Analysis of Complexity of One-dimensional Chaotic Maps with Entropy Characteristic

Sen Pei, Zhiming Zheng

Abstract—Chaotic phenomena are widespread in a variety of research fields. In this paper, we investigate the complexity of one-dimensional chaotic maps with entropy characteristic. First we analyze the dynamics of one-dimensional chaotic maps. Then we analogize the iteration of maps to a one-sided shift transformation, and give a probability explanation of Lyapunov exponent by Ergodic Theory. We use the entropy of corresponding shift to reflect the complexity of these maps, providing a new way to characterize complexity of chaotic systems.

Keywords—Chaos, Complexity, Entropy, Ergodicity, Lyapunov Exponent.

I. Introduction

SINCE the middle of 20th century, a large number of chaotic phenomena have emerged in the research of physics, astronomy, meteorology, life sciences, sociology, economy, etc. The Lorenz system in meteorology and Van der pol equations in electrology are two well-known examples. Because of its difficulty and great value in solving practical problems, the research on chaotic phenomena with different structures has continued to today since 1960s, and it is still a frontier and hot spot in non-linear science. The fundamental research on chaotic phenomena, which is based on application background widely, is of great significance not only in mathematical theory but also in practical applications. So in the last decades, there are many good research results of chaos appear [12]-[16].

One of the essential properties of chaos is the sensitive dependence on initial conditions. A sufficient condition for a system to have this property is to have a positive Lyapunov exponent. Therefore, to see whether a system has a positive Lyapunov exponent along critical orbits is an important method to study chaos. One-dimensional maps are the most fundamental maps, so the investigation in these maps has great value in non-linear science. The study in this field is started by considering maps which are induced from special maps. In 1983, Benedicks and Carleson [1] gave metric condition for $F(x;a)=1-ax^2,(-1\le x\le 1)$, where a is a parameter in (0,2), to be Collet-Eckmann (i.e. to have a positive Lyapunov exponent along the orbit of the critical point). In 1993, D.Sands [7] analyzed the mechanism of chaos on general S-unimodal maps in a topological point of view. Later in 2003, Zhang Tingting [6] studied a class of maps $F(x;a)=1-ax^2$, and gave topological conditions for positive Lyapunov exponent of these maps. In order to generalize their works, here we study more general maps, which are multi-modal, in a topological perspective.

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Complexity is an important property of a chaotic system. In some research fields, such as informatics, computer sciences, statistical mechanics and complex networks, entropy are used to characterize the complexity of a certain system. In this paper, we want to use entropy to depict the complexity of one-dimensional dynamical systems, so that we can compare the complexity of two different dynamical systems.

In this paper, we construct a topological tool which applies to general multi-modal maps. We partition the iteration interval [-1; 1] into infinite small open intervals with the pre-images of critical values. We depict a point's position by the small open interval which it falls into. Analyzing critical orbits by this tool, we find that the critical orbits behave somewhat periodically. Then we analogize the iteration of maps to a one-sided shift transformation. By Ergodic Theory, we give a probability explanation of Lyapunov exponent, which provides a numerical computation method to calculate this exponent. We use the entropy of corresponding shift to reflect the complexity of these maps, providing a new way to characterize complexity of chaotic systems.

II. Basic Definitions

In this section, we will introduce the definitions of regular family of maps and Lyapunov exponent, which are the main research objects in this paper. We will see that maps satisfying these conditions are ubiquitous. Assume $f:I\to I$ to be a C^2 map, where I = [-1; 1]. Our research work is based on the regular map family $\{f_a\}_{a\in P}$ with perturbable parameter a^*. We demand a is very close to a^*. In the following discussion, we denote $f^i(x)$ as x_i for any point $x\in I$.

Definition 2.1[2] A point c is called a critical point of f if $f'(c)=0$; and let $C(f)=\{c\in I\mid f'(c)=0\}$. A critical point $c\in C(f)$ is said to be non-flat, if there exists a neighborhood of c and a constant $\tau_c\ge 2$, such that f is C^{τ_c} in the neighborhood of c and $f^{(i)}(c)=0, i=1,2,\cdots,\tau_c-1$,

$f^{(\tau_c)}(c) \neq 0$, equivalently, if there exist positive constants c^* and c_* such that

$$c_* \rho^{\tau_c - 1} \leq | Df(x) | \leq c^* \rho^{\tau_c - 1}$$

where $\rho(x) = dist(y, C(f))$ =min{ $| x - c^t | \,|\, c^t \in C(f)$ }. We call $f : I \to I$ non-flat, if for every $c \in C(f)$, it is non-flat.

Lemma 2.2[2] Let $f : I \to I$ be a C^2 map without flat critical points. Then f consists of finite critical points.

Let $\{c^1, c^2, \cdots, c^k\}$ be the set of critical points of *f*, where $c^1 < c^2 < \cdots < c^k$. Define $c^0 = -1, c^{k+1} = 1$.

Definition 2.3[2] *f* is said to be a bounded distortion from linearity if it satisfies that for $x, y \in I$,

$$\left| \frac{Df(x)}{Df(y)} \right| \leq \exp\left(\frac{c_0}{\rho(y)} | x - y |\right)$$

where c_0 is a positive constant and $\rho(x) = dist(y, C(f))$.

Definition 2.4[2] Let P be an interval of parameter. A map family $f_a(x), x \in I, a \in P$ is regular, if $f_a(x)$ is C^2 w.r.t. (*x*, *a*) and for every $a \in P$, $f_a(x)$ is a bounded distortion from linearity without flat critical points.

Definition 2.5[2] Let $\{f_a\}_{a \in P}$ be a regular map family. A parameter $a^* \in P$ is said to be perturbable w.r.t. family $\{f_a\}_{a \in P}$, if

(M) There exists a neighborhood of $C(f_{a^*})$, $V = \bigcup_{c \in C(f_{a^*})} V_c$, where V_c is the neighborhood of $c \in C(f_{a^*})$, such that

$$\bigcup_{n \geq 1} f_{a^*}^n (C(f_{a^*})) \bigcap V = \varnothing$$

and f_{a^*} has no stable periodic point. That is, there exists $\varepsilon^* > 0$, such that

$$dist\left(\bigcup_{n \geq 1} f_{a^*}^n (C(f_{a^*})), C(f_{a^*})\right) \geq \varepsilon^*$$

and f_{a^*} has no stable periodic point.

(CE) There exists $\varepsilon^* > 0$, such that for every $\delta \in (0, \varepsilon^*)$ and *n*≥ 1, if $x \in I$ satisfies $f_{a^*}^i \notin V(\delta) = \bigcup_{c \in C(f_{a^*})} (c - \delta, c + \delta)$ for *i*=0, 1, 2,…,*n*-1 and $f_{a^*}^n(n) \in V(\delta)$, then

$$| Df_{a^*}^n (x) | \geq k^*.$$

(T)

$$\lim_{n \to \infty} \left| \frac{\partial_a f^n (c_0^{(i)}(a^*), a^*)}{\partial_x f^{n-1} (c_1^{(i)}(a^*), a^*)} \right| = Q(i) \neq 0,$$

where $c_l^i(a^*)$ denote the *lth* iteration of critical point $c^i(a^*)$ of map f_{a^*}, *i*=1, 2, …,*n*.

These maps are very common in one-dimensional maps. For example, for function $F(x; a) = 1 - ax^2$, we can prove that *a=2* is a perturbable parameter of the map.

We give the definition of Lyapunov exponent. Assume $f : I \to I$ is a map and let *C*(*f*) be the set of critical points of *f*. We define the Lyapunov exponent of f as

$$\liminf_{n \to \infty} \frac{\log | Df^n (f(C(f))) |}{n}.$$

If f has a positive Lyapunov exponent, we know that for each critical point c^t,

$$\liminf_{n \to \infty} \frac{\log | Df^n (f(c^t)) |}{n} > 0,$$

equivalently, there exist *k* > 0 and λ > 0, such that for all $n \geq 1$

$$| Df^n (c^t) | > k\lambda^n.$$

Notice that, by the chain rule,if the critical orbits fall into a near neighborhood of critical points, Lyapunov exponent will suffer a great down in value. So we need to investigate the dynamics of the critical orbits if we want to analyze Lyapunov exponent.

In *Ref* [4], it is proved that an interval will expand exponentially for a long time on the condition that the parameter *a* is very close to perturbable parameter a^*, which means the interval will cover critical points in finite iterations. This property guarantees the feasibility of the construction method of our topological tool introduced in the next section.

III. Position Function

In this section, we will partition the interval [-1*;* 1] into infinite subintervals, so that we can use the subinterval which a point locates to depict the distance between this point and critical points' set *C*(*f*).

We take one of the critical points c^t as an example. Obviously, the interval (c^t, c^{t+1}) is the largest monotone open interval under *f* on the right side of c^t. We partition this interval with a topological method, which is uniform to all the critical points. Without loss of generality, we only consider the situation of right side. We begin our partition process by induction.

Definition 3.1 For $t = 1, 2, \cdots, k$, define open interval $I(R, t, 0) = (c^t, c^{t-1})$. Let

$$I(R, t, m) = \begin{cases} f(I(R, t, m-1)) & C(f) \bigcap I(R, t, m-1) = \varnothing \\ f(J(R, t, m-1)) & C(f) \bigcap I(R, t, m-1) \neq \varnothing \end{cases}$$

where *J(R,t,m-1)* is the component of *I(R,t,m-1)\C(f)* which contains c^t_m. The parameter *t* means we consider the critical point c^t; *m* denotes the induction time; *R* means the interval we discuss is on the right side of c^t. If we discuss the left side, we use parameter L instead of R. We can also define *I(L ,t, m)* as above.

We know that when an interval covers critical point, the monotonicity under *f* will change. So the largest monotone interval on the right side of c^t will diminish. We only consider the largest monotone interval that contains critical points *J(R, t, m-1)* in the following induction. We call this manipulation a cutting process. If the interval *I(R, t, m)* covers any critical point, we call the induction time *m* a cutting-time, which is denoted by a function *F(R, t, n)*. Here, the parameter *n* means this is the *nth* cutting process since the induction began. Notice that *I(R, t, m)* may cover more than one critical point. We only concern the one which is the nearest to c^t_m. We denote this critical point by $c^{N(R,t,n)}$, so the endpoints of *J(R, t, m)* are c^t_m and $c^{N(R,t,n)}$. Consider the pre-image of *J(R, t, m)* under f^{-m}. Obviously, one endpoint is c^t, and we denote the other by $\gamma(R,t,n)$. In fact, $\gamma(R,t,n) = c^{N(R,t,n)}_{-F(R,t,n)}$. By the analysis above, we have *F(R, t, n* + 1) = max{*m*≥ *F(R; t; n)* | $f^m|_{(c^t,\gamma(R,t,n))}$ is monotone}, where $\gamma(R,t,n)$ = sup{*x*∈ [c^t , c^{t+1}] | $f^{F(R,t,n)+1}|_{(c^t,x)}$ is monotone}.

Definition 3.2 Define the cutting-interval of *C(f)* as follows:

$$V(n) = \bigcup_{t=1}^{k}(\gamma(L,t,n),\gamma(R,t,n)), n = 1,2,\cdots$$

We have finished the partition of interval [-1*;* 1]. The situation around c^t is shown in Figure 1 below.

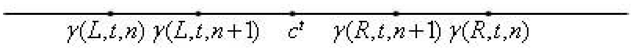

Figure 1

Now we introduce the Position Function *P*. We use this function to reflect the distance between a point and critical points. In the following discussion, we use (*a*; *b*) to denote the open interval whose endpoints are *a* and *b* for simplicity.

Definition 3.3 For *x*∈ [-1,1], define

$$P(t,x) = \max\{m \in N \mid f^m|_{(c^t;x)} \text{ is monotone}\}.$$

P(t, x) is a step function, which is non-decreasing as *x* gets close to c^t and can be arbitrarily large. The discontinuities of this function must be pre-images of critical points, but not all the pre-images are discontinuities. In fact, the discontinuities are corresponding γ. We have

$$P(t,x) = \min\{F \mid (c^t;\gamma) \subset (c^t;x)\}.$$

Here γ is corresponding to the cutting time *F*. By the definition of *F(R, t, n* + 1), we know that *F*(*R, t, n* + 1) =max{m≥ *F(R, t, n)*| $f^m_{(c^t,\gamma(R,t,n))}$ is monotone}. Considering $f^m|_I = f^i|_{f_I^{m-i}}$, we have that *F*(*R, t, n* + 1) = *F*(*R, t, n*) +max{m ≥ 0| $f^m|_{f^{F(R,t,n)}(c^t,\gamma(R,t,n))}$ is monotone }. Combine with the position function *P*, we can get

$$F(R,t,n+1) = F(R,t,n) + P(N(R,t,n), c^t_{F(R,t,n)}).$$

So the value of position function is a cutting time, and the difference between two successive cutting times is also a cutting time.

It is easy to know, when $x \in (\gamma(L,t,1),\gamma(R,t,1))$, *P(t, x)* ≥ 1, and *P(i, x)=0* for *i*≠ *t*. We only concern the nonzero one. So we can use *P*(*x*) to denote the largest value of all *P*(*t, x*).

Now we define some notations related to position function. For $x \in (c^t,\gamma(R,t,1))$, if *P*(*x*) = *F*(*R, t, n*), denote the corresponding γ as $\overline{P}(x) = c^{N(R,t,n)}_{-F(R,t,n)}$. Define $P^+(x)$ =*F*(*R, t, n* + 1), $P^-(x)$= *F*(*R, t, n*-1). Especially, if there exists *n* > 0 such that $f^i(x) \notin C(f)$, *i=0,1,…,n-1*, and $f^n(x) \in C(f)$, we denote *P*(*x*) = *n*, $\overline{P}(x) = x$.

IV. The Dynamics of Critical Orbits

We analyze the dynamics of critical orbits with position function. We only consider the right side of a critical point, the other side is just similar.

Recall ρ (*x*) is the distance between *x* and *C*(*f*). According to the definition of $\overline{P}(x)$, we know that for any $c^t \in C(f)$, $\rho(c^t_n) > \rho(\overline{P}(c^t_n))$. Besides, by the definition of *F*(*R, t, n*), the interval $(c^t,\gamma(R,t,n))$ is monotone under f^i, 1≤ *i*≤ *F*(*R, t, n* + 1).

Consider the definition of the function *F*, we have

Lemma 4.1 For $x \in (c^t,\gamma(R,t,1))$, if there exists a cutting point $\gamma = c^N_{-F} \in (c^t, x)$,, then *P*(*x*)≥ *F*. Here *N*∈ {1,2,… *k*}, *F* is a cutting time.

Lemma 4.2 Assume *F* = *F*(*R, t, n*), *N* = *N*(*R, t, n*). For *i* < *F*, we have $P(c^t_i) \ge P(f^i(c^N_{-F}))$.

Proof: If it is not in this case, we have Figure 2 below.

Figure 2

By the definition of $\overline{P}$, Figure 2 implies *F*-*i* > $P(c^t_i)$. However $\overline{P}(c^t_i) \in f^i|_{(c^N_{-F};c^t)}$, i.e. $c_{-P(c^t_i)-i} \in (c^N_{-F};c^t)$, which means that $P(c_i^t) + i > F$. So we get contradiction.

Lemma 4.3 For any $N_1, N_2 \in \{1,2,\cdots,k\}$, and $N_1 \neq N_2$, denote $F_i = F(R, N_2, m_i)$, i = 1, 2, where m_i are positive integers with $m_1 < m_2$. If the distribution of $c_{-F_1}^{N_1}, c_{-F_2}^{N_1}$ and c^{N_2} is like Figure 3 below,

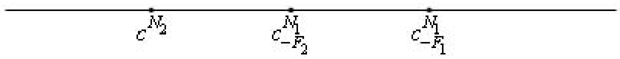

Figure 3

then $c_{F_1}^{N_2}, c_{F_2}^{N_2}$ must be in $(\gamma(L, N_1, 1), \gamma(R, N_1, 1))$. Besides, if $c_{F_1}^{N_2}, c_{F_2}^{N_2}$ are on the same side, $P(c_{F_2}^{N_2}) \geq P(c_{F_1}^{N_2})$.

Proof: By the definition of γ, the corresponding critical points of $c_{F_1}^{N_2}, c_{F_2}^{N_2}$ are all c^{N_1}. If $P(c_{F_2}^{N_2}) < P(c_{F_1}^{N_2})$ we have Figure 4.

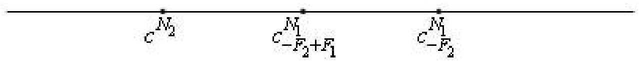

Figure 4

According to Figure 4, we have $F_2 - F_1 > F_2$, which contradicts with the definition of F.

Now we discuss the dynamics of the orbits of critical points. For any $c^t \in C(f)$, $I(R, t, m)$ will cover critical points infinitely according to the discussion in *Ref* [4]. But f has finite critical points. So there must be a critical point c^N covered by the interval for infinite times, which is shown in Figure 5 below.

Figure 5

Here, we have $F_1 < F_2 < F_3 < \cdots$. By Lemma 4.3, we have $P(c_{F_{i+1}}^t) \geq P(c_{F_i}^t)$, $i = 1,2,\ldots$. So c^t will get closer to c^N under the iteration.

Now let us discuss the iteration of the map. For any $x \in [-1,1] \backslash C(f)$, we assume x falls into the neighborhood of c^t c, $P(x) = F^+$, $\overline{P}(x) = c_{-F^+}^{N^+}$, as shown in the following figure.

Figure 6

Definition 4.4 For t = 1, 2, …, k, n = 1, 2,…, if $F = F(R, t, n)$, define $F^+ = F(R, t, n+1)$. For any cutting time F, denote $\Delta F = F^+ - F$. For every given F, denote $F_i = \{F(R, i, m) | F(R, i, m) \leq F;$ *and* $F(R, i, m+1) > F\}$.

Here, we denote the cutting time corresponding to c^i which is no bigger than F as F_i. In the next lemma, we will see this cutting time is an important parameter.

Lemma 4.5 For $F_i \neq F$, $P(x_{F_i}) = \Delta(F_i)$.

Proof: If $(F_i)^+ = F(R, t, l)$, denote $N^* = N(R, t, l)$. We have the figure below.

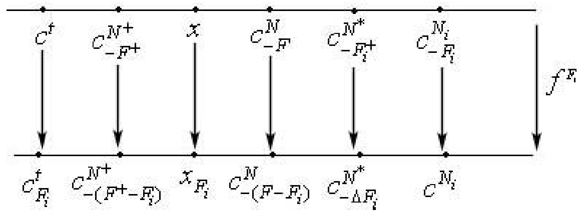

Figure 7

Though $c_{-(F_i)^+}^{N^*}$ may coincide with c_{-F}^N, this situation will not affect our proof. For $c_{-\Delta(F_i)}^{N^*} \in (x_{F_i}; c^{N_i})$, by Lemma 4.1, $P(x_{F_i}) \leq \Delta(F_i)$.

Because $x_{F_i} \in (c_{F_i}^t; c^{N_i})$, we have $P(x_{F_i}) \geq P(c_{F_i}^t)$ Apply Lemma 4.2 to c^t and $c_{-(F_i)^+}^{N^*}$, we have $P(c_{F_i}^t) \geq P(c_{-\Delta(F_i)}^{N^*})$.

If $P(c_{-\Delta(F_i)}^{N^*}) < \Delta F_i$, then there exists a cutting time $F_x < (F_i)^+$, such that $c_{-(F_x - F_i)}^j \in (c_{-\Delta(F_i)}^{N^*}; c^{N_i})$. So we have Figure 8.

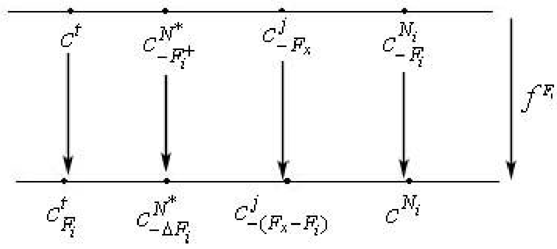

Figure 8

$F_x < (F_i)^+$, so $F_x \leq F_i$. No matter $F_x < F_i$ or $F_x = F_i$, they all contradict with the definition of $c_{-F_i}^{N_i}$. So we have $P(c_{-\Delta(F_i)}^{N^*}) \geq \Delta F_i$ and $P(x_{F_i}) \geq \Delta(F_i)$.

Finally, we have $P(x_{F_i}) = \Delta(F_i)$.

Lemma 4.6 If $F_i = F, P(x_F) \geq \Delta F = F^+ - F$. Especially when there exists $c_{-F^+}^j \in (x; c_F^N)$, $P(x_F) = \Delta F$.

Proof: The distribution of the points is shown in the figure below.

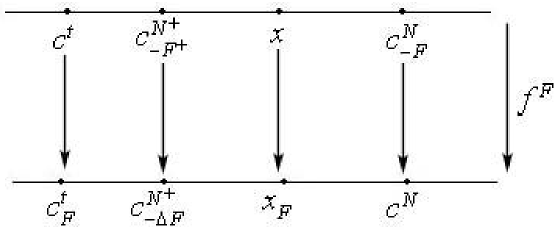

Figure 9

By Lemma 4.2, $P(x_F) \geq P(c_F^t) \geq P(c_{-\Delta F}^N)$. We can prove $P(c_{-\Delta F}^N) \geq \Delta F$ as above. So $P(x_F) \geq \Delta F$. Especially, when $c_{-F^+}^j \in (x; c_{-F}^N)$, we have $c_{-\Delta F}^j \in (x_F; c^N)$, by Lemma 4.1, $P(x_F) \leq \Delta F$, so $P(x_F) = \Delta F$.

Based on the analysis above, we have the next theorem.

Theorem 4.7 For any point $x \in (c^t, \gamma(R,t,1))$, assume $P^-(x) = F$, $F_m(x) = \max\{ \Delta(F_i) \mid i=1,2,\dots,k\}$. Then for i =1, 2,…, $P^-(x)-1$,

$$f^i(x) \cap \bigcup_{i=1}^{k} (\gamma(L,i,F_m(x)), \gamma(R,i,F_m(x))) = \varnothing.$$

We can see that, for i =1, 2,…, $P^-(x)-1$, $f^i(x)$ always stays outside of a neighborhood of critical points, but $f^{P^-(x)}(x)$ may fall into this neighborhood. By the chain rule, we know that if the critical orbit falls into a near neighborhood of critical points, Lyapunov exponent will suffer a great down in value. If critical orbits stay a distance from critical points for a long time, which makes an exponential increase to Lyapunov exponent, Lyapunov exponent will keep positive. Lyapunov exponent first increases exponentially under finite iterations, then decreases to much degree, followed by a gradual exponential increase again. So the great decrease and gradual increase alternate with each other. On the whole it keeps positive and the iteration of the function on the critical points behaves somewhat like a periodical movement.

V. The Ergodicity of Critical Orbits

First introduce some important definitions and results in ergodic theory.

Definition 5.1[3] Suppose (X_1, β_1, m_1) , (X_2, β_2, m_2) are probability spaces. A transformation $T: X_1 \to X_2$ is measurable if $T^{-1}(\beta_2) \subset \beta_1$ (i.e. $T^{-1}B_2 \in \beta_1$). A transformation $T: X_1 \to X_2$ is measure-preserving if T is measurable and $m_1(T^{-1}(B_2)) = m_2(B_2)$ for any $B_2 \in \beta_2$. We say that $T: X_1 \to X_2$ is an invertible measure-preserving transformation if T is measure-preserving, bijective and T^{-1} is also measure-preserving.

Definition 5.2[3] Let (X, β, m) be a probability space. A measure-preserving transformation T of (X, β, m) is called ergodic if the only members B of β with $T^{-1}B = B$ satisfy m(B) = 0 or m(B) = 1.

If T is measure-preserving, then the following statements are equivalent: (1) T is ergodic; (2) whenever f is measurable and f∘ T(x) = f(x) a.e. then f is constant a.e.

The first major result in ergodic theory was proved in 1931 by G.D.Birkhoff.

Theorem 5.3[3] (Birkhoff Ergodic Theorem).

Suppose $T: (X, \beta, m) \to (X, \beta, m)$ is measure-preserving and $f \in L^1(m)$. Then $(1/n)\sum_{i=0}^{n-1} f(T^i(x))$ converges a.e. to a function $f^* \in L^1(m)$. Also $f^* \circ T = f^*$ a.e. and if $m(X) < \infty$, then $\int f^* dm = \int f dm$.

If T is ergodic then f^* is constant a.e. and so if $m(X) < \infty$, f^* = $(1/m(X)) \int f dm$ a.e. If (X, β, m) is a probability space and T is ergodic we have

$$\forall f \in L^1, \lim_{n\to\infty} (1/n)\sum_{i=0}^{n-1} f(T^i x) = \int f dm \text{ a.e.}$$

Let T be a measure-preserving transformation of the probability space (X, β, m) and let $f \in L^1(m)$. We define the time mean of f at x to be

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) \text{ if the limit exists.}$$

The phase or space mean of f is defined to be

$$\int_X f(x) dm.$$

The ergodic theorem implies these means are equal a.e. for all $f \in L^1(m)$ if and only if T is ergodic.

A special type of product space will be important for us. Here each space (X_i, β_i, m_i) is the same space (Y, ℓ, μ) and Y is the finite set {0,1,…,k-1}, $\ell = 2^Y$ (the collection of all subsets of Y), and μ is given by a probability vector ($p_0, p_1, \dots p_{k-1}$) where $p_i = \mu(\{i\})$ .We shall denote the set $\{ (x_i)_{-\infty}^{\infty} \mid x_j = a_j$ for $|j| \leq n \}$ by $_{-n}[a_{-n}, \cdots, a_n]_n$ and call it a block with end points $-n$ and n. We have $m(_{-n}[a_{-n}, a_{-(n-1)}, \cdots, a_{n-1}, a_n]_n)$ = $\prod_{j=-n}^{n} p_{a_j}$ The measure m is called the ($p_0, p_1, \dots p_{k-1}$) -product measure. Sometimes we consider blocks with end points h and l where $h \leq l$. Such a set is one of the form $_h[a_h, \cdots, a_l]_l$ = $\{ (x_i)_{-\infty}^{\infty} \mid x_j = a_j$ for $h \leq i \leq l \}$.It has measure $\prod_{i=h}^{l} p_{a_i}$ .Let (X, β, m) = $\prod_0^\infty (Y, 2^Y, \mu)$. If we write points of X in the form ($x_0, x_1, \dots$), $x_i \in Y$, then define $T: X \to X$ by $T(x_0, x_1, x_2, \dots) = (x_1, x_2, \dots)$. This transformation is called the one-sided ($p_0, p_1, \dots p_{k-1}$)-shift. It has proved that T is measure-preserving, although it is not invertible. Moreover, T is ergodic[3].

We analogize the iteration of map to a one-sided shift. First we will construct a probability space (X, Ω, m).

We divide the interval [-1; 1] into m parts equally. Here we demand m is large enough so that the derivation in each part does not change much. We denote the symbol set $\{0,1,\dots,m-1\}$ as Y, and use 2^Y to denote the collection of all subsets of Y. μ is a probability measure, and we will introduce its definition later. $(Y, 2^Y, \mu)$ is a probability space. Let $(X, \Omega, m) = \prod_0^\infty (Y, 2^Y, \mu)$, where m means the corresponding product measure. Then (X, Ω, m) is the product space corresponding to $(Y, 2^Y, \mu)$.

Choose a critical point c^t , and denote its orbit as a one-sided infinite sequence $l_0 = (x_1, x_2, x_3, \dots, x_n, \dots)$. If $c_n^t \in [-1+2i/m, -1+2(i+1)/m]$, take $x_n = i$. We define probability measure μ by a probability vector ($p_0, p_1, \dots p_{m-1}$), where p_i means the probability that i appears in the sequence l_0. Then the probability measure space (X, Ω, m) is well defined.

We consider the one-sided shift transformation T on (X, Ω, m).We know that T is measure-preserving and ergodic. We can analogize the iteration of map to T.

Assume $f \in L^1(m)$, the time mean of f is

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) ,$$

and the phase mean is $\int_X f(x)dm$. By ergodic theorem, $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x))$ exists a.e. for any $f \in L^1(m)$, and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) = \int_X f(x)dm$$

We want to calculate the Lyapunov exponent. By chain rule, $\log|Df^n(c_1^t)| = \log|Df(c_1^t)| + \log|Df(c_2^t)| + \dots + \log|Df(c_n^t)|$, so what we need is to calculate $\log|Df(c_i^t)|$.

Define $g: X \to R$ as follows: for $l = (x_1, x_2, x_3, \dots, x_n, \dots)$, if $x_1 = i$, define $g(l)$ as the average value of $\log|Df(x)|$ on the interval $[-1+2i/m, -1+2(i+1)/m]$. If c_n^t falls into this interval, we use $g(l)$ to estimate $\log|Df(c_n^t)|$. Obviously, $g \in L^1(m)$, so we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(T^i(l_0)) = \int_X g(x)dm$$

For any $l \in X$, we denote the i_{th} coordinate of l as $l(i)$. Let $X_i = \{ l \in X \mid l(1) = i\}$, we have $X = \bigcup_{i=0}^{m-1} X_i$.

Now we analysis the meaning of $\int_X g(x)dm$.

$$\int_X g(x)dm = \sum_{i=0}^{m-1} \sum_{l \in X_i} g(l)m(l)$$

$$= \sum_{i=0}^{m-1} g(X_i) \sum_{l \in X_i} m(l) = \sum_{i=0}^{m-1} g(X_i)\mu(i).$$

The Lyapunov exponent is the synthesis of different values of $\log|Df(c_n^t)|$ with their probabilities, just analogous to the expectation in Probability Theory. $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(T^i(l_0))$ just reflects the specific form of this synthesis. By ergodic theorem, this formula equals to $\sum_{i=0}^{m-1} g(X_i)\mu(i)$, where $\mu(i)$ represents the probability that critical orbit falls into the interval $[-1+2i/m, -1+2(i+1)/m]$. We know that when the intervals locate near critical points, $g(i)$ is very small. So if we demand the Lyapunov exponent keep positive, the probability that critical orbit falls into these intervals should be very small. In other words, a positive Lyapunov exponent implies small probability that critical orbits fall near $C(f)$. From probability point of view, it should be quite a long time between two events of small probability. So the critical orbit will stay away from critical points for a long time before it comes close to critical points again. So we give a probability explanation for Lyapunov exponent. Besides, we can use this method to calculate Lyapunov exponent of a map numerically.

VI. Complexity and Entropy

Kolmogorov introduced the concept of entropy in ergodic theory in 1958. In some research fields, such as informatics, computer sciences, entropy are used to characterize the complexity of a system widely. There are a lot of works on the relation with complexity and entropy [17]. Here, as we analogize the iteration of map to a one-sided shift, we want to use the entropy of one-sided shift transformation to characterize the complexity of the dynamical systems.

The definition of the entropy of a measure-preserving transformation T of (X, β, m) is in three stages: the entropy of a finite sub-σ-algebra of β , the entropy of the transformation T relative to a finite-σ-algebra, and, finally, the entropy of T. The complete and rigorous definition of entropy of a measure-preserving transformation can be found in *Ref* [3]. We leave out the definition for simplicity.

In the section before, we analogize the iteration of map to a one-sided shift transformation, so we will focus on the entropy of one-sided shift. Usually, it is difficult to calculate the entropy of a measure-preserving transformation T *h(T)* from the definition directly. But for some special transformations, there are calculation methods. By the Kolmogorov-Sinai Theorem, it has proved that the entropy of one-sided ($p_0, p_1, \ldots, p_{k-1}$) shift transformation is $-\sum_{i=0}^{k-1} p_i \log p_i$ [3].

Obviously, the maximum of *h(T)* is reached if and only if $p_0 = p_1 = \ldots = p_{k-1}$. But in the discussion in section V, we have proved that for one-dimensional chaotic maps, a positive Lyapunov exponent means small probability that critical orbits fall near critical points. So the uniform distribution does not exist, and thus *h(T)* can't reach the maximum.

Definition 6.1 We denote the entropy of one-sided ($p_0, p_1, \ldots, p_{m-1}$) shift transformation as *h(T, m)*. In the partition of the interval, let $m \to \infty$, define *h(T)*= $\lim_{m\to\infty}$ *h(T, m)*. We call *h(T)* as the entropy of the dynamical system f^n.

We want to know, for every map *f*, whether the entropy of f^n exists.

First, for every *m*, it is obviously that *h(T, m)*>0. We refine the partition, dividing the interval [*-1+2i/m,-1+2(i+1)/m*] into two part, then the probability vector will become ($p_0, p_1, \ldots p_i', p_i'', \ldots, p_{m-1}$), where $p_i' + p_i'' = p_i$. Because $p_i \log p_i$ = ($p_i' + p_i''$) log ($p_i' + p_i''$)> $p_i' \log p_i' + p_i'' \log p_i''$, so the entropy will increase. We know for each *m*, *h(T, m)*<log *m*. When $m \to \infty$, the maximum can be arbitrarily large. So the entropy *h(T)* can be infinity.

In informatics, when we compare two different systems, if one system has larger entropy, we can tell this system is more complex than the other one. Here, we can also explain the meaning of entropy of dynamical systems. If a map has fixed points or it is attractive, we can say it is relatively simple. The critical orbits will always stay in several points or intervals, thus the entropy will keep unchanged or increase slowly as *m* increases. These maps have relatively small entropy, so they are relatively simple.

A very good example is the map $F(x;a) = 1 - ax^2$. When *a=2*, the critical orbits will come into the point -1. So the entropy will converge to 0. But when *a* is close to 2, the dynamics of critical orbits will be very complex, and the entropy will increase as *m* increases. For these two chaotic maps, we can say the first one is relatively simpler than the later one. In the informatics, entropy is a very important parameter to characterize complexity. If a system has zero entropy, it is a definite system, and it has no information. So in this example, when the parameter *a* is close to 2, the system has more complexity than the other one. In fact, in dynamical systems, when the parameter *a* is close to 2, the system is a non-hyperbolic dynamical system. It has more complex dynamical behaviors. This result is in accordance with the result in informatics.

VII. Conclusion

In this paper, we analyzed the dynamics of a class of one-dimensional chaotic maps. A new topological tool is constructed with pre-images of critical points, with which we can analyze the dynamics of critical orbits. It is shown that the critical orbits will come close to *C*(*f*), then move away from it. After that, critical orbits will come close to *C*(*f*) again, which behaves somewhat periodically. Then we analogize the iteration to a one-sided shift. By ergodic theory, we derive that positive Lyapunov exponent implies small probability that critical orbits fall near *C*(*f*). Also, we give a numerical method to calculate Lyapunov exponent of a map by considering the probability that critical orbits fall into the divided intervals. Finally, we analyze the relation between complexity and entropy. We use the entropy of corresponding shift to reflect the complexity of these maps, providing a new way to characterize complexity of chaotic systems.

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