Statistical causality and stochastic dynamic systems

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Abstract—In this paper we consider a problem (that follows directly from realization problem): how to find a possible states (even minimal) of a stochastic dynamic system S_1 with known outputs, provided it is in a certain causality relationship with another stochastic dynamic system S_2 whose states (or some information about them) are given. This paper is continuation of the papers (Gill and Petrović 1987), (Petrović 1996) and (Petrović 2005).

Keywords— Causality, Hilbert space, realization, stochastic dynamic system.

I. INTRODUCTION

In the first section of this paper we present different concepts of causality between flows of information that are represented by families of Hilbert spaces. Then we give a generalization of a causality relationship $\mathbf{G}$ is a cause of $\mathbf{E}$ within $\mathbf{H}$ which (in terms of σ-algebras) was first given in [6] and which is based on Granger's definition of causality (see [3]).

The study of Granger-causality has been mainly preoccupied with time series. We shall instead concentrated on continuous time processes. Many of systems to which it is natural to apply tests of causality, take place in continuous time. For example, this is generally the case within physics, medicine, finance and within economy

In the second section we relate concepts of causality to the stochastic realization problem. The approach adopted in this paper is that of [5]. However, since our results do not depend on probability distribution, we deal with arbitrary Hilbert spaces instead of those generated by Gaussian processes.

The given causality concept is shown to be equivalent to a generalization of the notion of weak uniqueness for weak solutions of stochastic differential equations (see [9] and [10]).

Also, in [11] it is shown that the given causality concept is closely connected to extremality of measures and martingale problem.

II. PRELIMINARIES AND NOTATIONS

Let $\mathbf{F} = (F_t), t \in R$ be a family of Hilbert spaces. We shall think about F_t as about the information available at time t, or as a current information. Total information $F_{<\infty}$ carried by $\mathbf{F}$ is defined by $F_{<\infty} = \bigvee_{t\in R} F_t$, while past and future information of $\mathbf{F}$ at t is defined as $F_{\leqq t} = \bigvee_{s\leqq t} F_s$ and $F_{\geqq t} = \bigvee_{s\geqq t} F_s$, respectively. It is to be understood that $F_{<t} = \bigvee_{s<t} F_s$ and $F_{>t} = \bigvee_{s>t} F_s$ do not have to coincide with $F_{\leqq t}$ and $F_{\geqq t}$ respectively; $F_{<t}$ and $F_{>t}$ are sometimes referred to as the real past and real future of $\mathbf{F}$ at t . Analogous notation will be used for families $\mathbf{H} = (H_t)$, $\mathbf{G} = (G_t)$ and $\mathbf{E} = (E_t)$

If F_1 and F_2 are arbitrary subspaces of a Hilbert space $\mathcal{H}$ then $P(F_1|F_2)$ will denote the orthogonal projection of F_1 onto F_2 and $F_1 \ominus F_2$ will denote a Hilbert space generated by all elements $x - P(x|F_2)$ where $x \in F_1$. If $F_2 \subseteq F_1$, then $F_1 \ominus F_2$ coincides with $F_1 \cap F_2^{\perp}$, where $F_2^{\perp}$ is the orthogonal complement of F_2 in $\mathcal{H}$.

Possibly the weakest form of causality can be introduced in the following way.

Definition 2.1. It is said that $\mathbf{H}$ is submitted to $\mathbf{G}$ (and written as $\mathbf{H} \subseteq \mathbf{G}$) if $H_{\leqq t} \subseteq G_{\leqq t}$ for each t.

It will be said that families H and G are equivalent (and written as $\mathbf{H} = \mathbf{G}$) if $\mathbf{H} \subseteq \mathbf{G}$ and $\mathbf{G} \subseteq \mathbf{H}$.

Definition 2.2. It is said that $\mathbf{H}$ is strictly submitted to $\mathbf{G}$ (and written as $\mathbf{H} \subseteqq \mathbf{G}$) if $H_t \subseteq G_t$ for each t.

It is easy to see that strict submission implies submission and that converse does not hold.

The notion of minimality of families of Hilbert spaces is specified in the following definition.

Definition 2.3. It will be said that $\mathbf{F}$ is a minimal (respectively, strictly minimal) family having a certain property if there is no family $\mathbf{F}^*$ having the same property which is submitted (respectively, strictly submitted) to $\mathbf{F}$.

It will be said that $\mathbf{F}$ is a maximal (respectively, strictly maximal) family having a certain property if there is no family $\mathbf{F}^*$ having the same property such that family $\mathbf{F}$ is submitted (respectively, strictly submitted) to $\mathbf{F}^*$.

It should be understood that a minimal (respectively, strictly minimal) and maximal (respectively, strictly maximal) family having a certain property are not necessarily unique.

The following results will be used later (for the proof see the given reference).

Theorem 2.1. ([5]) The space F is minimal one such that $F_1 \perp F_2|F$ if and only if $F = P(F_1|S)$ for some space S such that $F_2 \subseteq S \subseteq (F_2 \vee P(F_2|F_1)) \oplus (F_1 \vee F_2)^{\perp}$.

Corollary 2.1.1. ([5]) The space $F \subseteq F_1 \vee F_2$ is a minimal one such that $F_1 \perp F_2|F$ if $F = P(F_1|S)$ for some space S such that $F_2 \subseteq S \subseteq F_2 \vee P(F_2|F_1)$.

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In this paper the following definition of markovian property will be used.

Definition 2.4. ([14]) Family **G** will be called Markovian if $P(G_{\geqq t}|G_{\leqq t}) = G_t$ for each t.

Now we give a definition of a stochastic dynamic system in terms of Hilbert spaces. The characterizing property is the condition that past informations of outputs and states and future informations of outputs and states are conditionally orthogonal given the current state.

Definition 2.5. ([12]) A stochastic dynamic system (s.d.s.) is a set of two families: **H** (outputs) and **G** (states), that satisfy the condition

$$H_{<t} \vee G_{<t} \perp H_{>t} \vee G_{>t} | G_t. \qquad (1)$$

For given family of outputs H, any family **G** satisfying (1) is called a realization of a s.d.s. with those outputs.

It is clear that realization of a s.d.s. is Markovian.

The next intuitively justifiable notion of causality has been proposed in [7].

Definition 2.6. It is said that **G** is a cause of **E** within **H** (and written as $\mathbf{E}K\mathbf{G};\mathbf{H}$) if $E_{<\infty} \subseteq H_{<\infty}$, $\mathbf{G} \subseteq \mathbf{H}$ and $E_{<\infty} \perp H_{\leqq t}|G_{\leqq t}$ for each t.

Intuitively, $\mathbf{E}K\mathbf{G};\mathbf{H}$ means that, for arbitrary t, information about $E_{<\infty}$ provided by $H_{\leqq t}$ is not "bigger" than that provided by $G_{\leqq t}$, or that it is possible to reduce available information from $H_{\leqq t}$ to $G_{\leqq t}$ in order to predict $E_{<\infty}$. The meaning of this interpretation is specified in the next result.

Lemma 2.2. ([2]) $\mathbf{E}K\mathbf{G};\mathbf{H}$ if and only if $E_{<\infty} \subseteq H_{<\infty}$, $\mathbf{G} \subseteq \mathbf{H}$ and $P(E_{<\infty}|H_{\leqq t}) = P(E_{<\infty}|G_{\leqq t})$ for each t.

A definition, analogous to Definition 2.6, formulated in terms of σ-algebras, was first given in [6]: "It is said that **G** entirely causes **E** within **H** relative to P (and written as $\mathbf{E}K\mathbf{G};\mathbf{H};P$) if $\mathbf{E} \subseteq \mathbf{H}$, $\mathbf{G} \subseteq \mathbf{H}$ and $E_{<\infty} \perp H_{\leqq t}|G_{\leqq t}$ for each t. However, this definition (from [6]) contains the condition $\mathbf{E} \subseteq \mathbf{H}$ or equivalently for $E_t \subseteq H_t$ each t, (instead of $E_{<\infty} \subseteq H_{<\infty}$) which does not have intuitive justification. Since Definition 2.6 is more general than the definition given in [6], all results related to causality in the sense of Definition 2.6 will be true and in the sense of the definition from [6], when we add the condition $\mathbf{E} \subseteq \mathbf{H}$ to them.

If **G** and **H** are such that $\mathbf{G}K\mathbf{G};\mathbf{H}$, we shall say that **G** is its own cause within H (compare [6]). It should be mentioned that the notion of subordination (as introduced in [14]) is equivalent to the notion of being one's own cause, as defined here.

If **G** and **H** are such that $\mathbf{G}K\mathbf{G};\mathbf{G}\vee\mathbf{H}$ (where $\mathbf{G}\vee\mathbf{H}$ is a family determined by $(G \vee H)_t = G_t \vee H_t$), we shall say that **H** does not cause **G** . It is clear that the interpretation of Granger-causality is now that **H** does not cause **G** if $\mathbf{G}K\mathbf{G};\mathbf{G}\vee\mathbf{H}$ (see [6]). Without difficulty, it can be shown that this term and the term " **H** does not anticipate **G** " (as introduced in[14]) are identical.

These definitions can be applied to stochastic processes if we are talking about the corresponding induced Hilbert spaces.

Definition 2.7. It will be said that second order stochastic processes are in a certain relationship if and only if the Hilbert spaces they generate are in this relationship.

Remark 1. If stochastic process $Y(t), t \in R$ is a realization of a stochastic dynamic system with outputs $X(t)$, then there exists a stochastic process $Z(t)$ with orthogonal increments which is a realization of the same system. Stochastic process $Z(t)$ is not uniquely determined, but its spectral type is uniquely determined. This follows from the fact that realization of a stochastic dynamic system is Markovian, i. e. process with multiplicity one, so process $Y(t)$ is equivalent (in the sense that $F^Y_{\leqq t} = F^Z_{\leqq t}, t \in R$) to some process with orthogonal increments.

III. Causality and Stochastic Dynamic Systems

Suppose that a stochastic dynamic system S_1 causes, in a certain sense, changes of another stochastic dynamic system S_2. It is natural to assume that outputs **H** of system S_1 can be registered and that some information **E** about the states (or perhaps states themselves) of system S_2 is given. Results that we shall prove will tell us under which conditions concerning the relationships between **H** and **E** it is possible to find states **G** (i.e. Markovian representations) of system S_1 having certain causality relationship in the sense of Definition 2.6 with **H** and **E**.

More precisely, the following cases will be considered:

1° states of a s.d.s. S_1 are a cause of outputs of the same system within available information about s.d.s. S_2;

2° the available information about S_2 is a cause of outputs of S_1 within states of S_1.

We consider different kinds of causality between families **G**, **H** and **E**, while **G** and **H** are in the same relationship, that is, **G** is a realization of an s.d.s. with outputs **H** in all cases.

The first two theorems deal with case 1°, and the next two ones with case 2°.

The solutions of these problems follow from the next more general result which gives conditions under which it is possible to find minimal realizations of a s.d.s. S_1, that is a cause for **H** within a family $\mathbf{E^1} = \mathbf{H} \vee \mathbf{E}$.

Theorem 3.1. Let **H** and **E** be such that $P(E_t|H_{<\infty}) \subseteq E_{\leqq t}$ and $P(H_{<t} \vee E_{<t}|H_{<\infty}) \perp H_{>t}|P(H_t \vee E_t|H_{<\infty})$ for each t. If the family $\mathbf{E^1} = \mathbf{H} \vee \mathbf{E}$ is Markovian, then the family **G**, defined by

$$G_t = P(H_t \vee E_t|H_{<\infty}), t \in R \qquad (2)$$

is a minimal realization (of a s.d.s. with outputs **H**) that causes **H** within $\mathbf{E^1}$.

Proof. From $G_{\leqq t} = P(H_{\leqq t} \vee E_{\leqq t}|H_{<\infty})$ it follows that $H_{<\infty} \perp H_{\leqq t} \vee E_{\leqq t}|G_{\leqq t})$. Also, the definition of **G** and the assumption $P(E_t|H_{<\infty}) \subseteq E_{\leqq t}$ imply $G_{\leqq t} \subseteq H_{\leqq t} \vee E_{\leqq t}$, which together with the previous orthogonality relation means $\mathbf{H}K\mathbf{G};\mathbf{E^1}$. The minimality of **G** follows from Theorem 2.1 and Corollary 2.1.1.

From $G_{\leqq t} \subseteq E^1_{\leqq t}$, the fact that $P(G_{\geqq t}|G_{\leqq t}) = P(E^1_{\geqq t}|G_{\leqq t})$ which follows from $G_{<\infty} = H_{<\infty}$, equation (2) and the assumption that $\mathbf{E^1}$ is Markovian, we get

$$P(G_{\geqq t}|G_{\leqq t}) = P(E^1_{\geqq t}|G_{\leqq t}) = P(P(E^1_{\geqq t}|E^1_{\leqq t})|G_{\leqq t}) = P(E^1_t|G_{\leqq t}) \qquad (3)$$

The relation $\mathbf{H}K\mathbf{G};\mathbf{E^1}$ (because of $\mathrm{G}_{<\infty} = \mathrm{H}_{<\infty}$,) becomes $\mathrm{P}(\mathrm{G}_{\geqq t}|\mathrm{G}_{\leqq t}) = \mathrm{P}(\mathrm{E}_t^1|\mathrm{G}_{<\infty}) = \mathrm{P}(\mathrm{E}_t^1|\mathrm{H}_{<\infty}) = \mathrm{G}_t$, which means that $\mathbf{G}$ is Markovian. This fact with the assumption $\mathrm{G}_{<t} \perp \mathrm{H}_{>t}|\mathrm{G}_t$ gives $\mathrm{G}_{\leqq t} \perp \mathrm{H}_{>t} \vee \mathrm{G}_{\geqq t}|\mathrm{G}_t$. However, since $\mathrm{H}_{<t} \subseteq \mathrm{G}_{\leqq t}$ (which is an obvious consequence of $\mathbf{H}K\mathbf{G};\mathbf{E^1}$), the last relation implies that $\mathbf{G}$ is a realization of a s.d.s. with outputs $\mathbf{H}$.

The next example shows that family $\mathbf{G}$ defined by (2) is not strictly minimal realization of a s. d. s. with outputs $\mathbf{H}$ such that $\mathbf{H}K\mathbf{G};\mathbf{E^1}$.

Example 3.1. Let A and B be arbitrary Hilbert spaces and let $\mathbf{H} = (\mathrm{H}_t)$ and $\mathbf{E} = (E_t)$, $t \in \{1,2,3\}$ be defined by

$\mathrm{H}_1 = \mathrm{A}, \quad \mathrm{H}_2 = \mathrm{A}, \quad \mathrm{H}_3 = \mathrm{B},$

$\mathrm{E}_1 = \mathrm{A}, \quad \mathrm{E}_2 = \mathrm{B}, \quad \mathrm{E}_3 = \mathrm{B}.$

Then the family $\mathbf{E^1} = \mathbf{H} \vee \mathbf{E}$, is given by

$\mathrm{E}_1^1 = \mathrm{A}, \quad \mathrm{E}_2^1 = \mathrm{A} \vee \mathrm{B}, \quad \mathrm{E}_3^1 = \mathrm{B}.$

If the family $\mathbf{G}$ is defined by (2), then

$\mathrm{G}_1 = \mathrm{A}, \quad \mathrm{G}_2 = \mathrm{A} \vee \mathrm{B}, \quad \mathrm{G}_3 = \mathrm{B}.$

According to Theorem 3.1, $\mathbf{G}$ is a minimal realization (of a s. d. s. with outputs $\mathbf{H}$) and $\mathbf{H}K\mathbf{G};\mathbf{E^1}$. However, family $\mathbf{G}^* = (\mathrm{G}_t^*), t \in \{1,2,3\}$ defined by

$\mathrm{G}_1^* = \mathrm{A}, \quad \mathrm{G}_2^* = \mathrm{B}, \quad \mathrm{G}_3^* = \{0\},$

is another realization of the same s.d.s. and $\mathbf{H}K\mathbf{G}^*;\mathbf{E^1}$. Obviously, $\mathbf{G}^* \leqq \mathbf{G}$.

The following corollary to Theorem 3.1 gives a partial solution (under the condition that $\mathbf{E}$ is Markovian) of the problem 1° defined above.

Corollary 3.1.1. Let $\mathbf{H}$ and $\mathbf{E}$ be such that $\mathbf{H} \subseteq \mathbf{E}$ as well as $\mathrm{P}(\mathrm{E}_t|\mathrm{H}_{<\infty}) \subseteq \mathrm{E}_{\leqq t}$ and $\mathrm{P}(\mathrm{E}_{<t}|\mathrm{H}_{<\infty}) \perp \mathrm{H}_{>t}|\mathrm{P}(\mathrm{E}_t|\mathrm{H}_{<\infty})$ for each t. If the family $\mathbf{E}$ is Markovian, then the family $\mathbf{G}$, defined by

$$\mathrm{G}_t = \mathrm{P}(\mathrm{E}_t|\mathrm{H}_{<\infty}), t \in \mathrm{R}$$

is a minimal realization (of a s.d.s. with outputs $\mathbf{H}$) that causes $\mathbf{H}$ within $\mathbf{E}$.

The next example illustrates the last result.

Example 3.2. Let $Z_k, k > 0$, $(Z_0 = 0)$, be a sequence of uncorrelated random variables with the mean zero and let $X_{2k} = Z_{2k}$, $X_{2k+1} = Z_{2k}$, $k = 0,1,2,\ldots$. Then $\mathrm{F}_{\leqq k}^X \subseteq \mathrm{F}_{\leqq k}^Z, k \geqq 0$ and

$$\mathrm{P}(Z_k|\mathrm{F}_{<\infty}^{\mathrm{X}}) = \begin{cases} Z_k, & k = 2m \\ 0, & k = 2m+1 \end{cases},$$

$$\mathrm{P}\left(\mathrm{F}_{>2k}^X\middle|\mathrm{F}_{\leqq 2k}^{\mathrm{Y}}\right) = Z_{2k},\ \mathrm{P}\left(\mathrm{F}_{>2k+1}^X\middle|\mathrm{F}_{\leqq 2k+1}^{\mathrm{Y}}\right) = 0;$$

i.e. all conditions of Corollary 3.1.1. are satisfied. Therefore, the stochastic process

$$Y_k = P(Z_k|\mathrm{F}_{<\infty}^{\mathrm{X}})$$

is a minimal realization (of a s.d.s. with outputs X_k) that causes X_k within Z_k.

If $\mathbf{H}$ is its own cause within $\mathbf{E}$, we obtain a simpler version of the last result.

Corollary 3.1.2. Let $\mathbf{H}$ is its own cause within $\mathbf{E}$ and let for each t holds $\mathrm{P}(\mathrm{E}_{<t}|\mathrm{H}_{\leqq t}) \perp \mathrm{H}_{>t}|\mathrm{P}(\mathrm{E}_t|\mathrm{H}_{\leq t})$. If $\mathbf{E}$ is Markovian, then the family $\mathbf{G}$, defined by

$$\mathbf{G}_t = P(\mathrm{E}_t|\mathrm{H}_{\leqq t}), \quad t \in R$$

is a minimal realization (of a s.d.s. with outputs $\mathbf{H}$) that causes $\mathbf{H}$ within $\mathbf{E}$.

The following result does not require $\mathbf{E}^1 = \mathbf{H} \vee \mathbf{E}$ to be Markovian, but provides a realization whose present information at t is equal to its total information accumulated up to t.

Theorem 3.2. Let $\mathbf{H}$ and $\mathbf{E}$ be such that $P(\mathrm{E}_t|S) \subseteq \mathrm{E}_{\leqq t}$ for each t where S is some space such that

$$\mathrm{H}_{<\infty} \subseteq S \subseteq \mathrm{H}_{<\infty} \vee \mathrm{P}\left(\mathrm{H}_{<\infty}\middle|\mathrm{E}_{\leqq t}^1\right).$$

Family $\mathbf{G}$ is a minimal realization (of a s.d.s. with outputs) that causes $\mathbf{H}$ within $\mathbf{E}^1 = \mathbf{H} \vee \mathbf{E}$ if and only if it is defined by

$$\mathrm{G}_t = \mathrm{P}(\mathrm{H}_{\leqq t} \vee \mathrm{E}_{\leqq t}|S), \ t \in \mathrm{R} \qquad (4)$$

The next corollary gives all solutions to the problem 1° formulated above.

Corollary 3.2.1. Let $\mathbf{H}$ and $\mathbf{E}$ be such that $\mathrm{H}_{<\infty} \subseteq \mathrm{E}_{<\infty}$ as well as $\mathrm{P}(\mathrm{E}_t|S) \subseteq \mathrm{E}_{\leqq t}$ for each t where S is some space such that $\mathrm{H}_{<\infty} \subseteq S \subseteq \mathrm{H}_{<\infty} \vee \mathrm{P}(\mathrm{H}_{<\infty}|\mathrm{E}_{\leqq t})$. Family $\mathbf{G}$ is a minimal realization (of a s.d.s. with outputs $\mathbf{H}$) that causes $\mathbf{H}$ within $\mathbf{E}$ if and only if it is defined by

$$\mathrm{G}_t = \mathrm{P}(\mathrm{E}_{\leqq t}|\mathrm{H}_{<\infty}), t \in \mathrm{R}.$$

The next example illustrates the above result.

Example 3.3. Let

$$Z(t) = \sum_{n=1}^{2} \int_0^t \mathrm{g_n}(\mathrm{t},\mathrm{u}) \mathrm{d}Z_n(\mathrm{u}), \ \ \mathrm{t} \in [0,1]$$

be a proper canonical representation of the process $\{Z(t), \mathrm{t} \in [0,1]\}$ and let the process $\{X(t), \mathrm{t} \in [0,1]\}$ be defined by

$$X(t) = \int_0^t \mathrm{h}(\mathrm{u})\mathrm{d}Z_1(\mathrm{u}), \mathrm{t} \in [0,1].$$

Then $\mathrm{F}_{\leqq t}^X \subseteq \mathrm{F}_{\leqq t}^Z$ for each t. Further, for

$$Y(t) = \mathrm{P}(Z(\mathrm{t})|\mathrm{F}_{<\infty}^X) = \int_0^1 \mathrm{g}_1(\mathrm{t},\mathrm{u})\mathrm{d}Z_1(\mathrm{u}), \mathrm{t} \in [0,1]$$

we have that $\mathrm{F}_{\leqq t}^Y \subseteq \mathrm{F}_{\leqq t}^Z$. According to Corollary 3.2.1 it follows that family $\mathrm{F}_{\leqq t}^Y = \mathrm{P}\left(\mathrm{F}_{\leqq t}^Z\middle|\mathrm{F}_{<\infty}^X\right)$, $\mathrm{t} \in [0,1]$ is a minimal realization (of a s.d.s. with outputs $\mathrm{F}_{\leqq t}^X$) that causes $\mathrm{F}_{\leqq t}^X$ within $Z(t)$.

In the remaining part of paper we consider the problem 2° formulated above. The next theorem considers the problem of determining the possible states $\mathbf{G}$ (of a s.d.s. with outputs $\mathbf{H}$) such that the family $\mathbf{E}^1 = \mathbf{H} \vee \mathbf{E}$ is a cause of outputs $\mathbf{H}$ within $\mathbf{G}$.

Theorem 3.3. 1) Each Markovian family $\mathbf{G}$ such that $\mathbf{H} \vee \mathbf{E}\ K\ \mathbf{H} \vee \mathbf{E};\mathbf{G}$ and $P(\mathrm{H}_{>t}|\mathrm{G}_{<t}) \subseteq \mathrm{G}_t$ for each t is a realization (of a s. d. s. with outputs $\mathbf{H}$) and the family $\mathbf{H} \vee \mathbf{E}$ is a cause of $\mathbf{H}$ within $\mathbf{G}$.

2) If $\mathbf{J}$ is a Markovian family such that $\mathbf{H} \vee \mathbf{E}\ K\ \mathbf{H} \vee \mathbf{E};\mathbf{J}$ and $\mathrm{P}(\mathrm{J}_{<t}|\mathrm{H}_{\leqq t} \vee \mathrm{E}_{\leqq t}) \perp \mathrm{H}_{>t}|\mathrm{P}(\mathrm{J}_t|\mathrm{H}_{\leqq t} \vee \mathrm{E}_{\leqq t})$ for each t, then family $\mathbf{G}$, defined by

$$G_t = P(\mathrm{J}_t|\mathrm{H}_{\leqq} \vee \mathrm{E}_{\leqq t}), t \in \mathrm{R}, \qquad (5)$$

is minimal among the realizations (of a s. d. s. with outputs $\mathbf{H}$) such that the family $\mathbf{H} \vee \mathbf{E}$ is a cause of $\mathbf{H}$ within $\mathbf{G}$.

3) If families $\mathbf{E}$ and $\mathbf{H}$ are submitted to some given "framework" family $\mathbf{F}$ and if $\mathbf{H} \vee \mathbf{E}\ K\ \mathbf{H} \vee \mathbf{E};\mathbf{F}$ holds, then the family G, defined by

$$\mathrm{G}_{\mathrm{t}} = \mathrm{F}_{\leqq \mathrm{t}}, \mathrm{t} \in \mathrm{R} \quad (6)$$

is strictly maximal among the realizations (of a s. d. s. with outputs **H**) such that $\mathbf{H} \vee \mathbf{E}$ is a cause of **H** within **G**.

Proof. 1) According to Lemma 2.2, the assumption $\mathrm{G}_{<t} \perp \mathrm{H}_{>t} | \mathrm{G}_{\mathrm{t}}$ is equivalent to $P(\mathrm{H}_{>t} | \mathrm{G}_{\leqq \mathrm{t}}) \subseteq \mathrm{G}_{\mathrm{t}}$. From that and the assumption that **G** is Markovian family, we get $\mathrm{G}_{\mathrm{t}} = P(\mathrm{G}_{\geqq \mathrm{t}} | \mathrm{G}_{\leqq \mathrm{t}}) = P(\mathrm{G}_{\geqq \mathrm{t}} \vee \mathrm{H}_{>t} | \mathrm{G}_{\leqq \mathrm{t}})$
which is equivalent to $\mathrm{G}_{\leqq \mathrm{t}} \perp \mathrm{G}_{\geqq \mathrm{t}} \vee \mathrm{H}_{>t} | \mathrm{G}_{\mathrm{t}}$. However, since $\mathrm{H}_{<t} \subseteq \mathrm{G}_{\leqq \mathrm{t}}$ (which is an obvious consequence of $\mathbf{H} \vee \mathbf{E}\ K\ \mathbf{H} \vee \mathbf{E}; \mathbf{G}$ the last relation means that **G** is a realization of a s. d. s. with outputs **H**. From $\mathbf{H} \vee \mathbf{E}\ K\ \mathbf{H} \vee \mathbf{E}; \mathbf{G}$ it follows that $\mathbf{H}\ K\ \mathbf{H} \vee \mathbf{E}; \mathbf{G}$ holds.

2) From (5) it follows that $\mathrm{G}_{\leqq \mathrm{t}} = \mathrm{H}_{\leqq \mathrm{t}} \vee \mathrm{E}_{\leqq \mathrm{t}}$ and immediately we get $\mathbf{H}\ K\ \mathbf{H} \vee \mathbf{E}; \mathbf{G}$ and $\mathbf{G}\ K\ \mathbf{G}; \mathbf{J}$. According to Definition 2.6, it is clear that the family **G**, defined by (5), is a minimal family such that $\mathbf{H}\ K\ \mathbf{H} \vee \mathbf{E}; \mathbf{G}$. From the assumptions that $\mathbf{H} \vee \mathbf{E}\ K\ \mathbf{H} \vee \mathbf{E}; \mathbf{J}$ and the fact that **J** is Markovian we get $P(\mathrm{G}_{\geqq \mathrm{t}} | \mathrm{G}_{\leqq \mathrm{t}}) = P(\mathrm{J}_{\geqq \mathrm{t}} | \mathrm{H}_{\leqq t} \vee \mathrm{E}_{\leqq \mathrm{t}}) = P(P(\mathrm{J}_{\geqq \mathrm{t}} | \mathrm{J}_{\leqq \mathrm{t}}) | \mathrm{H}_{\leqq \mathrm{t}} \vee \mathrm{E}_{\leqq \mathrm{t}}) =$
$P(\mathrm{J}_{\mathrm{t}} | \mathrm{H}_{\leqq \mathrm{t}} \vee \mathrm{E}_{\leqq \mathrm{t}}) = \mathrm{G}_{\mathrm{t}}$,
which means that **G** is Markovian. Now, according to part 1) of this theorem, it follows that the family **G**, defined by (5) is a realization (of a s. d. s. with outputs **H**) such that $\mathbf{H}\ K\ \mathbf{H} \vee \mathbf{E}; \mathbf{G}$

3) Since $\mathrm{G}_{\leqq \mathrm{t}} = \mathrm{F}_{\leqq \mathrm{t}}$ the assumption $\mathbf{H} \vee \mathbf{E}\ K\ \mathbf{H} \vee \mathbf{E}; \mathbf{G}$ is equivalent to $\mathbf{H} \vee \mathbf{E}\ K\ \mathbf{H} \vee \mathbf{E}; \mathbf{G}$, so it follows $\mathbf{H} K\ \mathbf{H} \vee \mathbf{E}; \mathbf{G}$. From $\mathrm{G}_{\mathrm{t}} = \mathrm{G}_{\leqq \mathrm{t}}$, and $\mathbf{H} \subseteq \mathbf{G}$ immediately follows that **G** is a realization of a s. d. s. with outputs **H**.

From the fact that **F** is a framework family (i.e., $\mathbf{G} \subseteq \mathbf{F}$) it is clear that **G** is a strictly maximal realization with given properties.

It is easy to see that for given outputs **H** of a s.d.s. S_1 and information **E** about a s.d.s. S_2, the family **E**, defined by (5), is not an unique minimal realization (of a s. d. s. S_1) such that $\mathbf{H}\ K\ \mathbf{H} \vee \mathbf{E}; \mathbf{G}$. For each family $\mathbf{J}^* \subseteq \mathbf{F}$ which satisfies conditions from part 2) of Theorem 3.3, with $\mathrm{G}_{\mathrm{t}}^* = P(\mathrm{J}_{\mathrm{t}}^* | \mathrm{H}_{\leqq \mathrm{t}} \vee \mathrm{E}_{\leqq \mathrm{t}}), \mathrm{t} \in \mathrm{R}$ is defined a minimal realization (of a s. d.s. S_1) such that $\mathbf{H}\ K\ \mathbf{H} \vee \mathbf{E}; \mathbf{G}^*$**.** All these minimal realizations have the past information equivalent to $\mathrm{H}_{\leqq \mathrm{t}} \vee \mathrm{E}_{\leqq \mathrm{t}}$, $\mathrm{t} \in \mathbf{R}$, but their present information at *t* is different.

The problem of determining a strictly minimal realization **G** (of a s.d.s. with outputs **H**) such that $\mathbf{H}\ K\ \mathbf{H} \vee \mathbf{E}; \mathbf{G}$ is still open. If it would be possible to find a strictly minimal family $\mathbf{J}^{\mathbf{m}}$ between families $\mathbf{J}^* \subseteq \mathbf{F}$ that satisfy conditions from part 2) of Theorem 3.3, this strictly minimal family $\mathbf{J}^{\mathbf{m}}$ would define a strictly minimal family $\mathbf{G}^{\mathbf{m}}$ (with (5)) among all families **G** of part 2) of Theorem 3.3. It is clear that if there exists such strictly minimal family, it cannot be necessarily unique, so that a strictly minimal realization with given properties is not necessarily unique.

Especially, if $\mathbf{H} \subseteq \mathbf{E}$, Theorem 3.3. gives realizations of a s.d.s. with outputs **H** such that the family **E** is a cause of outputs **H** within **G**. More precisely, the next corollary to Theorem 3.3. gives a partial solution of the problem 2° formulated above.

Corollary 3.3.1. 1) Let **H** and **E** be such that $\subseteq \mathbf{E}$. Each Markovian family **G** such that $\mathbf{E}\ K\ \mathbf{E}; \mathbf{G}$ and $\mathrm{G}_{<t} \perp \mathrm{H}_{>t} | \mathrm{G}_{\mathrm{t}}$ for each t is a realization (of a s. d. s. with outputs **H**) and **E** is a cause of **H** within **G**.

2) Let **H**, **E** and **J** be such that $\mathbf{H} \subseteq \mathbf{E}$, as well as $\mathbf{E}\ K\ \mathbf{E}; \mathbf{J}$ and $P(\mathrm{J}_{<t} | \mathrm{E}_{\leqq \mathrm{t}}) \perp \mathrm{H}_{>t} | P(\mathrm{J}_{\mathrm{t}} | \mathrm{E}_{\leqq \mathrm{t}})$ for each t. If **J** is Markovian, then the family **G**, defined by

$$\mathrm{G}_{\mathrm{t}} = P(\mathrm{J}_{\mathrm{t}} | \mathrm{E}_{\leqq \mathrm{t}}), \quad \mathrm{t} \in \mathrm{R}$$

is minimal among the realizations (of a s. d. s. with outputs **H**) such that **E** is a cause of **H** within **G**.

3) If $\mathbf{H} \subseteq \mathbf{E}$ and if given "framework" family **F** is such that $\mathbf{E}\ K\ \mathbf{E}; \mathbf{F}$, then the family **G**, defined by

$$\mathrm{G}_{\mathrm{t}} = \mathrm{F}_{\leqq \mathrm{t}}, \ \mathrm{t} \in \mathrm{R}$$

is strictly maximal among the realizations (of a s. d. s. with outputs **H**) such that **E** is a cause of **H** within **G**.

Remark 2. It is clear that all results from this paper can be extended on the σ-algebras generated by finite dimensional Gaussian random variables. But, in the case that σ-algebras are arbitrary, the extensions of the proofs from this paper is nontrivial because one can not take an orthogonal complement with respect to a σ -algebra as one can with respect to subspaces in Hilbert space

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