Algorithms of approximate solving of some linear operator equations containing small parameters

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***Abstract*—**In the paper algorithms of approximate solving of some linear operator equations containing small parameters are described. In particular, an asymptotic method and an alternative variant of the asymptotic method are used. Algorithms and program products represent a new technology of approximate solving of system of linear algebraic equations, two–point boundary value problem, some linear nonhomogeneous integro-differential equations and singular integral equations containing an immovable singularity.

***Keywords*—**Nonhomogeneous operator equation, orthogonal series, asymptotic method, alternative method.

I. INTRODUCTION

PERTURBATION theory is a method to study and solve topical problems of science and technology. It comprises mathematical methods that are used to find an approximate solution to a problem which cannot be solved exactly; among them Poincare-Lyapunov's method, known as a small parameter method, is widely applied as one of the most powerful methods of research and calculation, but its convergence is asymptotic. While using the asymptotic method we obtain a two-point recurrent system of equations. In the present paper, problems of approximate solution of some boundary value problems are studied by means of a numerical-experimental method based on an approach, alternative to the asymptotic method. The method is developed by T. Vasha-kmadze ([1], pp. 124-127). It is based on presentation of the required vector with an orthogonal series instead of asymptotic one. In this case there is obtaining three-point recurrent system of operator equations of the special structure. A solution of this system is obtained inverting a relatively simple operator N-times and acting with the operator describing a perturbation degree over the known magnitudes N-times. The degree N of the polynomial defines exactness of the method. Algorithms of approximate solution of a linear nonhomogeneous operator equation are considered using both asymptotic and its alternative methods in [2]. There were shown general formulation of both methods and essential differences between them. Their comparative analysis was made. In the present paper the above-mentioned methods are approved for problems of finding an approximate solution of system of linear algebraic equations, a double-point boundary problem with variable coefficients, some linear nonhomogeneous integro-differential equation and singular integral equations containing an immovable singularity.

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II. ALGORITHMS OF APPROXIMATE SOLUTION OF A LINEAR OPERATOR EQUATION

Let us have a nonhomogeneous operator equation

$$Lu + \varepsilon Mu = f , \qquad (1)$$

where L and M are linear operators in any standardized space. In addition, there exist inverse operators L^{-1} and $(L + \varepsilon M)^{-1}$, where $\varepsilon \in [-1;1]$ is a small parameter.

Represent the solution $u(x)$ in the form of Fourier-Legandre series

$$u(x) = \gamma \sum_{k=0}^{\infty} \varepsilon^k v_k(x) + + (1-\gamma)\sum_{k=o}^{\infty} P_k(\varepsilon) w_k(x) \qquad (2)$$

where $\{P_k(\varepsilon)\}$ is a system of Legendre polynomials, $w_k(x)$ and $v_k(x)$ are unknown coefficients, γ is numerical parameter.

A method. In case $\gamma = 1$ we have an asymptotic method (Poincare-Lyapunov's method)

$$u(x) = \sum_{k=0}^{\infty} \varepsilon^k v_k(x) \qquad (3)$$

By putting series (3) into equation (1) and equating coefficients of terms with the same degrees of ε, we get a system of two-point recurrence operator equations having the following form:

$$\begin{cases} Lv_0 = f_0 \\ Lv_k = -Mv_{k-1} + f_k, \quad k = \overline{1,\infty} \end{cases} \qquad (4)$$

B method. When $\gamma = 0$, we have an approach alternative to the asymptotic method (see [1],[2])

$$u(x) = \sum_{k=0}^{\infty} P_k(\varepsilon) w_k(x) \tag{5}$$

If set series (5) into equation (1) use the main properties of Legandre polynomial, in case

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x),$$

with us

$$\varepsilon P_k(\varepsilon) = \frac{k+1}{2k+1} P_{k+1}(\varepsilon) + \frac{k}{2k+1} P_{k-1}(\varepsilon)$$

$$k = 1,2,...$$

and equate coefficients with equal degrees of ε, we shall get a system of three –point recurrence operator equations of the following form:

$$\begin{cases} Lw_0 + \frac{1}{3} Mw_1 = f_0 \\ Lw_k + \frac{k}{2k-1} Mw_{k-1} + \frac{k+1}{2k+3} Mw_{k+1} = f_k \end{cases} \tag{6}$$

$$k = 1,2,3,...$$

Suppose that the right hand side of equations f does not depend on small parameter ε therefore $f_k = 0$, $k = 1,2,3,...$, and even it depends on ε, this fact has no essential influence on realization of computing schemes.

Solutions of system of double-point operator equations (4) have the following simple form:

$$v_0 = L^{-1} f_0,$$
$$v_1 = -L^{-1}ML^{-1} f_0,$$
$$v_2 = L^{-1}ML^{-1}ML^{-1} f_0,$$
$$v_3 = -L^{-1}ML^{-1}ML^{-1}ML^{-1} f_0,$$

etc, i.e.

$$v_i = -L^{-1}Mv_{i-1} \qquad i = 1,2,3,... \tag{7}$$

Instead of infinite system of three-point operator equations (6), let us take its finite part, in addition even number of equations ($N = 2n$, $n \in \mathrm{N}$).

Let us introduce suitable designations to perform the algorithm:

$Lw_k = b_k$, $-Mw_k = t_k$, $k = 0,1,2,...$

and rewrite the (6) system of three-pointed operator equations in the foloving form:

$$b_0 = \frac{1}{3} t_1 + f_0,$$

$$b_k = \frac{k}{2k-1} t_{k-1} + \frac{k+1}{2k+3} t_{k+1}, \tag{8}$$

$$k = 0,1,2,..., N-1 \qquad (N = 2n).$$

The obtained (8) system of equations may be decomposed into two subsystems, when k is even (b with even numbers are mapped by t with odd numbers) and when k is odd (b with odd numbers are mapped by t with even numbers)

We find w_k, by means of regular process [1].

For clearness, let us consider special cases, when N=2, N=4 , N=6 and N=8.

When N=2 , then

$$\begin{cases} b_0 = f_0 & (t_1 \equiv 0), \\ b_1 = t_0 & (t_2 \equiv 0), \end{cases}$$

i.e.

$$\begin{cases} Lw_0 = f_0, \\ Lw_1 = -Mw_0. \end{cases} \tag{9}$$

The solution of system (9) has a form:

$$w_0 = L^{-1} f_0 = v_0,$$
$$w_1 = -L^{-1}ML^{-1} f_0 = -L^{-1}Mw_0 = v_1,$$

i.e.

When $N = 2$, we get $w_0 = v_0$, $w_1 = v_1$ and approximate solution of equation (1) has a form:

$$u_1(x) = w_0(x) + \varepsilon w_1(x). \tag{10}$$

Note: It is natural that asymptotic and alternative methods are coincide when $N = 2$, because

$$P_0(\varepsilon) = 1,$$
$$P_1(\varepsilon) = \varepsilon.$$

When $N = 4$, then we get:

$$\begin{cases} b_0 = \frac{1}{3}t_1 + f_0, \\ b_2 = \frac{2}{3}t_1 \quad (t_3 \equiv 0), \\ b_1 = t_0 + \frac{2}{5}t_2, \\ b_3 = \frac{3}{5}t_2 \quad (t_4 \equiv 0). \end{cases} \tag{11}$$

As a result of simple regular transformations we get from system (11)

$$w_0 = L^{-1}f_0 + \frac{1}{3}L^{-1}ML^{-1}ML^{-1}f_0 = v_0 + \frac{1}{3}v_2,$$

$$w_2 = \frac{2}{3}L^{-1}ML^{-1}ML^{-1}f_0 = \frac{2}{3}v_2,$$

$$w_1 = -L^{-1}ML^{-1}f_0 - \frac{3}{5}L^{-1}ML^{-1}ML^{-1}ML^{-1}f_0 = v_1 + \frac{3}{5}v_3,$$

$$w_3 = -\frac{2}{5}L^{-1}ML^{-1}ML^{-1}ML^{-1}f_0 = \frac{2}{5}v_3,$$

i.e.
When $N = 4$ then we have

$$w_0 = v_0 + \frac{1}{3}v_2,$$

$$w_2 = \frac{2}{3}v_2,$$

$$w_1 = v_1 + \frac{3}{5}v_3,$$

$$w_3 = \frac{2}{5}v_3$$

and approximate solution of equation (1) has a form

$$u_3 = w_0(x) + \varepsilon w_1(x) + \\ + P_2(\varepsilon)w_2(x) + P_3(\varepsilon)w_3(x), \tag{12}$$

where

$$P_2(\varepsilon) = \frac{1}{2}(3\varepsilon^2 - 1), \quad P_3(\varepsilon) = \frac{1}{2}(5\varepsilon^3 - 3\varepsilon).$$

When N=6 , then

$$\begin{cases} b_0 = \frac{1}{3}t_1 + f_0, \\ b_2 = \frac{2}{3}t_1 + \frac{3}{7}t_3, \\ b_4 = \frac{4}{7}t_3 \quad (t_5 \equiv 0), \\ b_1 = t_0 + \frac{2}{5}t_2, \\ b_3 = \frac{3}{5}t_2 + \frac{4}{9}t_4, \\ b_5 = \frac{5}{9}t_4 \quad (t_6 \equiv 0). \end{cases} \tag{13}$$

As a result of simple regular transformations we get from system (13)

$$\begin{cases} w_0 = v_0 + \frac{1}{3}v_2, \\ w_2 = \frac{2}{3}v_2 + \frac{3}{7}v_4, \\ w_4 = \frac{4}{7}v_4, \\ w_1 = v_1 + \frac{3}{5}v_3, \\ w_3 = \frac{2}{5}v_3 + \frac{5}{9}v_5, \\ w_5 = \frac{4}{9}t_5. \end{cases}$$

An approximate of equations (1) is calculated by the formula:

$$u_5(x) = \sum_{k=0}^{5} P_k(\varepsilon)w_k(x), \tag{14}$$

where

$$P_4(\varepsilon) = \frac{1}{8}(35\varepsilon^4 - 30\varepsilon^2 + 3),$$

$$P_5(\varepsilon) = \frac{1}{8}(63\varepsilon^5 - 70\varepsilon^3 + 15\varepsilon).$$

When N=8 , then

$$\begin{cases} b_0 = \frac{1}{3}t_1 + f_0, \\ b_2 = \frac{2}{3}t_1 + \frac{3}{7}t_3, \\ b_4 = \frac{4}{7}t_3 + \frac{5}{11}t_5, \\ b_6 = \frac{6}{11}t_5 \quad (t_7 \equiv 0), \\ b_1 = t_0 + \frac{2}{5}t_2, \\ b_3 = \frac{3}{5}t_2 + \frac{4}{9}t_4, \\ b_5 = \frac{5}{9}t_4 + \frac{6}{13}t_6, \\ b_7 = \frac{7}{13}t_6 \quad (t_8 \equiv 0). \end{cases} \tag{15}$$

As a result of simple regular transformations we get from system (15)

$$\begin{cases} w_0 = v_0 + \frac{1}{3}v_2, \\ w_2 = \frac{2}{3}v_2 + \frac{3}{7}v_4, \\ w_4 = \frac{4}{7}v_4 + \frac{5}{11}v_6, \\ w_6 = \frac{6}{11}t_6, \\ w_1 = v_1 + \frac{3}{5}v_3, \\ w_3 = \frac{2}{5}v_3 + \frac{5}{9}v_5, \\ w_5 = \frac{4}{9}v_5 + \frac{7}{13}v_7, \\ w_7 = \frac{6}{13}v_7. \end{cases}$$

An approximate of equations (1) is calculated by the formula:

$$u_7(x) = \sum_{k=0}^{7} P_k(\varepsilon) w_k(x) \tag{16}$$

where

$$P_6(\varepsilon) = \frac{1}{16}\left(231\varepsilon^6 - 315\varepsilon^4 + 105\varepsilon^2 - 5\right),$$

$$P_7(\varepsilon) = \frac{1}{16}\left(429\varepsilon^7 - 1989\varepsilon^5 + 315\varepsilon^3 - 35\varepsilon\right).$$

The above-mentioned method has been approved for approximate solution of system of linear algebraic equations, two- point linear boundary value problems, some linear nonhomogeneous integro-differential equations and singular integral equations containing an immovable singularity.

III APPROXIMATE SOLVING OF SYSTEM OF LINEAR ALGEBRIC

At the initial stage the above-mentioned method was approved in problems of approximate solving of system of algebraic equations and finding proper numbers. We mean that in operator equation (1) the operators L , M are matrices, at the same time L is tridiagonal matrix and M is filled matrix. In our opinion this method is more optimal with other known methods (see. E.g. [3],[4]).

To inverse the main operator we have an equation

$$Lx = D, \tag{17}$$

where

$$L = \begin{pmatrix} b_1 & c_1 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & 0 & 0 & . & . & . & 0 & a_n & b_n \end{pmatrix},$$

$x = (x_1, x_2, ..., x_n)^T$,

$D = (d_1, d_2, ..., d_n)^T$,

$|b_k| \ge |a_k| + |c_k|$, $k = \overline{1,n}$, $(a_1 = c_n = 0)$

We solve equation system (17) by well-known method of factorization. For direct run we have the formulae

$$p_k = a_k q_{k-1} + b_k, \qquad q_0 = 0,$$

$$q_k = -\frac{c_k}{p_k},$$

$$u_k = \frac{d_k - a_k u_{k-1}}{p_k},$$

$$u_0 = 0.$$

For reverse run we have

$$x_n = u_n \ ,$$

$$x_k = q_k x_{k+1} + u_k \ ,$$

$$k = n-1, n-2, ..., 2, 1 \, .$$

IV. APPROXIMATE SOLVING OF TWO-POINT BOUNDARY VALUE PROBLEM

Let us consider a two-point boundary value problem with variable coefficients

$$\begin{cases} u''(x) - q(x)u(x) = -f(x) \quad x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \tag{18}$$

where

$$q(x) \geq 0 \, , \; q(x), f(x) \in L_2[0,1] \, .$$

Problem (18) has an unique solution in the class $W_{2,0}^2(0,1)$. Using Green function it may be presented in the following form:

$$u(x) = \int_0^1 G(x, \xi, q(.)) f(\xi) d\xi \tag{19}$$

Notation $G(x, \xi, q(.))$ shows that Green function $G(x, \xi)$ of given boundary value problem depends on the function *q*(*x*) (on the variable coefficient), i.e. the Green function may be considered as nonlinear operator in respect with *q*. It is shown in [5] that at certain conditions $G(x, \xi, q(.))$ is analytic in the sense of Gateaux at the point *q*=0, the corresponding operator series is uniformly convergent, i.e. we can approach to the Green function for any *q*(*x*) by the well known Green functions constructed for *q*(.)=0.

$$G(x, \xi, 0) = \begin{cases} x(1-\xi), & 0 \leq x \leq \xi \\ \xi(1-x), & \xi \leq x \leq 1 \end{cases} \tag{20}$$

The above-mentioned boundary value problem was solved with an operator-interpolation approach [6]. In particular, new algorithms have been considered for an approximate solution of a two-point boundary value problem with variable coefficients. Approximation to the Green function of the given boundary value problem was made using operator interpolation polynomials of the Newton type (here linear combinations of the Heaviside function are used). Experimental convergence is revealed in respect to the degree of interpolation polynomials and a node number while approximate integration.

A new algorithm of the approximate solution of boundary-value problem (18) is given in [7].

In the present paper, boundary problem (18) is solved by asymptotic method (A) and by an approach, alternative to the asymptotic one (B). In our case the basic operator is $Lu = u''$ and operator describing the perturbation degree is $Mu = q_1 u$, $q_1 = q/\varepsilon$.

Approximate solution formulae for approximation of the inverted operator are constructed:

$$v_0(x) = \int_0^1 G(x, \xi, 0) f_0(\xi) d\xi \, ;$$

$$v_i(x) = -\int_0^1 G(x, \xi, 0) q_1(\xi) v_{i-1}(\xi) d\xi \, , \quad i = 1, 2, 3..... \tag{21}$$

$$w_0(x) = v_0(x) + \frac{1}{3} v_2(x), \qquad w_2(x) = \frac{2}{3} v_2(x),$$

$$w_1(x) = v_1(x) + \frac{3}{5} v_3(x), \qquad w_3(x) = \frac{2}{5} v_3(x).$$

V. APPROXIMATE SOLVING OF SOME LINEAR INTEGRO-DIFFERNTIAL EQUATIONS

Let us consider the following integro-differential equation:

$$\begin{cases} u''(x) - \varepsilon \int_0^1 K(x,t) u(t) dt = -f(x) \\ u(0) = u(1) = 0 \end{cases} \tag{22}$$

Where $f(x) \in L_2[0,1]$, $|K(x,t)| \leq M < +\infty$, $K(x,t) \in C([0,1] \times [0,1])$.

Problem (22) has a unique solution in the class $W_{2,0}^2(0,1)$.

It can be found by the successive approximation method(see, for example [3]). The approach elaborated in the §4 enables us to find and represent an approximate solution of equation (22) applying the Green formulae. Let us solve the above-mentioned problem by both asymptotic and alternative methods. In our case the basic operator is $Lu = u''$ and operator, describing the perturbation degree, is

$$Mu = \int_0^1 K(x,t) u(t) dt \, .$$

Analogously §4, here, in the case of asymptotic method the approximation formulae of the inverted operator are constructed:

$$v_0(x)=\int_0^1 G(x,\xi,0)f_0(\xi)d\xi$$

$$v_i(x)=\int_0^1 G(x,\xi,0)\int_0^1 K(\xi,t)v_{i-1}(t)dtd\xi$$

$$i=1,2,3.... \tag{23}$$

VI. APPROXIMATE SOLVING OF SOME SINGULAR INTEGRAL EQUATIONS

Let us consider the following singular integral equation containing an immovable singularity

$$\begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^1\left[\frac{1}{t-x}+\frac{\varepsilon}{t+x}\right]u(t)dt = f(x), \quad x\in[0,1] \\ \displaystyle\int_0^1 u(t)dt=0, \end{cases} \tag{24}$$

Where

$$u(t)\in H^*([0,1]), \varepsilon\in[-1,1],\ f(x)\in H_\mu[0,1],$$
$$0<\mu\le 1.$$

Antiplaned problems of elasticity theory, composed for orthotropic planes, weakened with cracks, are reduced to the following integral equation containing an immovable singularity (24) (see, for example [8],[9]).

Analysis of the above-mentioned integral equation and study of their exact and approximate solving methods are accompanied with some specific complexities due to the fact, that the solution has a composite asymptotic, which can be considered only in certain cases introducing weight functions.

As a rule, peculiarity of the solution of an integral equation nearby the boundary endpoints induces a strong convergence deceleration of the approximate solution to the exact one. An immovable singularity also brakes the degree of convergence of the approximate solution. In [9] problems of solution behavior are studied nearby the boundary endpoints. An order of peculiarity of the solution on the dividing line at the end of the crack depends on elasticity constants of the material and belong to the interval (0,1) at the other end we have a singularity of the square root type.

If we have square root type singularity on both ends of the integral, then Chebishev orthogonal polynoms can be used (see ,for example [10],[11] ,[12]).

IN the case of peculiarity of the square root while approximate solving singular integral equation we use:

a). The main integral equality Chebishev's polynoms:

$$\frac{1}{\pi}\int_{-1}^{+1}\frac{T_n(\eta)d\eta}{(\eta-\xi)\sqrt{1-\eta^2}}=u_{n-1}(\xi),$$

$$-1<\xi<1, \qquad n=1,2,...$$

where $T_n(x)$ are Chebishev's polynoms of the 1st kind, and $u_n(x)$ those of the 2nd kind..

In special case when $n=0$, we have

$$\int_{-1}^{+1}\frac{d\eta}{(\eta-\xi)\sqrt{1-\eta^2}}=0$$

and b). The condition of orthogonality for Chebishev's polynoms of the 2nd kind

$$\int_{-1}^{+1}u_{n-1}(\xi)u_{k-1}(\xi)\sqrt{1-\xi^2}d\xi=\begin{cases}0, & k\ne n\\ \dfrac{\pi}{2}, & k=n\end{cases}$$
$$(k,n=1,2,...).$$

The solution of equation (24) we can represent in the following form:

$$u(t)=u_0(t)/t^\alpha\sqrt{1-t} \tag{25}$$

where

$$u_0(t)\in H_0([0,1]), \quad u_0(0)=0,$$

α depends on material elasticity constants, $0<\alpha<1$.

Integral equation (24) can be solved by three approximate methods: spectral, collocation and asymptotic ones. In the work [13] we use the collocation method.

In the present paper, boundary problem (24) is solved by asymptotic method and by an approach, alternative to the asymptotic one. In our case the basic operator is

$$Lu=\frac{1}{\pi}\int_0^1\frac{1}{t-x}u(t)dt$$

and operator describing the perturbation degree is

$$Mu=\int_0^1\frac{1}{t+x}u(t)dt.$$

Let us discuss a case when there is a square root type singularity on each end of the interval . To invert main operators we use well–known formula (see [14]). Analogously, §5 here , in the case of asymptotic method the approximation formula of the inverted operator are constructed :

$$v_0(x)=\frac{\pi}{\sqrt{x(1-x)}}\int_0^1\frac{\sqrt{t(1-t)}}{t-x}f(t)dt, \quad v_i(x)=$$
$$\frac{\pi}{\sqrt{x(1-x)}}\int_0^1\frac{\sqrt{t(1-t)}}{t-x}\int_0^1\frac{1}{t+\xi}v_{i-1}(\xi)d\xi dt,$$
$$i=1,2,3.... \tag{26}$$

VII. NUMERICAL EXPERIMENTS

The algorithm proposed in [2] enables us to find approximate solutions of problems (17) , (18), (22) and (24) for *N*=2 and *N*=4 as by the asymptotic method so by the alternative approach. To calculate multiple integrals with predetermined exactness we use Simpson's quadrature formula.

For approximate solving boundary value problem the complex of programs in algorithm language *Turbo Pascal* is composed and many numerical experiments are carried out. The results obtained are good enough. In practical tasks it is often sufficient if we take interpolation polynomial or operator series of the order not more than 2 (see [5], [6]). Also, in the case of applying asymptotic and alternative methods N≤4 is quite enough.

Algorithms and program products represent a new technology of approximate solving of system of linear algebraic equations, double – point boundary value problem, some linear nonhomogeneous integro – differential equations and singular integral equations containing an immovable singularity.

At last, we give the results of numerical realization of the algorithms considered in §3, §4 ,§5 and §6.

A). Numerical experiments of system linear algebraic equations.

As an illustration let us consider the simplest test example:

$$Ax = B \tag{27}$$

where

$$A = \begin{bmatrix} 11 & -5 & 0.01 & 0.02 & 0.01 & 0.02 & 0.01 & 0.02 & 0.01 \\ -5 & 11 & -5 & 0.01 & 0.02 & 0.01 & 0.02 & 0.01 & 0.02 \\ 0.01 & -5 & 11 & -5 & 0.01 & 0.02 & 0.01 & 0.02 & 0.01 \\ 0.02 & 0.01 & -5 & 11 & -5 & 0.01 & 0.02 & 0.01 & 0.02 \\ 0.01 & 0.02 & 0.01 & -5 & 11 & -5 & 0.01 & 0.02 & 0.01 \\ 0.02 & 0.01 & 0.02 & 0.01 & -5 & 11 & -5 & 0.01 & 0.02 \\ 0.01 & 0.02 & 0.01 & 0.02 & 0.01 & -5 & 11 & -5 & 0.01 \\ 0.02 & 0.01 & 0.02 & 0.01 & 0.02 & 0.01 & -5 & 11 & -5 \\ 0.01 & 0.02 & 0.01 & 0.02 & 0.01 & 0.02 & 0.01 & -5 & 11 \end{bmatrix}$$

$$B = (16;-16;-0.01;16.02;-16.02;0.01;16;-16;5.01)^T$$

$$x = (1;-1;0;1;-1;0;1;-1;0)^T$$

System (27) can be reduced to the form (1) where, as noted above, L is a tridiagonal matrix, and M is a filled matrix, small parameter $\varepsilon = 0.01$. Results of calculations is given in the following table:

	uv_1	uw_1	uv_3	uw_3	u_{exact}
1	0.9985	0.9985	0.9985	0.9999	1.0000
2	-1.0003	-1.0003	-1.0003	-1.0002	-1.0000
3	-0.0006	-0.0006	-0.0006	-0.0004	0.0000
4	1.0011	1.0011	1.0011	1.0014	1.0000
5	-1.0011	-1.0011	-1.0011	-1.0006	-1.0000
6	0.0006	0.0006	0.0006	0.0013	0.0000
7	1.0004	1.0004	1.0004	1.0014	1.0000
8	-0.9998	-0.9998	-0.9998	-0.9981	-1.0000
9	0.0001	0.0001	0.0001	0.0027	0.0000

where

$$uv_1 = uw_1 = v_0 + \varepsilon v_1,$$

$$uv_3 = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \varepsilon^3 v_3,$$

$$uw_3 = w_0 + \varepsilon w_1 + P_2(\xi)w_2 + P_3(\xi)w_3,$$

$$u_{exact} = (1;-1;0;1;-1;0;1;-1;0)^T .$$

B). Numerical experiments of two-point boundary value problem**.**

We shall bring a test-problem as a describive example. The right hand side

$$f(x) = \left(q(x) + \left(\frac{\pi}{2}\right)^2\right)(1-x)\sin\frac{\pi}{2}x + \pi\cos\frac{\pi}{2}x$$

The exact solution

$$u(x) = (1-x)\sin\frac{\pi}{2}x .$$

a small $\varepsilon = 0.001$,
a node $x = 0.5$,
n is a number of interval divisions.
$(n = 10;20;40;80)$.

Let as introduce the designations:

$$u_1[w] = w_0 + \varepsilon w_1,$$

$$u_3[w] = w_0 + \varepsilon w_1 + P_2(\varepsilon)w_2 + P_3(\varepsilon)w_3, \tag{28}$$

$$u_1[v] = v_0 + \varepsilon v_1,$$

$$u_3[v] = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 .$$

u_{exact} - the exact value of function $u(x)$ at the given point.

a). When a variable coefficient is continuous then increasing a number of divisions of integration intervals we can get sufficiently high exaction.

Let a variable coefficient

$q(x) = 1 + x^2$.

The obtained numerical results are presented in the table

	$n=10$	$n=20$	$n=40$	$n=80$
$u_1[v]$	0.3421	0.3474	0.3495	0.3505
$u_1[w]$	0.3421	0.3474	0.3495	0.3505
$u_3[v]$	0.3582	0.3533	0.3534	0.3535
$u_3[w]$	0.3582	0.3533	0.3534	0.3535
u_{exect}	0.3536	0.3536	0.3536	0.3536

b). when a variable coefficient in discontinuous then increasing a number of integration interval divisions in not sufficient to get high discontinuity of the 1st kind. for example

$$q(x) = 1 + H(x-a)$$

where

$$H(x-a) = \begin{cases} 1 & , \quad x > a \\ 0 & , \quad x < a \end{cases}$$

is Heviside function.

In particular when

$$q(x) = 1 + H\left(x - \frac{1}{2}\right) = \begin{cases} 2 & , \quad x > 0.5 \\ 1 & , \quad x < 0.5 \end{cases}$$

we have the following numerical results:

	$n=10$	$n=20$	$n=40$	$n=80$
$u_1[v]$	0.3321	0.3304	0.3296	0.3236
$u_1[w]$	0.3321	0.3304	0.3296	0.3236
$u_3[v]$	0.3463	0.3447	0.3433	0.3411
$u_3[w]$	0.3463	0.3447	0.3433	0.3411
u_{exect}	0.3536	0.3536	0.3536	0.3536

C). Numerical experiments of some linear integro-differential equations. For clearness let us consider a test problem when

$K(x,t) = e^{xt}$,

an exact solution $u(x) = x(1-x)$,

the right hand side

$$f(x) = 2 + e^x \left[\frac{1}{x^2} - \frac{2}{x^3} \right] + \frac{1}{x^2} + \frac{2}{x^3};$$

a small parameter $\varepsilon = 0.001$,

a node $x = 0.5$;

n – number of the interval divisions.

Numerical results obtained are given in the Table below:

	$n=10$	$n=20$	$n=40$	$n=80$
$u_1[v]$	0.2401	0.2413	0.2412	0.2431
$u_1[w]$	0.2401	0.2413	0.2412	0.2431
$u_3[v]$	0.2464	0.2471	0.2473	0.2482
$u_3[w]$	0.2464	0.2471	0.2473	0.2482
u_{exect}	0.2500	0.2500	0.2500	0.2500

Where $u_1[v]$, $u_1[w]$, $u_3[v]$, $u_3[w]$ are calculated by formulae (28).

D). Numerical experiments of some singular integral equations.

The program is examined with testing problem, when an order of peculiarity of the solution $\alpha = \frac{1}{2}$, a small parameter $\varepsilon = 0.01$

the right side

$$f(x) = 2\pi\left(16(2x-1)^4 - 12(2x-1)^2 + 1\right),$$

To control the main part of the algorithm let us use the test:

when $\varepsilon = 0$,

the exact solution

$$u(x) = 16(2t-1)^5 - 20(2t-1)^3 + 5(2t-1)^1$$

a node $x = 0.5$

we have the following numerical results:

	$n=10$	$n=20$	$n=40$	$n=80$
$u_1[v]$	0.0234	0.0213	0.0206	0.0188
$u_1[w]$	0.0234	0.0213	0.0206	0.0188
$u_3[v]$	0.0167	0.0145	0.0123	0.0102
$u_3[w]$	0.0167	0.0145	0.0123	0.0102
u_{exect}	0.0000	0.0000	0.0000	0.0000

Several numerical experiments give the satisfactory results. We can conclude, that experimental convergence of approximate solution in the exact one is detected.

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