About the diophantine equations \( \frac{x^5 + y^5}{x + y} = 5z^5 \) and \( \frac{x^5 + y^5}{x + y} = z^5 \), in special conditions

Abstract— In this paper we solve the Diophantine equations \( \frac{x^5 + y^5}{x + y} = 5z^5 \) and \( \frac{x^5 + y^5}{x + y} = z^5 \), in special conditions.

Keywords—Cyclotomic fields, Diophantine equations.

I. INTRODUCTION

Other equations related of the equation from the title have just been studied. For example in 1998 B. Poonen studied the Diophantine equation \( x^5 + y^5 = z^3 \). For this purpose he used the properties of elliptic curves, concretely the properties of Frey curves.

In 1999, Y. Bugeaud and later Bugeaud and M. Mignotte studied the Diophantine equation \( \frac{x^5 + y^5}{x + y} = \alpha r+s \). They proved that the only solution \((x, y, z, n, q)\) of the equation \( x^5 + y^5 = z^3 \), in special conditions.

Main Theorem. The Diophantine equation \( \frac{x^5 + y^5}{x + y} = 5z^5 \) does not have integer solutions, \( x,y,z \in \mathbb{Z}, (x, y) = 1 \).

In our proof we used techniques of cyclotomic fields and Dirichlet’s theorem.

First we recall some theoretical results, necessary in the proof of the Main Theorem.

Proposition 1.1. ([1], [5]). Let \( r \) be a prime odd positive integer, \( \xi \) be a primitive root of order \( r \) of unity and the cyclotomic field \( \mathbb{Q}(\zeta) \). The following statements hold:

i) \( 1 - \zeta \) is a irreducible element in \( \mathbb{Z}[\zeta] \).

ii) \( 1 - \zeta^k = u_k(1 - \zeta) \), where \( k \notin r\mathbb{Z}, u_k \in \mathbb{U}(\mathbb{Z}[\xi]) \).

iii) \( r = u(1 - \zeta)^{r-1} \), where \( u \in \mathbb{U}(\mathbb{Z}[\xi]) \).

Proposition 1.2. ([1]). Let \( r \) be a prime odd positive integer, \( \xi \) be a primitive root of order \( r \) of unity and the cyclotomic field \( \mathbb{Q}(\zeta) \). Let \( m,n \) be rational positive integers, \( x, y \) be rational integers and suppose that \( r \) does not divide \( m-n \). Then, the ideals

\[
(x + \zeta^m y) \text{ and } (x + \zeta^n y)
\]

are coprime ideals of the ring \( \mathbb{Z}[\xi] \) if and only if \( \text{g.c.d.}(x, y) = 1 \) and \( x+y \) is not divisible by \( r \).

Proposition 1.3 ([8]). Let \( r \) be a prime odd positive integer, \( \xi \) be a primitive \( r \)-th root of unity and the cyclotomic field \( \mathbb{Q}(\zeta) \). Let \( m,n \) be rational positive integers, \( x, y \) be rational integers such that \( \text{g.c.d.}(x,y) = 1 \), \( r \) divides \( x+y \) and \( r \) does not divide \( m-n \). Then, the greatest common divisor of the elements \( x + \zeta^m y \) and \( x + \zeta^n y \) (in the ring \( \mathbb{Z}[\xi] \)) is \( 1 - \zeta \).

Proposition 1.4. Let \( r \) be a prime odd positive integer, \( \xi \) be a primitive \( r \)-th root of unity and the cyclotomic field \( \mathbb{Q}(\zeta) \). Let \( G \) be the Galois group of the cyclotomic field \( \mathbb{Q}(\zeta) \) over \( \mathbb{Q} \). Then, the following statements holds:

i) for any \( \sigma \in G \) and for any \( P \in \text{Spec}(\mathbb{Z}[\xi]) \), we have \( \sigma(P) \in \text{Spec}(\mathbb{Z}[\xi]) \).

ii) the Galois group \( G \) is isomorphic with the multiplicative group \( \mathbb{U}(\mathbb{Z}_r) \).

Let \( K \) be a field of algebraic numbers, \( [K: \mathbb{Q}] = n \) and \( \alpha \in K \) be such that \( K = \mathbb{Q}(\alpha) \). We denote with \( \mathbb{Z}_K \) the ring of integers of the field \( K \).

Let \( \alpha_1 = \alpha, \alpha_2, ..., \alpha_n \) be the conjugates of \( \alpha \) over \( \mathbb{Q} \). We consider \( \alpha_1, \alpha_2, ..., \alpha_r \in \mathbb{R}, \alpha_{r+1}, \alpha_{r+2}, ..., \alpha_{r+s} \in \mathbb{C}, \alpha_{r+s} = \alpha_1, r \leq s \), with \( r + 2s = n \).

Let us denote with:

\[
\phi_j : \mathbb{K} \to \mathbb{R}, \quad \phi_j(\alpha) = \alpha_{r+j}, j = 1, r
\]

the embeddings of \( K \) in \( \mathbb{R} \), and
\[ \varphi_j : K \to C \quad \varphi_j(\alpha) = \alpha_{j \mod r+1} \]

the embeddings of \( K \) in \( C \).

We can define:

\[ \psi : U(\mathbb{Z}_K) \to \mathbb{R}^{r+s}, \]

\[ \psi(\mathcal{E}) = (\log |\varphi_1(\mathcal{E})|, \log |\varphi_2(\mathcal{E})|, \ldots, \log |\varphi_{r+s}(\mathcal{E})|), \]

\( \forall \mathcal{E} \in U(\mathbb{Z}_K) \).

\( \psi \) is called logarithmic representation of algebraic numbers.

**Dirichlet’s units theorem.**

i) \( \text{Ker} \psi \) is a finite group and \( \text{Ker} \psi = W_K \),

where

\[ W_K = \{ \beta \in K \mid \beta \text{ is a root of unity} \}. \]

We denote with:

\[ \omega = |W_K|. \]

ii) \( \text{Im} \psi \) is a discrete abelian group, free of rank \( r+s-1 \) and there exists the isomorphism

\[ U(\mathbb{Z}_K) \cong \mathbb{Z} / \omega \mathbb{Z} \times \mathbb{Z}^{r+s-1}. \]

If \( u \in \mathbb{Z}_K \), it results from Dirichlet’s units theorem that there exists \( \xi \in W_K \) and \( t = r+s-1 \) units of finite order \( u_1, u_2, \ldots, u_t \) such that

\[ u = \xi^a \cdot u_1^h \cdots u_t^h, \]

where \( a, h_1, h_2, \ldots, h_t \in \mathbb{N}, a \leq \omega \).

The set \( \{u_1, u_2, \ldots, u_t\} \) is called fundamental system of units of the field \( K \).

**Lemma 1.5.** An element \( a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3 \in \mathbb{Z}[\xi] \) is divisible by 5 if and only if \( 5 \mid a_i \) (in \( \mathbb{Z} \)), \( (\forall) i \in \{0,1,2,3\} \).

II. Results

At the beginning we made some calculations in Mathematica 6.0 on the expression

\[ \frac{x^5 + y^5}{x + y}. \]

We gave \( x = \sqrt[5]{200}, y = \sqrt[5]{200} \) and we remarked that does not exist rational integers \( z \) such that

\[ \frac{x^5 + y^5}{x + y} = 5 \xi^3. \]

We show in the following the calculus in Mathematica 6.0 for \( x = \sqrt[5]{20}, y = \sqrt[5]{20} \).

\[ \text{In}[1] := \text{Do}[\text{Print}[\text{FactorInteger}[(i^5 + j^5)/(i+j)], \{i, 20\}, \{j, 20\}]] \]

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Proof of the Main Theorem.

Let \((x, y, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}\) a solution of the Diophantine equation
\[
\frac{x^5 + y^5}{x + y} = 5z^5,
\]
with \((x, y) = 1\).

Applying Small Fermat’s Theorem we have
\[
x^5 \equiv x \pmod{5}, \ y^5 \equiv y \pmod{5},
\]
so
\[
x^5 + y^5 \equiv x+y \pmod{5}.
\]

If 5 does not divide \(x+y\), it results that 5 does not divide \(x^5 + y^5\). A contradiction was obtained, so 5 divides \(x+y\).

We denote with \(1\) the biggest positive integer with the property:
\[
5^1 \mid (x+y).
\]

So \(5^k \mid (x^5 + y^5)\).

Let \(\zeta\) be a primitive 5-th root of unity and \(G\) be the Galois group of the cyclotomic field \(\mathbb{Q}(\zeta)\) over \(\mathbb{Q}\).

It is known that the ring \(\mathbb{Z}[\zeta]\) is principal. Since \(\frac{1}{5}\) is a generator of the group \((\mathbb{Z}^*, \cdot)\) it results that \(\sigma \in G\).

\[
\sigma (\zeta^5) = \zeta^5
\]
is a generator of the group \(G\).

Our equation is equivalent to
\[
(\sigma(x^5) + \sigma(y^5))\sigma(x + y^5) = u(1 - \zeta^5)z^5,
\]
with \(u \in U(\mathbb{Z}[\zeta])\).

Since
\[
\sigma (x + \zeta^5 y) = x + \zeta^4 y,
\]
\[
\sigma (\sigma (x + \zeta^5 y)) = x + \zeta^4 y,
\]
applying Proposition 1.4 and Proposition 1.3 we obtain
\[
x + \zeta^4 y = u(1 - \zeta^5) \alpha j,
\]
where \(\alpha_j \in \mathbb{Z}[\zeta]\), \(j = 1, \ldots, 5\), \(\alpha\) and \(\alpha_j\) are coprime for any \(i, j\) \(1, \ldots, 5\), \(u \in U[\mathbb{Z}[\zeta]], j = 1, 4\).

We have:
\[
\alpha_j = a_0 + a_1 \zeta + a_2 \zeta^2 + a_3 \zeta^3 + a_4 \zeta^4,
\]
with \(a_i \in \mathbb{Z}\), \(i = 1, 2, 3\).

\[
\alpha = b_0 + b_1 \zeta + b_2 \zeta^2 + b_3 \zeta^3 + b_4 \zeta^4,
\]
with \(b_i \in \mathbb{Z}\), \(i = 1, 2, 3\).

We obtain
\[
\alpha 5 \equiv a_0^5 + a_1^5 \zeta^5 + a_2^5 \zeta^{10} + a_3^5 \zeta^{15} \equiv a_0 + a_1 + a_2 + a_3 (\pmod{5}).
\]
Analogously
\[
\alpha 5 \equiv b_0 + b_1 + b_2 + b_3 (\pmod{5}).
\]

Applying Dirichlet’s units theorem we have
\[
u_1 = \zeta v_1, \text{ with } v_1 \text{ a real unity from } \mathbb{Z}[\zeta],
\]
\[
u_2 = \zeta v_2, \text{ with } v_2 \text{ a real unity from } \mathbb{Z}[\zeta].
\]

We obtain
\[
x + \zeta^4 y \equiv (1 - \zeta^5) \zeta v_1 (a_0 + a_1 + a_2 + a_3) (\pmod{5}),
\]
\[
x + \zeta^4 y \equiv (1 - \zeta^5) \zeta v_2 (b_0 + b_1 + b_2 + b_3) (\pmod{5}).
\]

After calculus we get:
\[
v_1, v_2 \in \left\{ \pm \left( \frac{1 + \sqrt{5}}{2} \right) \right\}_k \ / \ k \in \mathbb{Z} \right\}.
\]

Denoting with:
\[
\beta_j = v_1 (a_0 + a_1 + a_2 + a_3),
\]
\[
\beta_j = v_2 (b_0 + b_1 + b_2 + b_3),
\]
the last congruences become
\[
x + \zeta^4 y \equiv \beta_1 (1 - \zeta^5) \zeta v_1 (\pmod{5}),
\]
\[
x + \zeta^4 y \equiv \beta_2 (1 - \zeta^5) \zeta v_2 (\pmod{5}).
\]

We obtain:
\[
y \equiv \gamma \zeta^{5 \cdot h - 1} (\beta_1 - \beta_2 \zeta^{h - 1} \zeta^5) (\pmod{1 - \zeta^5}),
\]
\[
x \equiv \gamma \zeta^{5 \cdot h} (\beta_2 - \beta_1 \zeta^{5 \cdot h - 1} \zeta - \beta) (\pmod{1 - \zeta^5}).
\]

It results
\[
x + y \equiv \beta_1 \gamma \zeta^{5 \cdot h - 1} (1 - \zeta^5) - \beta_2 \zeta^{5 \cdot h - 1} (1 - \zeta^5) (\pmod{1 - \zeta^5}).
\]

The last congruence is equivalent with
\[
x + y \equiv (1 - \zeta^5) [ \beta_1 \zeta^{5 \cdot h - 1} (1 + \zeta^5) - \beta_2 \zeta^{5 \cdot h - 1} ] (\pmod{1 - \zeta^5}).
\]
Since
\[ x+y \equiv 0 \pmod{(1-\zeta)^3} \]
we obtain
\[ (1-\zeta) \beta_i \xi^{h_i} (1+\zeta) \equiv (1-\zeta) \beta_i \xi^{h_i} \pmod{(1-\zeta)^3}. \]

Multiplying the last congruence with \(\zeta\) it results
\[ (x+\zeta y)(1+\xi) \equiv x+\zeta y \pmod{(1-\zeta)^3} \]
(1).

But
\[ x + \zeta y = \sigma(x+\zeta y), \]
so
\[ x + \zeta^2 y \equiv \sigma(x+\zeta y) \pmod{5}. \]
The last congruence is equivalent to
\[ x+\zeta^2 y \equiv \pmod{(1-\zeta)^2} \]
\[ \equiv (1-\zeta)^2 \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^k \right] \]
\[ \equiv (a_0 + a_1 + a_3 + a_5) \zeta^2 \pmod{5}, \]
therefore
\[ x + \zeta^2 y \equiv (x+\zeta y)(1+\zeta) \xi^{h_i} \]
\[ \equiv (x+\zeta y)(1+\zeta) \xi^{h_i} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^k \right] \pmod{(1-\zeta)^3}. \]
(2).

From the relations (1) and (2) we obtain
\[ (x+\zeta y)(1+\zeta) \equiv (x+\zeta y)(1+\zeta) \xi^{h_i} \]
\[ \equiv (x+\zeta y)(1+\zeta) \xi^{h_i} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^k \right] \pmod{(1-\zeta)^3}. \]

The last congruence is equivalent to
\[ (x+\zeta y)(1+\zeta) \left[ 1 - \xi^{h_i} \left( \frac{1 + \sqrt{5}}{2} \right)^k \right] \equiv 0 \pmod{(1-\zeta)^3}. \]

This congruence is equivalent with
\[ (x+\zeta y)(1+\zeta) \left[ 1 - \xi^{h_i} + \xi^{h_i} \pmod{5} \left( \frac{1 + \sqrt{5}}{2} \right)^k \right] \equiv 0 \pmod{(1-\zeta)^3}. \]

Applying Proposition 1.1 we get
\[ (x+\zeta y)(1+\zeta) \left[ (1-\xi)\xi^{h_i} + \xi^{h_i} \left( \frac{1 + \sqrt{5}}{2} \right)^k \right] \equiv 0 \pmod{(1-\zeta)^3}, \]
with \( u \in \mathbb{U}(\mathbb{Z}[\zeta]). \)

If \((1-\zeta)\) divides \(1 + \left( \frac{1 + \sqrt{5}}{2} \right)^k \), since \(1 + \left( \frac{1 + \sqrt{5}}{2} \right)^k \) is a real number it results that 5 divides \(1 + \left( \frac{1 + \sqrt{5}}{2} \right)^k \). So the biggest positive integer \( l \) with the property \((1-\zeta)^l \) divides
\[ (x+\zeta y)(1+\zeta) \left[ (1-\xi)\xi^{h_i} + \xi^{h_i} \left( \frac{1 + \sqrt{5}}{2} \right)^k \right] \]
is \( l = 1 \).

If \((1-\zeta)\) does not divide \(1 + \left( \frac{1 + \sqrt{5}}{2} \right)^k \), it results that \( (1-\zeta) \)
does not divide
\[ (x+\zeta y)(1+\zeta) \left[ (1-\xi)\xi^{h_i} + \xi^{h_i} \left( \frac{1 + \sqrt{5}}{2} \right)^k \right] \]

It results that \((1-\zeta)^j \) divides \((x+\zeta y)(1+\zeta)\).

Knowing that \(1-\zeta\) is a prime element in the ring \( \mathbb{Z}[\zeta] \) and \(1-\zeta\) does not divide \(1+\zeta\), we obtain that \((1-\zeta)^j \) divides \(x+y\). We obtain a contradiction with the fact that \(x+y\) are coprime.

### III. The Diophantine Equation \( \frac{x^5 + y^5}{x + y} = 5z^5 \) in Some General Conditions

In the following we study the Diophantine equation \( \frac{x^5 + y^5}{x + y} = 5z^5 \), in the case when g.c.d.(x,y) \( \neq 1 \).

Let \( q \) a prime positive integer such that \( q \mid x \) and \( q \mid y \). We denote with \( k \) the biggest positive integer with the property \( q^k \mid x \) and \( q^k \mid y \).

Case I: if \( q \neq 5 \), simplifying with \( q^k \) we obtain an equation of the same type (that is \( \frac{x^5 + y^5}{x + y} = 5z^5 \), with g.c.d.(x,y)=1).

Case II: if \( q=5 \), simplifying with \( 5^k \) we obtain the...
Diophantine equation \( x^5 + y^5 = z^5 \), where \( x=5^5a, y=5^5b \),

\[
\begin{align*}
\frac{4k-1}{z} &= \frac{5}{5} - z_1, \\
\text{We remark that } g.c.d.(x,y) &= 1.
\end{align*}
\]

We change the notations: \( x_1 \rightarrow x, y_1 \rightarrow y, z_1 \rightarrow z \) and we study the Diophantine equation \( x^5 + y^5 = z^5 \) if we put the restrictions \( g.c.d.(5,x)=1, g.c.d.(5,y)=1, x \) is not congruent with \( y \) modulo 5.

It is well known that if \( p \) is a prime positive integer and \( x \) and \( y \) are coprime integers, then every prime factor of \( x^p + y^p \) is \( p \) and it appears with exponent 1 or it is congruent with 1 modulo \( p \).

It is clear that 5 does not divide \( z \), so 5 does not divide \( x^5 + y^5 \).

Since \( x^5 + y^5 \equiv x + y \pmod{5} \), it results that 5 does not divide \( x + y \).

The last congruence can be written in the form \( x^5 + y^5 \equiv z^5 \pmod{5} \). Applying Lemma 1.5 we obtain that \( x + y \equiv 0 \pmod{5} \), contradiction.

We remark that if an element \( z \in Z[\xi], \) with \( v_1 \) a real unity from \( Z[\xi], \) the previous proposition we have that \( x, y \equiv \xi, \) or \( \xi^2, \) or \( \xi^3, \).

Analogously with the proof of Main Theorem we obtain that

\[
\begin{align*}
\alpha^1_i &= a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3, \\
\text{with } a_i &\in Z, \ (\forall) i \notin \{0,1,2,3\}.
\end{align*}
\]

\[
\text{Applying Dirichlet’s units theorem we have } u_i = \xi^m v_i, \text{ with } v_i \text{ a real unity from } Z[\xi].
\]

Denoting with:

\[
\begin{align*}
\beta_i &= v_1 (a_0 + a_1 + a_2 + a_3),
\end{align*}
\]

we obtain

\[
\begin{align*}
x + \xi y \equiv \beta \xi^i \pmod{5}.
\end{align*}
\]

The last congruence can be written in the form \( \xi^i \equiv \beta \pmod{5} \).

Knowing that, if an element \( z \in Z[\xi], \) it results \( \xi \in Z[\xi], \) from the last congruence is results \( \xi \equiv \beta \pmod{5} \).

We obtain

\[
\begin{align*}
x^{5} + y^{5} - x^{5} - y^{5} - \xi^5 &= 0 \pmod{5}.
\end{align*}
\]

We study the exponents of \( \xi \) from the congruence (3). If some exponent of \( \xi \) from the congruence (3), for example \( s_1 \) (analogous to proceed if instead of \( s_1 \) we consider any other exponent of \( \xi \)) is congruent with 4 modulo 5, replacing \( \xi^{s_1} = \xi - \xi - \xi^{s_1} \), congruence (3) becomes

\[
\begin{align*}
x - 2x - (x + y) \xi^{s_1} + (y - x) \xi^{s_1} &= 0 \pmod{5}.
\end{align*}
\]

Applying Lemma 1.5 we obtain that \( x + y \equiv 0 \pmod{5} \), contradiction.

We show that any two from the exponents of \( \xi \) from the congruence (3) are not congruent modulo 5. It is clear that \( s_1 \) and \( 1 - s_1 \) are not congruent modulo 5. If \( s_1 \equiv 1 - s_1 \pmod{5} \) results that \( 5 \mid s_1 \) and congruence (3) becomes

\[
\begin{align*}
x^5 + y^5 - x^5 - y^5 - \xi^5 &= 0 \pmod{5}.
\end{align*}
\]

This congruence is equivalent with \( x^5 + 2x^5 - x^5 - y^5 \equiv 0 \pmod{5} \).

Applying Lemma 1.5 we obtain that \( x \equiv 0 \pmod{5} \), contradiction.

If \( s_1 \equiv 1 \pmod{5} \) it results that \( s_1 \equiv 1 \pmod{5} \) and congruence (3) becomes

\[
\begin{align*}
x^5 + y^5 - x^5 - y^5 - \xi^5 &= 0 \pmod{5}.
\end{align*}
\]

This congruence is equivalent with \( x^5 + 2x^5 - x^5 - y^5 \equiv 0 \pmod{5} \).

Applying Lemma 1.5 we obtain that \( x \equiv 0 \pmod{5} \), contradiction.

From the previously proved we obtain that the Diophantine equation \( x^5 + y^5 = 5z^5 \) does not have integer solutions, with \( g.c.d.(x, y) = 1, g.c.d.(5,x) = 1, g.c.d.(5,y) = 1, x \) is not congruent with \( y \) modulo 5.

\[ \text{IV. Conclusion} \]

From the previously proved we obtain that the Diophantine equation \( x^5 + y^5 = 5z^5 \) does not have solutions \( (x, y, z) \in Z \times Z \times Z, \) with \( g.c.d. (x, y) = 1. \)

In the case when \( g.c.d.(x, y) \neq 1, \) our problem reduces to study the Diophantine equation \( x^5 + y^5 = z^5, \) which we have shown that does not have solutions \( (x, y, z) \in Z \times Z \times Z, \) with \( g.c.d.(x, y) = 1, g.c.d.(5,x) = 1, g.c.d.(5,y) = 1, x \) is not congruent with \( y \) modulo 5.
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