# Logic Functions of Complementary Arrays

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***Abstract*****—** Complementary arrays mean a pair of multi-dimensional sequences, that at the same phase-shift the sum of these aperiodic auto-correlation functions takes zero except zero-shift. This paper clarifies logic functions of complementary arrays with dimension $n$, whose length is a power of two. The complementary arrays include binary complementary sequences with $n = 1$ and polyphase complementary arrays consisting of complex elements with unit magnitude. Complete complementary arrays, that the sum of these aperiodic cross-correlation functions for a pair of complementary arrays takes zero at any shift, are also investigated. These logic functions can easily give a lot of complementary arrays, and show that the number of complementary arrays is irrelevant to dimension.

***Keywords*****—** sequence design, complementary sequences, correlation function, logic function.

## I. Introduction

COMPLEMENTARY sequences [1]-[4] generally mean a pair of binary sequences with the same length characterized by a good correlation property such that at the same phase-shift the sum of these aperiodic auto-correlation functions takes zero except zero-shift, that is, the impulse characteristic. Some kinds of complementary sequences were proposed and applied in communications [4]-[8]. Families called complementary set were proposed and discussed for applications to CDMA communication [10]-[15]. The above families were extended to two dimension and three dimension, which can be applied to CDMA communication and digital water marking[16]-[19].

Recently complementary sequences generalized to multi-dimension, called complementary arrays, were discussed [20]-[22]. However the first author already reported complementary arrays, and considered these logic functions, which had been not considered yet [23]. The logic functions include ones of complementary sequences [3],[8]. Even-shift orthogonal arrays, which are closely related to complementary arrays, also discussed [24]-[26]

This paper concentrates on logical functions of complementary arrays including polyphase arrays consisting of complex elements with unit magnitude and complete complementary arrays that at the same phase-shifts the sum of the aperiodic cross-correlation functions for two complementary arrays takes zero for any shift.

In section 2, complementary arrays are defined. In section 3, a method that binary complementary arrays of long length can be systematically derived from complementary sequences of the shortest length called kernels, are given. It seems that the method can generate all complementary arrays. In section 4, logic functions of binary complementary arrays of length $2^l$ are formulated successfully. The logic functions can give not only a sequence generator consisting of binary counters and a feed forward logic, but also the number of the complementary arrays. In section 5, binary complementary arrays are extended to polyphase ones with $2^m$ phases, and give the logic functions of polyphase complementary arrays of length $2^l$ Quadriphase complementary arrays included in them may be practicable as well as binary complementary arrays. In section 6, the results derived in this paper are summarized and further studies are mentioned.

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## II. Complementary Arrays

Let $a$ be a complex array of length $L = L_1 L_2 \cdots L_n$ with dimension $n$, defined by

$$a = \{a_{i_1,i_2,\cdots,i_j,\cdots,i_n} \in C |\ 0 \le i_j < L_j\},$$

where if $i_j < 0$ or $i_j \ge L_j$, any element is regarded as 0, i.e., $a_{i_1,\cdots,i_n} = 0$. Concentrate on a polyphase array with $|a_{i_1,\cdots,i_n}| = 1$, which includes a binary array consisting of elements 1 and $-1$.

Let $C_{ab}(\tau_1, \tau_2, \cdots, \tau_n)$ be the aperiodic correlation function between polyphase arrays $a$ and $b$ of length $L$ with dimension $n$ at shifts $\tau_j (|\tau_j| < L_j)$ defined by

$$C_{ab}(\tau_1, \tau_2, \cdots, \tau_n) = \sum_{i_1=0}^{L_1-1} \sum_{i_2=0}^{L_2-1} \cdots \sum_{i_n=0}^{L_n-1} a_{i_1,i_2,\cdots,i_n} b^*_{i_1+\tau_1,i_2+\tau_2,\cdots,i_n+\tau_n},$$

where $x^*$ denotes the complex conjugate of $x$.

*Definition 1* Let $a$ and $b$ be polyphase arrays. If the sum of these auto-correlation functions takes zero at the same shift except zero-shift, that is,

$$C_{aa}(\tau_1, \tau_2, \cdots, \tau_n) + C_{bb}(\tau_1, \tau_2, \cdots, \tau_n) = \begin{cases} 2L & \text{for } \tau_1 = \tau_2 = \cdots = \tau_n = 0, \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

the arrays expressed by $[a, b]$ are called complementary arrays.

Let $[a, b]$ and $[c, d]$ be complementary arrays. If the sum of these cross-correlation functions is zero for any shift, that is,

$$C_{ac}(\tau_1, \tau_2, \cdots, \tau_n) + C_{bd}(\tau_1, \tau_2, \cdots, \tau_n) = 0,$$

these are called complete complementary arrays.

Note that complementary sequences are discussed as the special case of $n = 1$.

## III. Construction of Binary Complementary Arrays

This section concentrates on binary complementary arrays of length $2^n$.

Construction methods of complementary arrays are given as the following theorems.

*Theorem 1* Let $[a, b]$ be complementary arrays.

1) Interchanging $a$ and $b$ gives complementary arrays. i.e., $[b, a]$.
2) Inversion of $a$ gives complementary arrays, i.e., $[-a, b]$.
3) Interchanging some directions gives complementary arrays expressed as
$$[\{a_{i_{k_1},i_{k_2},\cdots,i_{k_n}}\},\{b_{i_{k_1},i_{k_2},\cdots,i_{k_n}}\}]$$
with $1 \le k_m(\ne k_j) \le n$.
4) Reversing of $a$, $\tilde{a}$ is a mate of $b$, i.e.,
$$[\tilde{a} = \{a_{L_1-i_1-1,L_2-i_2-1,\cdots,L_n-i_n-1,}\}, b].$$
5) Reversing for a direction gives complementary arrays, i.e.,
$$[\{a_{i_1,\cdots,L_j-i_j-1,\cdots,i_n}\},\{b_{i_1,\cdots,L_j-i_j-1,\cdots,i_n}\}].$$

*Proof:* The proofs of the items 1, 2, and 3 are trivial. The item 4 is proven by

$$\begin{aligned}&C_{\tilde{a},\tilde{a}}(\tau_1,\tau_2,\cdots,\tau_n)\\ &= \sum_{i_1=0}^{L_1-1}\sum_{i_2=0}^{L_2-1}\cdots\sum_{i_n=0}^{L_n-1} a_{N_1-i_1-1,N_2-i_2-1,\cdots,N_n-i_n-1}\\ &a^*_{N_1-i_1-1+\tau_1,N_2-i_2-1+\tau_2,\cdots,N_n-i_n-1+\tau_n}\\ &= \sum_{j_1=0}^{L_1-1}\sum_{j_2=0}^{L_2-1}\cdots\sum_{j_n=0}^{L_n-1} a_{j_1,j_2,\cdots,j_n}a^*_{j_1+\tau_1,j_2+\tau_2,\cdots,j_n+\tau_n}\\ &= C_{aa}(\tau_1,\tau_2,\cdots,\tau_n),\end{aligned}$$

where $j_k = N_k - i_k - 1$. The item 5 is also proven by a method similar to the proof of the item 4. □

Note that Theorem 1 can produce a lot of complementary arrays. For example, use of 2, 1, 2, and 1 in Theorem 1, in order, gives complementary arrays, $[-a, -b]$.

*Theorem 2* Let $[a, b]$ be complementary arrays of length $L$ and dimension $n$. Arrays $[\hat{a}, \hat{b}]$ expressed by

$$\hat{a}_{i_1,i_2,\cdots,i_n,i_{n+1}} = \begin{cases} a_{i_1,i_2,\cdots,i_n} & \text{for } i_{n+1}=0,\\ b_{i_1,i_2,\cdots,i_n} & \text{for } i_{n+1}=1,\end{cases}$$
$$\hat{b}_{i_1,i_2,\cdots,i_n,i_{n+1}} = \begin{cases} a_{i_1,i_2,\cdots,i_n} & \text{for } i_{n+1}=0,\\ -b_{i_1,i_2,\cdots,i_n} & \text{for } i_{n+1}=1.\end{cases}$$

are complementary arrays of length $2L$ and dimension $n + 1$.

*Proof:* It is enough to show that the correlation function for $[\hat{a}, \hat{b}]$ satisfies (1) at shifts $-1 \le \tau_{n+1} \le 1$. For $\tau_{n+1} = -1$

$$\begin{aligned}&C_{\hat{a}\hat{a}}(\tau_1,\cdots,\tau_n,-1) + C_{\hat{b}\hat{b}}(\tau_1,\cdots,\tau_n,-1)\\ =\ & C_{ba}(\tau_1,\cdots,\tau_n) - C_{ba}(\tau_1,\cdots,\tau_n)\\ =\ & 0.\end{aligned}$$

For $\tau_{n+1} = 0$

$$\begin{aligned}&C_{\hat{a}\hat{a}}(\tau_1,\cdots,\tau_n,0) + C_{\hat{b}\hat{b}}(\tau_1,\cdots,\tau_n,0)\\ =\ & C_{aa}(\tau_1,\cdots,\tau_n) + C_{bb}(\tau_1,\cdots,\tau_n)\\ & + C_{aa}(\tau_1,\cdots,\tau_n) + C_{bb}(\tau_1,\cdots,\tau_n)\\ =\ & 2(C_{aa}(\tau_1,\cdots,\tau_n) + C_{bb}(\tau_1,\cdots,\tau_n)).\end{aligned}$$

For $\tau_{n+1} = 1$

$$\begin{aligned}&C_{\hat{a}\hat{a}}(\tau_1,\cdots,\tau_n,1) + C_{\hat{b}\hat{b}}(\tau_1,\cdots,\tau_n,1)\\ =\ & C_{ba}(\tau_1,\cdots,\tau_n) - C_{ba}(\tau_1,\cdots,\tau_n)\\ =\ & 0.\end{aligned}$$

□

*Theorem 3 Let $[a, b]$ be complementary arrays of length $L$ and dimension $n$. Arrays $[\hat{a}, \hat{b}]$ expressed by*

$$\hat{a}_{i_1,\cdots,i'_j,\cdots,i_n} = \begin{cases} a_{i_1,\cdots,i_j,\cdots,i_n} & \text{for } i'_j = 2i_j,\\ b_{i_1,\cdots,i_j,\cdots,i_n} & \text{for } i'_j = 2i_j+1,\end{cases}$$
$$\hat{b}_{i_1,\cdots,i'_j,\cdots,i_n} = \begin{cases} a_{i_1,\cdots,i_j,\cdots,i_n} & \text{for } i'_j = 2i_j,\\ -b_{i_1,\cdots,i_j,\cdots,i_n} & \text{for } i'_j = 2i_j+1,\end{cases}$$

*or*

$$\hat{a}_{i_1,\cdots,i'_j,\cdots,i_n} = \begin{cases} a_{i_1,\cdots,i_j=i'_j,\cdots,i_n} \\ \quad \text{for } 0 \le i'_j < L_j,\\ b_{i_1,\cdots,i_j=i'_j-L_j,\cdots,i_n} \\ \quad \text{for } L_j \le i'_j < 2L_j,\end{cases}$$
$$\hat{b}_{i_1,\cdots,i'_j,\cdots,i_n} = \begin{cases} a_{i_1,\cdots,i_j=i'_j,\cdots,i_n} \\ \quad \text{for } 0 \le i'_j < L_j,\\ -b_{i_1,\cdots,i_j=i'_j-L_j,\cdots,i_n} \\ \quad \text{for } L_j \le i'_j < 2L_j.\end{cases}$$

are cpmplementary arrays of length $2L$ with dimension $n$.

*Proof:* The sum of the correlation functions of the arrays $\hat{a}$ and $\hat{b}$ produced by the former method, which are extension of length $L = L_1L_2\cdots L_j\cdots L_n$ to $2L = L_1\cdots L_{j-1}2L_jL_{j+1}\cdots L_n$ can be written as follows. For $\tau_j = 2k + 1$ with $|k| < L_j$,

$$\begin{aligned}&C_{\hat{a}\hat{a}}(\tau_1,\cdots,\tau_j,\cdots,\tau_n) + C_{\hat{b}\hat{b}}(\tau_1,\cdots,\tau_j,\cdots,\tau_n)\\ =\ & C_{ab}(\tau_1,\cdots,\tau_j=k,\cdots,\tau_n)\\ & + C_{ba}(\tau_1,\cdots,\tau_j=k+1,\cdots\tau_n)\\ & - C_{ab}(\tau_1,\cdots,\tau_j=k,\cdots,\tau_n)\\ & - C_{ba}(\tau_1,\cdots,\tau_j=k+1,\cdots,\tau_n)\\ =\ & 0.\end{aligned}$$

For $\tau_j = 2k$ with $|k| < L_j$,

$$\begin{aligned}&C_{\hat{a}\hat{a}}(\tau_1,\cdots,\tau_j,\cdots,\tau_n) + C_{\hat{b}\hat{b}}(\tau_1,\cdots,\tau_j,\cdots,\tau_n)\\ =\ & C_{aa}(\tau_1,\cdots,\tau_j=k,\cdots,\tau_n)\\ & + C_{bb}(\tau_1,\cdots,\tau_j=k,\cdots\tau_n)\\ & + C_{aa}(\tau_1,\cdots,\tau_j=k,\cdots,\tau_n)\\ & + C_{ba}(\tau_1,\cdots,\tau_j=k,\cdots,\tau_n)\\ =\ & 2(C_{aa}(\tau_1,\cdots,\tau_j=k,\cdots,\tau_n)\\ & + C_{bb}(\tau_1,\cdots,\tau_j=k,\cdots,\tau_n)).\end{aligned}$$

Therefore the arrays are complementary arrays.

The arrays $\hat{a}$ and $\hat{b}$ produced by the latter method, which are extension of length L to 2L, can be written as the following correlation functions.

For $0 < \tau_j \le L_j$

$$\begin{aligned} & C_{\hat{a}\hat{a}}(\tau_1, \cdots, \tau_n) + C_{\hat{b}\hat{b}}(\tau_1, \cdots, \tau_n) \\ = \quad & C_{aa}(\tau_1, \cdots, \tau_j, \cdots, \tau_n) \\ & +C_{ab}(\tau_1, \cdots, \tau_j - L_j, \cdots \tau_n) \\ & +C_{bb}(\tau_1, \cdots, L_j - \tau_j, \cdots, \tau_n) \\ & +C_{aa}(\tau_1, \cdots, \tau_j, \cdots, \tau_n) \\ & -C_{ab}(\tau_1, \cdots, \tau_j - L_j, \cdots, \tau_n) \\ & +C_{bb}(\tau_1, \cdots, \tau_j, \cdots, \tau_n) \\ = \quad & 2(C_{aa}(\tau_1, \cdots, \tau_j, \cdots, \tau_n) \\ & +C_{bb}(\tau_1, \cdots, \tau_j, \cdots, \tau_n)). \end{aligned}$$

For $L_j < \tau_j < 2L_j$

$$\begin{aligned} & C_{\hat{a}\hat{a}}(\tau_1, \cdots, \tau_n) + C_{\hat{b}\hat{b}}(\tau_1, \cdots, \tau_n) \\ = \quad & C_{ab}(\tau_1, \cdots, \tau_j - L_j, \cdots, \tau_n) \\ & -C_{ab}(\tau_1, \cdots, \tau_j - L_j, \cdots \tau_n) \\ = \quad & 0. \end{aligned}$$

Similarly it is shown that the correlation function at $-2L_j < \tau_j \le 0$ is almost the same as the above equations. Therefore this proof is complete. □

The formar and latter constructions are called as the interleaving method and concatenation method, respectively.

A special method of complementary arrays of length $2^l$ is given as the following conjecture, newly.

*Conjecture 1* Let $[a, b]$ be complementary arrays of length $L = 2^l$ with dimension $n$. Let $K = 2^k \le L$. Complementary arrays $[\hat{a}, \hat{b}]$ of length $2L$ with dimension $n$ is expressed by

$$\hat{a}_{i_1,\cdots,i'_j,\cdots,i_n} = \begin{cases} a_{i_1,\cdots,i_j=m_2K+m_1,\cdots,i_n} \\ \qquad \text{for } i'_j = 2m_2K + m_1, \\ b_{i_1,\cdots,i_j=m_2K+m_1,\cdots,i_n} \\ \qquad \text{for } i'_j = (2m_2+1)K + m_1, \end{cases}$$

$$\hat{b}_{i_1,\cdots,i'_j,\cdots,i_n} = \begin{cases} a_{i_1,\cdots,i_j=m_2K+m_1,\cdots,i_n} \\ \qquad \text{for } i'_j = 2m_2K + m_1, \\ -b_{i_1,\cdots,i_j=m_2K+m_1,\cdots,i_n} \\ \qquad \text{for } i'_j = (2m_2+1)K + m_1, \end{cases}$$

with $0 \le m_1 < K$, $0 \le m_2 < L/K$, $0 \le i'_j < 2L_j$, and $0 \le l_k < L_k$.

The cases of $K = 1$ and $K = L$ are true, since these are corresponding to Theorem 3. The other case can be certainly confirmed by a computer. Therefore it seems that Conjecture 5 is true. Note that a lot of complementary arrays of length $L = S2^l$ can be derived from complementary sequences of length $S = 1, 10$ or $26$ called kernels[1][4].

*Theorem 4* Let $[a, b]$ be complementary arrays. The mate of $[a, b]$ for complete complementary arrays can be given as $[c, d]$ or $[-c, -d]$, expressed by

$$\begin{aligned} c_{i_1,\cdots,i_n} &= b_{L_1-i_1-1,\cdots,L_n-i_i-1}, \\ d_{i_1,\cdots,i_n} &= -a_{L_1-i_1-1,\cdots,L_n-i_i-1}. \end{aligned}$$

*Proof:* From Theorem 1, $[c, d]$ are complementary arrays. The sum of aperiodic cross-correlation functions is written as

$$\begin{aligned} & C_{ac}(\tau_1, \tau_2, \cdots, \tau_n) + C_{bd}(\tau_1, \tau_2, \cdots, \tau_n) \\ & = C_{ab}(\tau_1, \tau_2, \cdots, \tau_n) - C_{ab}(\tau_1, \tau_2, \cdots, \tau_n) \\ & = 0. \end{aligned}$$

□

*Example 1* Let $[(+,+),(+,-)]$ be complementary sequences of length 2, where $+$ and $-$ denote 1 and $-1$, respectively. Theorem 2 gives complementary arrays of length $2 \times 2$ with dimension 2, written as

$$\left[ a = \begin{pmatrix} + & + \\ + & - \end{pmatrix}, b = \begin{pmatrix} + & + \\ - & + \end{pmatrix} \right].$$

These aperiodic auto-correlation functions $C_{aa}(\tau_1, \tau_2)$ and $C_{bb}(\tau_1, \tau_2)$ for $-1 \le \tau_1, \tau_2 \le 1$ are respectively expressed by

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Since the sum of $C_{aa}(\tau_1, \tau_2)$ and $C_{bb}(\tau_1, \tau_2)$ is written as

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

it is confirmed that $[a, b]$ are complementary arrays.

The concatenation method in Theorem 3 also gives complementary arrays of length $4 \times 2$ written as

$$\left[ \begin{pmatrix} + & + & + & + \\ + & - & - & + \end{pmatrix}, \begin{pmatrix} + & + & - & - \\ + & - & + & - \end{pmatrix} \right].$$

Since these aperiodic auto-correlation functions $C_{aa}(\tau_1, \tau_2)$ and $C_{bb}(\tau_1, \tau_2)$ for $-3 \le \tau_1 \le 3$, $-1 \le \tau_2 \le 1$ are respectively expressed by

$$\begin{pmatrix} 1 & 0 & -1 & 0 & -1 & 0 & 1 \\ 2 & 0 & 2 & 8 & 2 & 0 & 2 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ -2 & 0 & -2 & 8 & -2 & 0 & -2 \\ -1 & 0 & 1 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

the sum of them is written as

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 4 gives the mates for complete complementary arrays

$$\left[ \pm \begin{pmatrix} - & + & - & + \\ - & - & + & + \end{pmatrix}, \mp \begin{pmatrix} + & - & - & + \\ + & + & + & + \end{pmatrix} \right].$$

Since one on the aperiodic cross correlations is written as

$$\begin{pmatrix} 1 & 2 & 1 & 0 & -1 & -2 & -1 \\ 2 & 0 & -2 & 0 & 2 & 0 & -2 \\ 1 & -2 & 1 & 0 & -1 & 2 & -1 \end{pmatrix},$$

and the other is done as

$$\begin{pmatrix} -1 & -2 & -1 & 0 & 1 & 2 & 1 \\ -2 & 0 & 2 & 0 & -2 & 0 & 2 \\ -1 & 2 & -1 & 0 & 1 & -2 & 1 \end{pmatrix},$$

the sum of them takes zero at any shift.

The interleaving method in Theorem 3 also gives complementary arrays of length $4 \times 4$ written as

$$\left[\begin{pmatrix} + & + & + & + \\ + & + & - & - \\ + & - & - & + \\ + & - & + & - \end{pmatrix}, \begin{pmatrix} + & + & + & + \\ - & - & + & + \\ + & - & - & + \\ - & + & - & + \end{pmatrix}\right].$$

Theorem 4 give the mate of complete complementary arrays as

$$\left[\begin{pmatrix} + & - & + & - \\ + & - & - & + \\ + & + & - & - \\ + & + & + & + \end{pmatrix}, \begin{pmatrix} + & - & + & - \\ - & + & + & - \\ + & + & - & - \\ - & - & - & - \end{pmatrix}\right].$$

And also Theorem 3 gives complementary arrays of length $4 \times 4 \times 2$ with dimension 3

$$\left[\left(\begin{array}{cccc|cccc} + & + & + & + & + & + & + & + \\ + & + & - & - & - & - & + & + \\ + & - & - & + & + & - & - & + \\ + & - & + & - & - & + & - & + \end{array}\right),\right.$$
$$\left.\left(\begin{array}{cccc|cccc} + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - \\ + & - & - & + & - & + & + & - \\ + & - & + & - & + & - & + & - \end{array}\right)\right].$$

These mates for complete complementary arrays are given as

$$\left[\pm\left(\begin{array}{cccc|cccc} - & + & - & + & - & + & - & + \\ - & + & + & - & + & - & - & + \\ - & - & + & + & - & - & + & + \\ - & - & - & - & + & + & + & + \end{array}\right),\right.$$
$$\left.\mp\left(\begin{array}{cccc|cccc} + & - & + & - & - & + & - & + \\ + & - & - & + & + & - & - & + \\ + & + & - & - & - & - & + & + \\ + & + & + & + & + & + & + & + \end{array}\right)\right].$$

Note that theorems can produce a lot of complementary arrays of length $S2^{l_1} \times 2^{l_2} \times \cdots \times 2^{l_n}$ with $S = 1, 10$ or 26, in order.

## IV. Logic Functions of Binary Complementary Arrays

Assume that Conjecture 1 is true and consider binary complementary arrays of length $L = 2^l$ given by Theorems 1-3 and Conjecture 1.

Let $\vec{x} = (x_1, x_2, \cdots, x_m)$ be a binary vector of order $m$, whose elements are coefficients expressed by binary expansion of an integer $x(0 \leq x \leq N-1)$ expressed by

$$x = x_1 2^0 + x_2 2^1 + \cdots + x_m 2^{m-1}.$$

*Theorem 5* Complementary arrays $[a, b]$ of length $L = 2^l$ with dimension $n$ are produced by logic functions expressed as

$$\begin{aligned} a_{i_1, i_2, \cdots, i_n} &= (-1)^{f_a(\vec{i_1}, \vec{i_2}, \cdots, \vec{i_n})}, \\ b_{i_1, i_2, \cdots, i_n} &= (-1)^{f_b(\vec{i_1}, \vec{i_2}, \cdots, \vec{i_n})}, \end{aligned}$$

with

$$\begin{aligned} & f_a(\vec{i_1}, \cdots, \vec{i_n}) \\ &\quad = \lambda_1\lambda_2 \oplus \lambda_2\lambda_3 \oplus \cdots \oplus \lambda_{l-1}\lambda_l \\ &\qquad \oplus a_{1,1} i_{1,1} \oplus \cdots \oplus a_{1,l_1} i_{1,l_1} \\ &\qquad \oplus a_{2,1} i_{2,1} \oplus \cdots \oplus a_{2,l_2} i_{2,l_2} \\ &\qquad \vdots \\ &\qquad \oplus a_{n,1} i_{n,1} \oplus \cdots \oplus a_{n,l_n} i_{n,l_n} \\ &\qquad \oplus e_a, \end{aligned} \tag{2}$$

$$f_b(\vec{i_1}, \cdots, \vec{i_n}) = f_a(\vec{i_1}, \cdots, \vec{i_n}) \oplus \lambda_l \oplus e_b, \tag{3}$$

where $\oplus$ denotes addition over GF(2) (modulo 2), $L_k = 2^{l_k}$, $l = l_1 + l_2 + \cdots + l_n$, $\vec{i_j} = (i_{j,1}, i_{j,2}, \cdots, i_{j,l_j})$, $\lambda_k (1 \leq k \leq l)$ is one of elements $\{i_{j,1}, \cdots, i_{j,l_j}\}$ of $\vec{i_j} (1 \leq j \leq n)$ with $\lambda_k \neq \lambda_m$ for $k \neq m$, and $a_{1,1}, \cdots, a_{n,l_n}, e_a$ and $e_b$ are parameters to give different complementary arrays.

*Proof:* Induction method is adapted to this proof. For $n = 1$ and $l = 1$, the logic functions

$$\begin{aligned} f_a(\vec{i_1}) &= a_{1,1} i_{1,1} \oplus e_a, \\ f_b(\vec{i_1}) &= f_a(\vec{i_1}) \oplus \lambda_1 \oplus e_b, \end{aligned}$$

with $\vec{i_1} = i_{1,1} = \lambda_1$ are true, because these can produce all the complementary sequences for $n = 1$. And also, it can be confirmed that

$$\begin{aligned} f_a(\vec{i_1}) &= \lambda_1\lambda_2 \oplus a_{1,1} i_{1,1} \oplus \cdots \oplus a_{1,l_1} i_{1,l_1} \oplus e_a, \\ f_b(\vec{i_1}) &= f_a(\vec{i_1}) \oplus \lambda_2 \oplus e_b, \end{aligned}$$

can be produced all the complementary sequences for $l = 2$.

Assume that Eqs (2) and (3) for $L = L_1 L_2 \cdots L_n$ are true. Theorem 3 gives the logic functions of complementary arrays $[\hat{a}, \hat{b}]$ of length $2L = L_1 L_2 \cdots L_n L_{n+1}$ with dimension $n+1$ as

$$\begin{aligned} & f_{\hat{a}}(\vec{i_1}, \cdots, \vec{i_n}, i_{n+1,1}) \\ &\quad = (i_{n+1,1} \oplus 1) f_a(\vec{i_1}, \cdots, \vec{i_n}) \\ &\quad \oplus i_{n+1,1} f_b(\vec{i_1}, \cdots, \vec{i_n}) \oplus e_f \\ &\quad = i_{n+1,1} f_a(\vec{i_1}, \cdots, \vec{i_n}) \oplus f_a(\vec{i_1}, \cdots, \vec{i_n}) \\ &\quad \oplus i_{n+1,1} f_a(\vec{i_1}, \cdots, \vec{i_n}) \oplus i_{n+1,1}\lambda_l \oplus i_{n+1,1} e_b \oplus e_f \\ &\quad = f_a(\vec{i_1}, \cdots, \vec{i_n}) \oplus i_{n+1,1}\lambda_l \oplus i_{n+1,1} e_b \oplus e_f \\ &\quad = \lambda_1\lambda_2 \oplus \cdots \oplus \lambda_{l-1}\lambda_l \oplus \lambda_l i_{n+1,1} \\ &\quad \oplus a_{1,1} i_{1,1} \oplus \cdots \oplus a_{1,l_1} i_{1,l_1} \\ &\quad \vdots \\ &\quad \oplus a_{n,1} i_{n,1} \oplus \cdots \oplus a_{n,l_n} i_{n,l_n} \\ &\quad \oplus e_b i_{n+1,1} \oplus e_f, \\ & f_{\hat{b}}(\vec{i_1}, \cdots, \vec{i_n}, i_{n+1,1}) \\ &\quad = (i_{n+1,1} \oplus 1) f_a(\vec{i_1}, \cdots, \vec{i_n}) \\ &\quad \oplus i_{n+1,1} (f_b(\vec{i_1}, \cdots, \vec{i_n}) \oplus 1) \oplus e_g \\ &\quad = f_a(\vec{i_1}, \cdots, \vec{i_n}) \oplus i_{n+1,1}\lambda_l \oplus i_{n+1,1} e_b \\ &\quad \oplus i_{n+1,1} \oplus e_g \end{aligned}$$

Setting $i_{n+1,1} = \vec{i_{n+1}} = \lambda_{l+1}$, $e_b = a_{n+1,1}$, and $e_g = e_b$, and rewriting $e_a \oplus e_f$ as $e_a$, the above equations are corresponding to Eqs (2) and (3).

Consider the logic functions of complementary arrays $[\hat{a}, \hat{b}]$ of length $2L = L_1 L_2 \cdots 2L_k \cdots L_n$ with dimension $n$ derived from Conjecture 1.

Let $\vec{\hat{i}}_k$ be a vector of order $l_k+1$. Let $\vec{\tilde{i}}_k$ be a vector of order $l_k$, which removes an element $i_{k,m_k}$ from $\vec{\hat{i}}_k$, written as $\vec{\tilde{i}}_k = (i_{k,1}, \cdots, i_{k,m_k-1}, i_{k,m_k+1}, \cdots, i_{k,l_k})$ with $1 \leq m_k \leq l_k+1$. These logic functions are written as

$$\begin{aligned}
& f_{\hat{a}}(\vec{i}_1, \cdots, \vec{\hat{i}}_k, \cdots, \vec{i}_n) \\
&\quad = (i_{k,m_k} \oplus 1) f_a(\vec{i}_1, \cdots, \vec{\tilde{i}}_k, \cdots, \vec{i}_n) \\
&\quad \oplus i_{k,m_k} f_b(\vec{i}_1, \cdots, \vec{\tilde{i}}_k, \cdots, \vec{i}_n) \oplus e_f \\
&\quad = i_{k,m_k} f_a(\vec{i}_1, \cdots, \vec{\tilde{i}}_k, \cdots, \vec{i}_n) \\
&\quad \oplus f_a(\vec{i}_k, \cdots, \vec{\tilde{i}}_k, \cdots, \vec{i}_n) \\
&\quad \oplus i_{k,m_k} f_a(\vec{i}_1, \cdots, \vec{\tilde{i}}_k, \cdots, \vec{i}_n) \oplus i_{k,m_k}\lambda_l \\
&\quad \oplus i_{k,m_k} e_b \oplus e_f \\
&\quad = f_a(\vec{i}_1, \cdots, \vec{\tilde{i}}_k, \cdots, \vec{i}_n) \oplus i_{k,m_k}\lambda_l \oplus i_{k,m_k} e_b \oplus e_f \\
&\quad = \lambda_1\lambda_2 \oplus \cdots \oplus \lambda_{l-1}\lambda_l \oplus \lambda_l i_{k,m_k} \\
&\quad \oplus a_{1,1} i_{1,1} \oplus \cdots \oplus a_{1,j_1} i_{1,j_1} \\
&\quad \vdots \\
&\quad \oplus a_{k,1} i_{k,1} \oplus \cdots \oplus a_{k,m_{k-1}} i_{k,m_{k-1}} \oplus e_b i_{k,m_k} \\
&\quad \oplus a_{k,m_{k+1}} i_{k,m_{k+1}} \oplus \cdots \oplus a_{k,l_k} i_{k,l_k} \\
&\quad \vdots \\
&\quad \oplus a_{n,1} i_{n,1} \oplus \cdots \oplus a_{n,l_n} i_{n,l_n} \\
&\quad \oplus e_f,
\end{aligned}$$

$$\begin{aligned}
& f_{\hat{b}}(\vec{i}_1, \cdots, (\vec{\tilde{i}}_k, i_{k,m_k}), \cdots, \vec{i}_n) \\
&\quad = (i_{k,m_k} \oplus 1) f_a(\vec{i}_1, \cdots, \vec{\tilde{i}}_k, \cdots, \vec{i}_n) \\
&\quad \oplus i_{k,m_k}(f_b(\vec{i}_1, \cdots, \vec{\tilde{i}}_k, \cdots, \vec{i}_n) \oplus 1) \oplus e_g \\
&\quad = f_a(\vec{i}_1, \cdots, \vec{\tilde{i}}_k, \cdots, \vec{i}_n) \oplus i_{k,m_k}\lambda_l \oplus i_{k,m_k} e_b \\
&\quad \oplus i_{k,m_k} \oplus e_g.
\end{aligned}$$

Setting $i_{k,m_k} = \lambda_{l+1}$, $e_b = a_{k,m_k}$ and $e_g = e_b$, the above equations are corresponding to Eqs (2) and (3). Therefore the final result is obtained. □

*Theorem 6* Let $f_a(\cdot)$ and $f_b(\cdot)$ be functions of complementary arrays $[a,b]$ of length $L = 2^l$ with dimension $n$. The mates $[c,d]$ of $[a,b]$ for complete complementary arrays can be produced by

$$\begin{aligned}
f_c(\vec{i}_1, \cdots, \vec{i}_n) &= f_a(\vec{i}_1, \cdots, \vec{i}_n) \oplus \lambda_1 \oplus e_b, \\
f_d(\vec{i}_1, \cdots, \vec{i}_n) &= f_a(\vec{i}_1, \cdots, \vec{i}_n) \oplus \lambda_1 \oplus \lambda_l \oplus 1.
\end{aligned}$$

where $e_b$ is the given parameter in Eq.(3).

*Proof:* In Theorem 4, $c$ and $d$ are the reverse of $b$ and the inversion of the reverse of $a$, respectively. Note that the reverse of $\vec{i}_j$ can be expressed by the inversion of $\vec{i}_j$, i.e.,

$$\vec{i}_j{}' = (i_{j,1} \oplus 1, i_{j,2} \oplus 1, \cdots, i_{j,l_j} \oplus 1).$$

Therefore these logic function can be expressed as

$$\begin{aligned}
f_c(\vec{i}_1, \cdots, \vec{i}_n) &= f_b(\vec{i}_1{}', \cdots, \vec{i}_n{}') \\
&= f_a(\vec{i}_1{}', \cdots, \vec{i}_n{}') \oplus (\lambda_l \oplus 1) \oplus e_b, \\
f_d(\vec{i}_1, \cdots, \vec{i}_n) &= f_a(\vec{i}_1{}', \cdots, \vec{i}_n{}') \oplus 1.
\end{aligned}$$

where $e_b$ is the given parameter in Eq.(3). As well as the proof of Theorem 5 substituting Eq.(2) to the above equations, and arranging parameters give the final result. □

Note that the given logic functions can provide a sequence generator, whch consists of a feed forward logic circuit derived from $f(\cdot)$ and binary counters generating the input vectors $\vec{i}_1, \vec{i}_2, \cdots, \vec{i}_n$ .

*Example 2* Let $[a,b]$ be complementary arrays of length $4 \times 2$ with dimension 2. From Theorem 5, these logic functions are given as

$$\begin{aligned}
f_a(\vec{i}_1, \vec{i}_2) &= \lambda_1\lambda_2 \oplus \lambda_2\lambda_3 \\
&\quad \oplus a_{1,1} i_{1,1} \oplus a_{1,2} i_{1,2} \oplus a_{2,1} i_{2,1} \oplus e_a, \\
f_b(\vec{i}_1, \vec{i}_2) &= f_a(\vec{i}_1, \vec{i}_2) \oplus \lambda_3 \oplus e_b,
\end{aligned}$$

where $\vec{i}_1 = (i_{1,1}, i_{1,2})$ and $\vec{i}_2 = (i_{2,1})$. Let $\lambda_1 = i_{1,1}, \lambda_2 = i_{2,1}, \lambda_3 = i_{1,2}$ and $a_{1,1} = a_{1,2} = a_{2,1} = e_a = e_b = 0$. The logic functions of $a$ and $b$ are written as

$$\begin{aligned}
f_a(\vec{i}_1, \vec{i}_2) &= i_{1,1} i_{2,1} \oplus i_{2,1} i_{1,2}, \\
f_b(\vec{i}_1, \vec{i}_2) &= f_a(\vec{i}_1, \vec{i}_2) \oplus i_{1,2}.
\end{aligned}$$

Similarly, from Theorem 6 the mate $[c,d]$ of $[a,b]$ for complete complementary arrays is also expressed as

$$\begin{aligned}
f_c(\vec{i}_1, \vec{i}_2) &= f_a(\vec{i}_1, \vec{i}_2) \oplus i_{1,1}, \\
f_d(\vec{i}_1, \vec{i}_2) &= f_a(\vec{i}_1, \vec{i}_2) \oplus i_{1,1} \oplus i_{1,2} \oplus 1.
\end{aligned}$$

Table I shows the truth table of the above logic functions.

TABLE I
TRUTH TABLE OF LOGIC FUNCTIONS OF BINARY COMPLETE COMPLEMENTARY ARRAYS.

| $i_{2,1}$ | $i_{1,2}$ | $i_{1,1}$ | $f_a(\cdot)$ | $f_b(\cdot)$ | $f_c(\cdot)$ | $f_d(\cdot)$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 0 | 1 | 1 | 1 |

Therefore the binary complementary arrays defined by

$$s_{i_1,i_2} = (-1)^{f_a(\vec{i}_1, \vec{i}_2)},$$

where $s$ is $a$, $b$, $c$ or $d$, are easily generated. Note that these are the complementary arrays of length $4 \times 2$ in Example 1.

Note that selection of the parameters $\lambda_1, \lambda_2, \lambda_3 \in \{i_{1,1}, i_{1,1}, i_{2,1}\}$ ($\lambda_1 \neq \lambda_2 \neq \lambda_3$) and $a_{1,1}, a_{1,2}, a_{2,1}, e_a, e_b \in 0, 1$ can give a lot of different complementary arrays of length $4 \times 2$.

So that, Theorems 5 and 6 can easily produce a lot of complementary arrays of length $2^l$ with dimension $n$, which include complementary sequences with $n = 1$.

*Theorem 7* The number of complementary arrays of length $2^l$ with $l \geq 1$, is

$$\#ab = 2^{l+2} l(l-1)(l-2)\cdots 1.$$

*Proof:* For the logic function $f_a(\cdot)$ of Eq.(2), the number of different expressions of $\lambda_1\lambda_2 \oplus \cdots \oplus \lambda_{l-1}\lambda_l$ is $l!/2$, the number of patterns $(a_{1,1}, \cdots, a_{n,l_n}, e_a)$ is $2^{l+1}$. Therefore, the number of different sequences $a$'s produced by $f_a(\cdot)$ is $l!2^l$. Since the number of different $b$'s for $a$, which is produced by $f_b(\cdot)$ of Eq.(3), obeys the selection of $\lambda_1$ or $\lambda_l$ and $e_b \in \{0,1\}$, it is $2^2$. Therefore the final result is obtained. □

Note that dimension $n$ is not related to the number of complementary arrays.

The number of different complementary arrays of length $2^l$ are listed in Table II.

TABLE II
THE NUMBER OF DIFFERENT COMPLEMENTARY ARRAYS WITH LENGTH $2^l$ AND $2^m$ PHASES.

| l | $m=1$ | $m=2$ | $m=3$ | $m=4$ |
|---|---|---|---|---|
| 1 | 8 | 64 | 512 | 2048 |
| 2 | 32 | 512 | 8192 | $\approx 1.3\times 10^5$ |
| 3 | 192 | 6144 | $\approx 1.9\times 10^5$ | $\approx 6.2\times 10^6$ |
| 4 | 1536 | 98304 | $\approx 6.2\times 10^5$ | $\approx 4.1\times 10^8$ |
| 5 | 15360 | $\approx 1.9\times 10^6$ | $\approx 2.6\times 10^8$ | $\approx 3.2\times 10^{10}$ |
| 6 | 184320 | $\approx 4.8\times 10^7$ | $\approx 1.2\times 10^9$ | $\approx 3.0\times 10^{12}$ |

## V. POLYPHASE COMPLEMENTARY ARRAYS

This section clarifies polyphase complementary arrays with $M = 2^m$ phases, whose elements takes

$$\omega_M^k = \cos\left(\frac{2\pi k}{M}\right) + j\sin\left(\frac{2\pi k}{M}\right)$$

with $j = \sqrt{-1}$ and $0 \le k < m$. Especially quadriphase complementary arrays with elements $\{\pm 1, \pm j\}$ included in them, are interesting, because in general quadriphase sequences are practicable as well as binary ones.

The following theorems can be given without proofs, since these are similar to proofs of theorems for binary complementary arrays.

*Theorem 8* If $[a,b]$ be polyphase complementary arrays, $[\omega_M^k a, b]$ is also polyphase complementary arrays.

Complementary arrays of the same length can be derived from Theorems 1 and 8. For example, $[\omega_M^{k_1} a, \omega_M^{k_2} b]$ are also complementary arrays.

*Theorem 9* If $[a,b]$ be complementary arrays, the mate of complete complementary arrays $[c,d]$ is given as

$$\begin{aligned} c_{i_1,\cdots,i_n} &= \omega_M^k b_{L_1-i_1-1,\cdots,L_n-i_n-1}, \\ d_{i_1,\cdots,i_n} &= -\omega_M^k a_{L_1-i_1-1,\cdots,L_n-i_n-1}. \end{aligned}$$

*Theorem 10* Logic functions of polyphase complementary arrays $[a,b]$ of length $L = L_1 \cdots L_n = 2^l$ with dimension $n$ and $2^m$ phases, which are mapping from $V_2^m$ to integers modulo $2^m$, can be expressed as

$$\begin{aligned} a_{i_1,i_2,\cdots,i_n} &= \omega_M^{f_a(\vec{i_1},\vec{i_2},\cdots,\vec{i_n})}, \\ b_{i_1,i_2,\cdots,i_n} &= \omega_M^{f_b(\vec{i_1},\vec{i_2},\cdots,\vec{i_n})}. \end{aligned}$$

with

$$\begin{aligned} & f_a(\vec{i_1},\cdots,\vec{i_n}) \\ &\quad = 2^{m-1}(\lambda_1\lambda_2 \oplus \lambda_2\lambda_3 \oplus \cdots \oplus \lambda_{l-1}\lambda_l \\ &\quad \oplus a_{1,1}^1 i_{1,1} \oplus \cdots \oplus a_{1,l_1}^1 i_{1,l_1} \oplus \cdots \oplus a_{n,l_n}^1 i_{n,l_n} \oplus e_a^1) \\ &\quad \oplus 2^{m-2}(a_{1,1}^2 i_{1,1} \oplus \cdots \oplus a_{1,l_1}^2 i_{1,l_1} \oplus \cdots \oplus a_{n,l_n}^2 i_{n,l_n} \oplus e_a^2) \\ &\quad \vdots \\ &\quad \oplus 2^0(a_{1,1}^m i_{1,1} \oplus \cdots \oplus a_{1,l_1}^m i_{1,l_1} \oplus \cdots \oplus a_{n,l_n}^m i_{n,l_n} \oplus e_a^m), \end{aligned}$$

$f_b(\vec{i_1}, \cdot\cdot$

where $\oplus$ denotes addition over modulo $2^m$, $L_k = 2^{l_k}$, $l = l_1 + l_2 + \cdots + l_n$, $\vec{i_j} = (i_{j,1},\cdots,i_{j,l_j})$, $\lambda_k (1 \le k \le l)$ is one of elements $\{i_{j,1}\cdots i_{j,l_j}\}$ of $\vec{i_j}(1 \le j \le n)$ with $\lambda_k \ne \lambda_m$ for $k \ne m$, $a_{1,1}^1,\cdots,a_{1,l_n}^1,\cdots,a_{n,l_n}^m, e_a^1,\cdots,e_a^m$ and $e_b^1,\cdots,e_b^m$ are parameters to give different complementary arrays.

The mate $[c,d]$ of $[a,b]$ for complete complementary arrrays can be generated by

$$\begin{aligned} & f_c(\vec{i_1},\cdots,\vec{i_n}) = f_a(\vec{i_1},\cdots,\vec{i_n}) \\ &\qquad \oplus 2^{m-1}(\lambda_1 \oplus e_b^1) \oplus 2^{m-2}e_c^2 \cdots \oplus 2^{m-m}e_c^m, \end{aligned}$$

$$\begin{aligned} & f_d(\vec{i_1},\cdots,\vec{i_n}) = f_a(\vec{i_1},\cdots,\vec{i_n}) \\ &\qquad \oplus 2^{m-1}(\lambda_1 \oplus \lambda_l \oplus 1) \oplus 2^{m-2}e_d^2 \cdots \oplus 2^{m-m}e_d^m, \end{aligned}$$

where $e_c^2,\cdots,e_c^{m-1}$ and $e_d^2,\cdots,e_d^{m-1}$ are elements of $\{0,1\}$ to give different complementary arrays.

*Example 3* Let $[a,b]$ be complementary arrays of length $4\times 2$ with dimension 2. From Theorem 10, these logic functions are given as

$$\begin{aligned} f_a(\vec{i_1},\vec{i_2}) &= 2(\lambda_1\lambda_2 \oplus \lambda_2\lambda_3 \\ &\quad \oplus a_{1,1}^1 i_{1,1} \oplus a_{1,2}^1 i_{1,2} \oplus a_{2,1}^1 i_{2,1} \oplus e_a^1), \\ &\quad \oplus a_{1,1}^2 i_{1,1} \oplus a_{1,2}^2 i_{1,2} \oplus a_{2,1}^2 i_{2,1} \oplus e_a^2 \\ f_b(\vec{i_1},\vec{i_2}) &= f_a(\vec{i_1},\vec{i_2}) \oplus 2(\lambda_3 \oplus e_b^1) \oplus e_b^2, \end{aligned}$$

where $\vec{i_1} = (i_{1,1}, i_{1,2})$ and $\vec{i_2} = (i_{2,1})$. Let $\lambda_1 = i_{1,1}, \lambda_2 = i_{2,1}, \lambda_3 = i_{1,2}$ and $a_{1,1}^1 = 1.a_{1,2}^1 = a_{2,1}^1 = 0$, $a_{1,1}^2 = 0.a_{1,2}^1 = a_{2,1}^1 = 1$,$e_a^1 = e_a^2 = e_b^1 = e_b^2 = 0$. The logic functions of $a$ and $b$ are written as

$$\begin{aligned} f_a(\vec{i_1},\vec{i_2}) &= 2(i_{1,1}i_{2,1} \oplus i_{2,1}i_{1,2} \oplus i_{1,1}) \oplus i_{1,2} \oplus i_{2,1}, \\ f_b(\vec{i_1},\vec{i_2}) &= f_a(\vec{i_1},\vec{i_2}) \oplus 2i_{1,2}. \end{aligned}$$

Similarly, the mate $[c,d]$ of $[a,b]$ for complete complementary arrays is also expressed as

$$\begin{aligned} f_c(\vec{i_1},\vec{i_2}) &= f_a(\vec{i_1},\vec{i_2}) \oplus i_{1,1}, \\ f_d(\vec{i_1},\vec{i_2}) &= f_a(\vec{i_1},\vec{i_2}) \oplus i_{1,1} \oplus i_{1,2} \oplus 1. \end{aligned}$$

Table III shows the truth table of the above logic functions of polyphase (quadri-phase) complementary arrays. Therefore polypahse complementary arrays defined by

$$s_{i_1,i_2} = \exp(j\frac{\pi f_s(\vec{i_1},\vec{i_2})}{2}),$$

where $s$ is $a$, $b$, $c$ or $d$, are easily given as

$$\left[a = \begin{pmatrix} 1 & -1 & j & -j \\ j & j & 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 & -1 & -j & j \\ j & j & -1 & -1 \end{pmatrix}\right],$$

and

$$\left[c = \begin{pmatrix} 1 & 1 & j & j \\ j & -j & 1 & -1 \end{pmatrix}, d = \begin{pmatrix} -1 & -1 & j & j \\ -j & j & 1 & -1 \end{pmatrix}\right],$$

Since these aperiodic auto-correlation functions $C_{aa}(\tau_1,\tau_2)$ and $C_{bb}(\tau_1,\tau_2)$ for $-3 \le \tau_1 \le 3$, $-1 \le \tau_2 \le 1$ are respectively expressed by

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 2j & 0 & 2j & 8 & -2j & 0 & -2j \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 & 1 & 0 & -1 & 0 & 1 \\ -2j & 0 & -2j & 8 & 2j & 0 & 2j \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

the sum of them is written as

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that selection of the parameters $\lambda_1,\lambda_2,\lambda_3 \in \{i_{1,1}, i_{1,1}, i_{2,1}\}$ ($\lambda_1 \ne \lambda_2 \ne \lambda_3$) and $a_{1,1}, a_{1,2}, a_{2,1}, e_a, e_b \in 0, 1$ can give a lot of different complementary arrays of length $4 \times 2$.

TABLE III

TRUTH TABLE OF LOGIC FUNCTIONS OF POLYPHASE COMPLEMENTARY ARRAYS.

| $i_{2,1}$ | $i_{1,2}$ | $i_{1,1}$ | $f_a(\cdot)$ | $f_b(\cdot)$ | $f_c(\cdot)$ | $f_d(\cdot)$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| 0 | 0 | 1 | 2 | 2 | 0 | 2 |
| 0 | 1 | 0 | 1 | 3 | 1 | 1 |
| 0 | 1 | 1 | 3 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 | 1 | 3 |
| 1 | 0 | 1 | 1 | 1 | 3 | 1 |
| 1 | 1 | 0 | 0 | 2 | 0 | 0 |
| 1 | 1 | 1 | 0 | 2 | 2 | 2 |

*Theorem 11* The number of polyphase complementary arrays with length $2^l$ and $2^m$ phases, is

$$\#ab = 2^{ml+2m} l(l-1)(l-2)\cdots 1.$$

Note that the case of $m = 1$ is equal to Theorem 7.

The number of different complementary arrays with length $2^l$ and $2^m$ phases are listed in Table IV.

TABLE IV

THE NUMBER OF DIFFERENT COMPLEMENTARY ARRAYS OF LENGTH $2^l$.

| l | # ab | l | # ab |
|---|---|---|---|
| 1 | 8 | 9 | $\approx 7.4 \times 10^8$ |
| 2 | 32 | 10 | $\approx 1.5 \times 10^{10}$ |
| 3 | 192 | 11 | $\approx 3.3 \times 10^{11}$ |
| 4 | 1536 | 12 | $\approx 7.8 \times 10^{12}$ |
| 5 | 15360 | 13 | $\approx 2.0 \times 10^{14}$ |
| 6 | 184320 | 14 | $\approx 5.7 \times 10^{15}$ |
| 7 | 2580480 | 15 | $\approx 1.7 \times 10^{17}$ |
| 8 | $\approx 4.1 \times 10^7$ | 16 | $\approx 5.5 \times 10^{18}$ |

## VI. CONCLUSION

This paper has clarified the construction of complementary arrays with length $L = S2^l$ and the phase number $2^m$ with $S = 1, 10, 26$ and $m \ge 0$, which include complete complementary arrays. The functions generating complementary arrays with $S = 1$ have been formulated. It has been shown that these number relates to length and the phase number, but not the dimension.

The fact that the complementary arrays can be systematically derived from complementary sequences may be useful for extension to multi-dimension for the other families of complementary sequences. Complementary arrays will be used as secret information in the digital watermark for the copyright protection, because of possession of good characteristics, such that the number is a lot, and these aperiodic auto-correlation property is optimal.

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