques in order to process this vast amount of information.

We consider for each asset the values of five characteristics described above on 01. 08. 2011.

TABLE 1: THE VALUE OF THE 5 CHARACTERISTICS

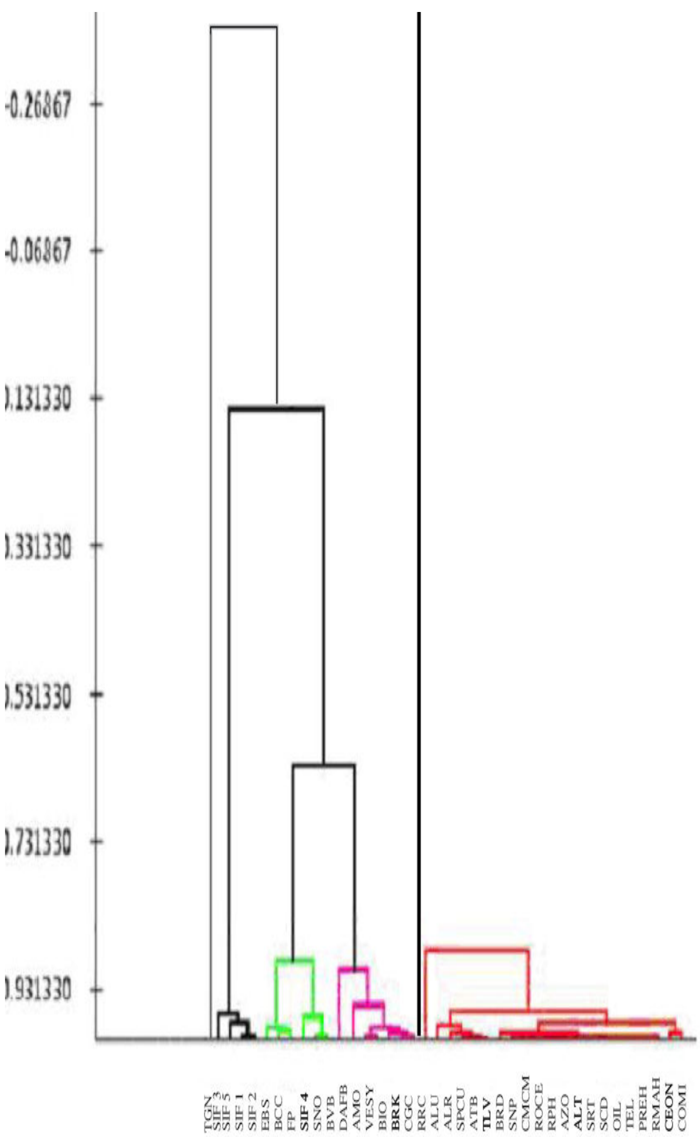
No	Simbol	BV	PBR	PSR	Min/ P	Max/P
1	FP	1.192	0.43	0.9	0.88	1.25
2	SIF5	2.904	0.46	0.9	0.76	1.31
3	SIF2	2.313	0.51	0.9	0.69	1.13
4	BRD	6.83	2.01	0.9	0.8	1.17
5	TLV	1.12	1.05	0.9	0.84	1.14
6	BVB	11.471	3.3	0.9	0.76	1.24
7	BCC	0.08	0.92	0.9	0.85	1.53
8	SIF1	2.629	0.51	0.9	0.7	1.26
9	SIF3	1.338	0.36	0.9	0.87	1.47
10	SIF4	1.697	0.39	0.9	0.89	1.32
11	DAFR	0.166	0.62	0.5	0.77	1.06
12	TEL	34.33	0.6	0.58	0.71	1.16
13	COMI	0.556	0.510	0.41	0.97	1.05
14	TGN	236.23	0.95	2.00	0.73	1.33
15	BRK	0.289	0.71	0.9	0.89	1.23
16	SNP	0.325	1.15	1.08	0.83	1.02
17	ATB	0.577	0.86	0.93	0.8	1.05
18	AZO	1.484	0.7	0.33	0.63	1.15
19	BIO	0.138	1.51	2.59	0.71	1.16
20	PREH	0.166	0.72	1.06	0.85	1.23
21	ALR	2.212	1.64	1.32	0.8	1.06
22	SNO	7.72	0.41	0.53	0.98	1.44
23	SCD	0.86	1.15	1.6	0.68	1.05
24	VESY	0.195	0.31	0.15	0.83	1.4
25	OIL	0.59	0.41	1.06	0.86	1.26
26	AMO	0.077	0.25	0.33	0.97	1.7
27	COTR	130.4	0.25	0.91	0.81	1.4
28	RMAH	0.28	0.7	0.11	0.82	1.06
29	SPCU	0.544	0.48	0.48	0.93	1.25
30	CEON	0.179	0.51	0.98	0.97	1.44
31	EBS	156.02	0.9	0.9	0.7	1.26
32	PTR	0.433	0.8	1.16	0.97	1.42
33	RPH	0.299	2.87	0.69	0.9	1.22
34	ALT	0.132	0.32	0.29	0.87	1.13
35	MPN	0.328	0.7	0.29	0.88	1.27
36	RRC	0.045	1.39	0.17	0.78	1.12
37	CMCM	0.323	0.07	1.16	0.77	1.02
38	ROCE	0.559	0.36	0.47	0.91	1.54
39	ALU	2.055	0.7	0.64	0,78	1.78
40	CGC	0,323	0.07	0.15	0.98	1.66

SOURCE: WWW.BVB.RO

4.2 PRINCIPAL COMPONENT ANALYSIS

We apply data analysis techniques to discover the similarities and differences between the stocks of the Bucharest Stock Exchange, using the package XL STAT 1.8. Figure 1 contains the tree resulted from PCA. Stocks belonging to the same cluster are similar in terms of characteristicss taken into account. In order to build a diversified portfolio, we first choose the number of clusters (for our study, we chose 2), which will be taken into account.

FIGURE 1 GROUP OF STOCKS



SOURCE: PACKAGE OF PROGRAMMES XL STAT

Remarks**:** We observe the 2 classes in which the stocks were grouped. Those classes are presented still.

4.3 RISK ESTIMATION

We used the closing price values daily for each stock, corresponding to a time horizon of 50 days to measure VaR for each stock. We used the data available on the Bucharest Stock Exchange from May 2011 – June 2011. The following tables contain values of VaR for each stock and three levels of probability values

TABLE2 : VAR FOR EACH STOCK

Class 1	**0.9**	**0.95**	**0.99**
ALU	0.0570	0.0378	0.0876
AMO	0.0721	0.089	0.135
BCC	0.0721	0.0891	0.1325
BIO	0.0235	0.0372	0.0434
BRK	0.0223	0.0424	0.0486
BVB	0.0270	0.0299	0.0676
COTR	0.0496	0.0702	0.1054
CGC	0.0512	0.0747	0.1289
DAFR	0.0328	0.0400	0.0468
EBS	0.031	0.082	0.2134
FP	0.0572	0.0650	0.1074
ROCE	0.0415	0.0705	0.1025
RPH	0.0428	0.0612	0.136
SIF1	0.0276	0.0317	0.0506
SIF2	0.0225	0.027	0.0329
SIF3	0.0255	0.0305	0.0543
SIF4	0.0237	0.028	0.0588
SIF5	0.0242	0.026	0.0495
SNO	0.0218	0.0623	0.1042
TGN	0.0167	0.0221	0.0527
VESY	0.0543	0.0I58	0.1057

Class 2	**0.90**	**0.95**	**0.99**
ALR	0.0415	0.0705	0.1025
AZO	0.0265	0.0405	0.0669
ALR	0.0139	0.0171	0.0519
ATB	0.0221	0.0293	0.033
BRD	0.0257	0.0301	0.0275
CEON	0.0287	0.0455	0.0995
CMCM	0.0348	0.0383	0.0584
COMI	0.0282	0.0313	0.042
OIL	0.0497	0.0723	0.1043
PREH	0.0568	0.0807	0.1476
RMAH	0.0465	0.055	0.1003
ROCE	0.0317	0.0485	0.0584
RPH	0.0421	0.0833	0.1034
RRC	0.0531	0.0742	0.0969
SCD	0.0248	0.0471	0.075
SNP	0,0201	0.0248	0.0325
SPCU	0.0418	0.0585	0.2624
TEL	0.0247	0.0347	0.0492
TLV	0.0134	0.0204	0.0259

SOURCE: PACKAGE OF PROGRAMMES XL STAT

4.4 CONSTRUCTION OF AN OPTIMAL PORTFOLIO MADE OF 2 STOCK

We start from the classes we formed above and we choose from each of the stock which has minimal VaR for the probability 0.99.

We get the two stocks (TLV, BIO) which, together with the risk-free asset, will form the portfolio to which we will build the efficient frontier.

4.5 EFFICIENT FRONTIER FOR PORTFOLIO OF TWO RISKY ASSETS AND A RISK-FREE ASSET

We selected the closing prices for the period 9.04.2011-9.05.2011 for those 2 selected stocks:TLV and BIO We will transform the prices in anual rentability by the formula (final price-initial price)x360/ initial price and we get

$$X_{TLV}:\begin{pmatrix} -28.8 & -18 & -10.8 & -8.3 & 0 & 5.8 & 14.4 & 28.8 \\ \frac{1}{20} & \frac{3}{20} & \frac{4}{20} & \frac{2}{20} & \frac{5}{20} & \frac{2}{20} & \frac{2}{20} & \frac{1}{20} \end{pmatrix}$$

$$X_{BIO}:\begin{pmatrix} -7.2 & -1.8 & 0 & 1.8 & 3.6 & 7.2 & 10.8 & 14.4 \\ \frac{1}{20} & \frac{3}{20} & \frac{7}{20} & \frac{4}{20} & \frac{2}{20} & \frac{1}{20} & \frac{1}{20} & \frac{1}{20} \end{pmatrix}$$

We apply the formulae specific to statistics and we obtain:

$\mu_1 = 3.1$ and deviation $\sigma_1 = 16$;

$\mu_2 = 1.7$ and deviation $\sigma_2 = 4.7$;

$\mathrm{cov}_{12} = 12$

We consider $\mu_3 = 8$ and deviation $\sigma_3 = 0$

We have $\mathrm{cov}_{13} = 0$; $\mathrm{cov}_{23} = 0$

Let x.y.z be the weights of the 3 stocks. Following the relations from section 3 we obtain:

$m_p = 3.1\,x + 1.7\,y$; $\sigma_p^2 = 256\,x^2 + 22\,y^2 + 2\,12\,xy$

We therefore have the following problem of nonlinear optimization:

$$\text{Min } \sigma_p^2 = 256\,x^2 + 22\,y^2 + 24\,xy$$

$$\begin{cases} 3.1\,x + 1.7\,y + 8\,z = m & (1) \\ x + y + z = 1 & (2) \\ x, y, z > 0 & (3) \end{cases}$$

We replace x and y in (1) and (2) in the goal function and we get :

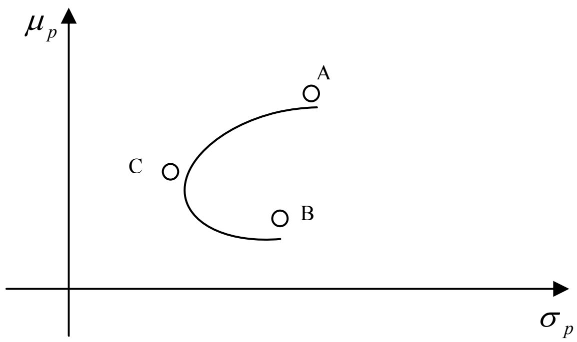
$$\sigma_p^2 = 3960z^2 - 2\,(64 - 722m)z + 132m^2 - 386\,m + 418$$

Minimum of this function is achieved for

$$z = \frac{64 - 722m}{3960} = 0.016 - 0.18\,m$$

We replace z in the goal function and we get $\sigma_p^2 = 250\,m_p^2 - 703\,m_p + 417$

In the profitability-risk plane the previous relation is the frontier of profitability for the portfolio made of the BIO, TLV and one risk-free asset. Graphicly, we get the representation below, with the remark that the eficient frrontier is just the curve AC.



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