Nonlocal symmetries for a family Benjamin-Bona-Mahony-Burgers equations. Some exact solutions

M. S. Bruzón and M.L. Gandarias

***Abstract*—In this work the nonlocal symmetries of a family Benjamin-Bona-Mahony-Burgers equations are studied. The partial differential equation written as a conservation law can be transformed into an equivalent system by introducing a suitable potential. The nonlocal symmetry group generators of original partial differential equation can be obtained through their equivalent system. We have proved that the nonclassical method applied to this system leads to new symmetries, which are not solutions arising from potential symmetries of the Benjamin-Bona-Mahony-Burgers equations.**

We also have derived traveling wave solutions for the Benjamin-Bona-Mahony-Burgers equations by using a direct method. Among them we find a solution which describes a kink solution.

***Index Terms*—Nonlocal symmetries, Partial differential equations, Nonlinear equations, MAXIMA Software, kink solutions**

I. Introduction

We consider the family of Benjamin-Bona-Mahony-Burgers equations (GBBMB)

$$u_t + bu_x + a\left(u^m\right)_x + \left(u^n\right)_{xxt} + c\left(u^k\right)_{xx} = 0, \qquad (1)$$

where c, m, n, k are parameters different to zero and a, b are arbitrary constants, which was introduced by Bruzón and Gandarias in [18]. The authors in [18] made a complete Lie group classification for the equation (1) and the corresponding reduced equations were derived from the optimal system of subalgebras. They considered the ordinary differential equations (ODE) $y'' + F(y)\,y' + G(y) = 0$ and they determined the functional forms $G(y)$ when $F(y)$ is any arbitrary function for which this equation admits solutions in terms of the Jacobi elliptic functions. Due to the fact that a reduced equation of (1) is of this form, equation (1) admits solutions in terms of the Jacobi elliptic functions.

Conservation laws are important in the study of evolutionary partial differential equations since they lead to constants of motion for the time evolution of field variables. Nonlinear partial differential equations (PDEs) that admit conservation laws arise in many disciplines of the applied sciences including physical chemistry, fluid mechanics, particle and quantum physics, plasma physics, elasticity, gas dynamics, electromagnetism, magneto-hydro-dynamics, nonlinear optics, and the bio-sciences, [1], [25].

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Manuscript received March 23, 2011; revised, 2011.

Equation (1) includes important evolution equations employed in mathematical physics, engineering and fluid mechanics. For instance, Benjamin-Bona-Mahony equation

$$u_t + u_x + uu_x - u_{xxt} = 0,$$

was propused as an alternative model to the Korteweg–de Vries equation for the long wave motion in nonlinear dispersive systems. Bruzón, Gandarias and Camacho [7] studied similarity reductions of equation (1) for $c = 0$.

There is no existing general theory for solving nonlinear PDEs. These solutions are obtained by using group invariants to reduce the number of independent variables. Most of the required theory and description of the method can be found in for example [30], [31], [32]. Due to the great advance in computation in the last years a great progress has been made in the development of methods and their applications to nonlinear PDEs for finding exact solutions. For instance, classical Lie method [8], [21], nonclassical method [15], simplest equation method [26], (G'/G)-expansion method [9], [11], [14], extended simplest equation method [27], among other.

An obvious limitation of group-theoretic methods based in local symmetries, in their utility for particular PDEs, is that there exists PDEs of physical interest possessing few symmetries or none at all [30]. It turns out that PDEs can admit nonlocal symmetries whose infinitesimal generators depend on integrals of the dependent variables in some specific manner.

In [2], [3] Bluman introduced a method to find a new class of symmetries for a PDE. By writing a given PDE, denoted by R$\{x, t, u\}$ in a conserved form a related system denoted by S$\{x, t, u, v\}$ as additional dependent variables is obtained. Any Lie group of point transformations admitted by S$\{x, t, u, v\}$ induces a symmetry for R$\{x, t, u\}$; when at least one of the generators of the group depends explicitly of the potential, then the corresponding symmetry is neither a point nor a Lie-Bäcklund symmetry. These symmetries of R$\{x, t, u\}$ are called *potential* symmetries. The existence of potential symmetries leads to the construction of corresponding invariant solutions as well as to the linearization of nonlinear PDEs by non-invertible mappings. In [22] Gandarias introduced a new classes of symmetries for a PDE, which can be written in the form of conservation laws. These symmetries, called *nonclassical potential* symmetries, are obtained as nonclassical symmetries of an associated system.

In this work we also show how the free software MAXIMA program symmgrp2009.max, derived by W. Heremann, can be used to calculate the determining equations for the nonlocal symmetries of the GBBMB equation (1).

The structure of the work is as follows: In Sec. II we analyze the classical potential symmetries of the GBBMB equation. In Sec. III we study the nonclassical potential symmetries of the GBBMB equation. We apply a direct method and we obtain travelling wave solutions in Sec. IV. Finally, in Sec. V we give conclusions.

II. Analysis

In order to find potential symmetries of (1) we write the equation in a conserved form and the associated auxiliary system is given by

$$\left\{\begin{array}{l} v_x = u, \\ v_t = -bu - au^m - (u^n)_{xt} - c\left(u^k\right)_x . \end{array}\right. \tag{2}$$

A Lie point symmetry admitted by $S(x,t,u,v)$ is a symmetry characterized by an infinitesimal transformation of the form

$$\begin{array}{rl} x^* = & x_i + \epsilon\xi(x,t,u,v) + \mathcal{O}(\epsilon^2), \\ t^* = & t + \epsilon\tau(x,t,u,v) + \mathcal{O}(\epsilon^2), \\ u^* = & u + \epsilon\psi(x,t,u,v) + \mathcal{O}(\epsilon^2) \\ v^* = & v + \epsilon\varphi(x,t,u,v) + \mathcal{O}(\epsilon^2) \end{array} \tag{3}$$

admitted by system (2). In the present work, we will study if the point symmetries of (2) induce potential symmetries of equation (1). These symmetries are such that

$$\xi_v^2 + \tau_v^2 + \psi_v^2 \neq 0.$$

If the above relation does not hold, then the point symmetries of (2) project into point symmetries of (1). System (2) admit Lie symmetries if and only if

$$\mathrm{pr}^{(2)}X(v_x - u) = 0,$$

$$\mathrm{pr}^{(2)}X\left(v_t + bu + au^m + (u^n)_{xt} + c(u^k)_x\right) = 0,$$

where $\mathrm{pr}^{(2)}V$ is the second extended generator of vector field

$$\begin{array}{rl} X = & \xi(x,t,u,v)\partial_x + \tau(x,t,u,v)\partial_t \\ & +\psi(x,t,u,v)\partial_u + \varphi(x,t,u,v)\partial_v . \end{array} \tag{4}$$

In other words, we require that the infinitesimal generator leaves invariant the set of solutions of (2) and we obtain an overdetermined system of equations called determining equations.

A. Symbolic manipulation programs

In this section we show how the free software MAXIMA program symmgrp2009.max derived by W. Heremann can be used to calculate the determining equations for the potential symmetries of the GBBMB equation (1). To use symmgrp2009.max, we have to convert (1) into the appropriate MAXIMA syntax: x[1] and x[2] represent the independent variables x and t, respectively, u[1] and u[2] represent the dependent variables u and v, respectively, u[1, [1, 0]] represents u_x, u[1, [1,1]] represents u_{xt}, u[2, [1, 0]] represents v_x, and u[2, [0, 1]] represents v_t. Hence (2) is rewritten as

$$\begin{array}{l} u[2,[1,0]] - u[1]; \\ u[2,[0,1]] + b*u[1] + a*u[1]^m \\ +c*k*u[1]^{(k-1)}*u[1,[1,0]] \\ +c*(n-1)*n*u[1]^{(n-2)}*(u[1,[0,1]]) \\ (u[1,[1,0]]) + c*n*u[1]^{(n-1)}*(u[1,[1,1]]); \end{array}$$

The infinitesimals ξ, τ, ψ and φ are represented by eta1, eta2, phi1 and phi2, respectively. The program symmgrp2009.max automatically computes the determining equations for the infinitesimals. The batchfile batch containing the MAXIMA commands to implement the program symmgrp2009.max, which we have called GBBMBPotcls.mac is

```
kill(all);

batchload("C:\\CLA\\symmgrp2009.max");

/*potential symmetries GBBMB eq.*/

batch("C:\\GBBMB\\GBBMBPotcls.dat");
symmetry(1,0,0);

printeqn(lode);

save("lodegnlh.lsp",lode);

for j thru q do (x[j]:=concat(x,j));

for j thru q do (u[j]:=concat(u,j));

ev(lode)$

gnlhode:ev(%,x1=x,x2=t,u1=u,u2=v);

grind:true$

stringout("gnlhode",gnlhode);

derivabbrev:true;
```

The first lines of this file are standard to symmgrp. max and explained in [19]. The last lines are in order to create an output suitable for solving the determining equations. This changes x[1], x[2], u[1] and u[2] to x,t, u and v, respectively. The file GBBMBPotcls.mac in turn batches the file GBBMBPotcls.dat which contains the requisite data about (2).

```
p:2$
q:2$
m:2$

parameters:[a,b,c,n,k,s]$

warnings:true$

sublisteqs:[all]$

subst_deriv_of_vi:true$

info_given:true$

highest_derivatives:all$

depends([eta1,eta2,phi1,phi2],
[x[1],x[2],u[1],u[2]]);
```

```
e1:u[2,[1,0]]-u[1];
e2:u[2,[0,1]]+b*u[1]+a*u[1]^s
+c*k*u[1]^(k-1)*u[1,[1,0]]
+c*(n-1)*n*u[1]^(n-2)*(u[1,[0,1]])
*(u[1,[1,0]])+c*n*u[1]^(n-1)
*(u[1,[1,1]]);

v1:u[2,[1,0]];
v2:u[2,[0,1]];
```

The program symmgrp2009.max generates a system of thirty determining equations which is given in Appendix 1.

B. Potential symmetries of GBBMB equation

Solving the system obtained using symmgrp2009.max and converting MAXIMA syntax we obtain that $\xi = \xi(x)$, $\tau = \tau(t)$, $\varphi = \varphi(x,t,v)$ and

$$\psi = (\varphi_v - \xi_x)\, u + \varphi_x$$

where ξ, τ and φ must satisfy the following equations

$$\begin{aligned}
&cnu^{2n+1}\left(n\varphi_{vv}u + n\varphi_{vx} - \varphi_{vx}\right) = 0,\\
&u^{n+2}\left(cn\varphi_{vvv}u^{n+2} + 2cn\varphi_{vvx}u^{n+1} + cn\varphi_{vxx}u^{n}\right.\\
&\quad\left. -n\varphi_v u + \varphi_v u + \xi_x n u + \xi_x u - n\varphi_x + \varphi_x\right) = 0,\\
&c\,(n-1)\,nu^{n}\left(n\varphi_{vv}u + n\varphi_{vx} - \varphi_{vx}\right) = 0,\\
&cknu^{n+k}\left(n\varphi_{vv}u + n\varphi_{vx} - \varphi_{vx}\right) = 0,\\
&cnu^{n}\left(n\varphi_{vv}u^2 + \varphi_{vv}u^2 + 2n\varphi_{vx}u - \xi_{xx}nu + n\varphi_{xx}\right.\\
&\quad\left. -\varphi_{xx}\right) = 0,\\
&-\left(acn\varphi_{vvv}u^{m+n+2} + 2acn\varphi_{vvx}u^{m+n+1}\right.\\
&\quad +acn\varphi_{vxx}u^{m+n} - a\varphi_v m u^{m+1} + a\xi_x m u^{m+1}\\
&\quad +a\varphi_v u^{m+1} - a\tau_t u^{m+1} - a\varphi_x m u^{m} + bcn\varphi_{vvv}u^{n+3}\\
&\quad +2bcn\varphi_{vvx}u^{n+2} - cn\varphi_{tvv}u^{n+2} + bcn\varphi_{vxx}u^{n+1}\\
&\quad -2cn\varphi_{tvx}u^{n+1} - cn\varphi_{txx}u^{n} - ck\varphi_{vv}u^{k+2}\\
&\quad -2ck\varphi_{vx}u^{k+1} + c\xi_{xx}ku^{k+1} - ck\varphi_{xx}u^{k} - b\tau_t u^2\\
&\quad \left. +b\xi_x u^2 - b\varphi_x u - \varphi_t u\right) = 0,\\
&an^2\varphi_{vv}u^{m+n+1} + an^2\varphi_{vx}u^{m+n}\\
&\quad -an\varphi_{vx}u^{m+n} + ckn\varphi_{vvv}u^{n+k+2} + 2ckn\varphi_{vvx}u^{n+k+1}\\
&\quad +ckn\varphi_{vxx}u^{n+k} + bn^2\varphi_{vv}u^{n+2} + bn^2\varphi_{vx}u^{n+1}\\
&\quad -bn\varphi_{vx}u^{n+1} - n^2\varphi_{tv}u^{n+1} - n^2\varphi_{tx}u^{n} + n\varphi_{tx}u^{n}\\
&\quad -k^2\varphi_v u^{k+1} + k\varphi_v u^{k+1} + \xi_x k^2 u^{k+1} - \tau_t k u^{k+1}\\
&\quad +\xi_x k u^{k+1} - k^2\varphi_x u^{k} + k\varphi_x u^{k} = 0,\\
&(n-1)\,u^{n+1}\left(cn\varphi_{vvv}u^{n+2} + 2cn\varphi_{vvx}u^{n+1}\right.\\
&\quad +cn\varphi_{vxx}u^{n} - n\varphi_v u + \varphi_v u + \xi_x n u + \xi_x u\\
&\quad \left. -n\varphi_x + 2\varphi_x\right) = 0.
\end{aligned} \tag{5}$$

If a, b, c and n, k, m are arbitrary constants,

$$\xi = k_1, \qquad \tau = k_2, \qquad \psi = 0, \qquad \varphi = k_3,$$

where k_1, k_2 and k_3 are arbitrary constants. So the only symmetries admitted by (1) are defined by the infinitesimal generators

$$\mathbf{X}_1 = \partial_x, \qquad \mathbf{X}_2 = \partial_t, \qquad \mathbf{X}_3 = \partial_v.$$

If $a = b = 0$ or $a = 0$ and $n = 2k-1$ or $b = 0$ and $n = 2(k-m)+1$ we obtain new symmetries of the system (2): From system (5) the infinitesimals are:

$$\xi = \frac{(n-1)k_1}{2}x + k_2,$$

$$\tau = (n-k)k_1\, t + k_3,$$

$$\psi = k_1 u,$$

$$\varphi = \frac{k_1\,(n+1)\,v}{2} + k_4.$$

The previous generators do not correspond to a potential symmetry of the equation (1) because $(\xi_v)^2 + (\tau_v)^2 + (\psi_v)^2 = 0$. In this case we obtain a new symmetry which is given by the infinitesimal generator

$$\mathbf{X}_4 = \frac{(n-1)}{2}x\partial_x + (n-k)t\partial_t + u\partial_u + \frac{(n+1)}{2}\,v\partial_v.$$

We do not consider the case $k = m = n = 1$ because in this case equation (1) is a linear PDE.

C. Reduction equations

As in the case of Lie point symmetries, potential symmetries may be used to derive similarity transformations. Such transformations reduce the number of independent variables of a system of PDEs. In order to find similarity solutions for system (2) we need to solve the invariant surface conditions

$$\xi u_x + \tau u_t = \psi, \qquad \xi v_x + \tau v_t = \varphi,$$

where ξ, τ, ψ and φ are the infinitesimal of the transformation (3). The similarity solutions can be found by solving the corresponding characteristic equations

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\psi} = \frac{dv}{\varphi}. \tag{6}$$

We present the similarity solutions which are produced by the symmetries that were obtained in Section II.
Case 1. Substituting the generator $X_1 + X_2 + X_3 = \lambda\partial_x + \mu\partial_t + \gamma\partial_v$ into (6) we get

$$z = \mu x - \lambda t, \quad u(x,t) = h(z), \quad v = \gamma x - \lambda w(z). \tag{7}$$

Similarity transformation (7) reduces system (2) to the nonlinear system

$$\begin{aligned}
&h\,\lambda - \gamma + \mu\, w' = 0,\\
&h^{n-2}\,(h')^2\,\mu\, n^2\,\lambda + h^{n-1}\,h''\,\mu\, n\,\lambda\\
&-h^{n-2}\,(h')^2\,\mu\, n\,\lambda - w' - c\,h^{k-1}\,h'\,k\,\mu\\
&-a\,h^m - b\,h = 0.
\end{aligned} \tag{8}$$

System (8) can be transformed into the nonlinear ODE

$$\begin{aligned}
&h^{n-2}\,(h')^2\,\mu\, n^2\,\lambda + h^{n-1}\,h''\,\mu\, n\,\lambda\\
&-h^{n-2}\,(h')^2\,\mu\, n\,\lambda + \tfrac{h\,\lambda}{\mu} - \tfrac{\gamma}{\mu} - c\,h^{k-1}\,h'\,k\,\mu\\
&-a\,h^m - b\,h = 0.
\end{aligned} \tag{9}$$

Equation (9) can be written in the form

$$h'' + F(h)(h')^2 + G(h)h' + H(h) = 0, \tag{10}$$

where $F(h) = \dfrac{n-1}{h}$, $G(h) = -\dfrac{c\,h^{k-n}\,k}{n\,\lambda}$ and $H(h) = -\dfrac{h^{1-n}\,(\gamma + a\,h^m\,\mu + b\,h\,\mu)}{\mu^2\,n\,\lambda}$.

Case 2. From X_4 substituting the infinitesimals into equations (6) we get

$$z = \frac{x^{\frac{2(n-k)}{n-1}}}{t}, \qquad u(x,t) = x^{\frac{2}{n-1}}\,h(z), \quad v = x^{\frac{n+1}{n-1}}\,w(z). \tag{11}$$

Similarity transformation (11) reduces system (2) with $a = b = 0$ to the nonlinear system

$$\begin{aligned} &2nw'z - 2kw'z + nw + w - hn + h = 0,\\ &2h^n\left(h'\right)^2 n^3z^3 - 2h^n\left(h'\right)^2 kn^2z^3 + 2h^{n+1}h''n^2z^3\\ &-2h^n\left(h'\right)^2 n^2z^3 - 2h^{n+1}h''knz^3\\ &+2h^n\left(h'\right)^2 knz^3 + h^2nw'z^2 - h^2w'z^2\\ &+4h^{n+1}h'n^2z^2 - 2h^{n+1}h'knz^2 - 2ch^{k+1}h'knz\\ &+2ch^{k+1}h'k^2z - 2ch^{k+2}k = 0. \end{aligned} \tag{12}$$

Similarity transformation (11) reduces system (2) with $a = 0$ and $n = 2k-1$ to the nonlinear system

$$\begin{aligned} &2\,(2k-1)\,w_zz - 2kw_zz + (2k-1)\,w + w\\ &-h\,(2k-1) + h = 0,\\ &4h^{2k}\left(h'\right)^2 k^3z^3 + 2h^{2k+1}h''k^2z^3\\ &-3h^{2k+1}h''kz^3 + 8h^{2k}\left(h'\right)^2 kz^3 + h^{2k+1}h''z^3\\ &-2h^{2k}\left(h'\right)^2 z^3 + h^3kw'z^2 - h^3w'z^2\\ &+6h^{2k+1}h'k^2z^2 - 7h^{2k+1}h'kz^2 + 2h^{2k+1}h'z^2\\ &-ch^{k+2}h'k^2z + ch^{k+2}h'kz - ch^{k+3}k\\ &-bh^4k + bh^4 - 10h^{2k}\left(h'\right)^2 k^2z^3 = 0. \end{aligned} \tag{13}$$

Similarity transformation (11) reduces system (2) with $b = 0$ and $n = 2(k-m)+1$ to the nonlinear system

$$\begin{aligned} &2\,(2\,(k-m)+1)\,w'z + (2\,(k-m)+1)\,w\\ &+w - h\,(2\,(k-m)+1) + h - 2kw'z = 0,\\ &4h^{2k}\left(h'\right)^2 k^3z^3 + 2h^{2k+1}h''k^2z^3\\ &8h^{2k}\left(h'\right)^2 m^3z^3 - 20h^{2k}\left(h'\right)^2 km^2z^3\\ &-4h^{2k+1}h''m^2z^3 - 8h^{2k}\left(h'\right)^2 m^2z^3\\ &+16h^{2k}\left(h'\right)^2 k^2mz^3 + 6h^{2k+1}h''kmz^3\\ &+14h^{2k}\left(h'\right)^2 kmz^3 + 4h^{2k+1}h''mz^3\\ &+2h^{2k}\left(h'\right)^2 mz^3 - 4h^{2k}\left(h'\right)^2 k^3z^3\\ &-2h^{2k+1}h''k^2z^3 - 6h^{2k}\left(h'\right)^2 k^2z^3\\ &-3h^{2k+1}h''kz^3 - 2h^{2k}\left(h'\right)^2 kz^3\\ &-h^{2k+1}h''z^3 + h^{2m+1}mw'z^2\\ &-h^{2m+1}kw'z^2 - 8h^{2k+1}h'm^2z^2\\ &+14h^{2k+1}h'kmz^2 + 8h^{2k+1}h'mz^2\\ &-6h^{2k+1}h'k^2z^2 - 7h^{2k+1}h'kz^2 - 2h^{2k+1}h'z^2\\ &-2ch^{2m+k}h'kmz + ch^{2m+k}h'k^2z + ch^{2m+k}h'kz\\ &-ah^{3m+1}m + ah^{3m+1}k + ch^{2m+k+1}k = 0. \end{aligned} \tag{14}$$

III. Nonclassical Potential Symmetries

To obtain *nonclassical potential* symmetries of the GBBMB equation, we apply the nonclassical method to system (2). The basic idea is that the potential system $S\{x,t,u,v\}$ is augmented with the invariance surface conditions

$$\xi u_x + \tau u_t - \psi = 0, \tag{15}$$

$$\xi v_x + \tau v_t - \varphi = 0, \tag{16}$$

which is associated with the vector field (4)

$$\begin{aligned} X = \; &\xi(x,t,u,v)\partial_x + \tau(x,t,u,v)\partial_t\\ &+\psi(x,t,u,v)\partial_u + \varphi(x,t,u,v)\partial_v. \end{aligned}$$

In the case $\tau \neq 0$, without loss of generality, we may set $\tau(x,t,u) = 1$. By requiring that both (2), (15) and (16) are invariant under the transformations with infinitesimal generator (4) one obtains overdetermined, nonlinear system of equations for the infinitesimals $\xi(x,t,u,v)$, $\tau(x,t,u,v)$, $\psi(x,t,u,v)$ and $\varphi(x,t,u,v)$.

We use the program symmgrp2009.max. We implement the program GBBMBPotNcls.mac of similar form to program GBBMBPotcls.mac. We implement the program GBBMBPotNcls.mac of the following form

```
p:2$
q:2$
m:2$
parameters:[a,b,c,n,k,s]$

warnings:true$

sublisteqs:[all]$

subst_deriv_of_vi:true$

info_given:true$

highest_derivatives:all$

depends([eta1,eta2,phi1,phi2],
[x[1],x[2],u[1],u[2]]);

ut:phi1-eta1*u[1,[1,0]];

vt:phi2-eta1*u[2,[1,0]];

uxt:diff(phi1,x[1])
+diff(phi1,u[1])*u[1,[1,0]]
-diff(eta1,x[1])*u[1,[1,0]]
-diff(eta1,u[1])*u[1,[1,0]]^2
-eta1*u[1,[2,0]];

eta2:1;

e1:u[2,[1,0]]-u[1];
e2:vt+b*u[1]+a*u[1]^s
+c*k*u[1]^(k-1)*u[1,[1,0]]
+c*(n-1)*n*u[1]^(n-2)*(ut)*(u[1,[1,0]])
+c*n*u[1]^(n-1)*(uxt);


v1:u[2,[1,0]];
v2:u[1,[2,0]];
```

The program symmgrp2009.max generates a system of six determining equations which is given in Appendix 2. From this system, if we require that $\xi_u = \varphi_u = 0$, we obtain that

$$\psi = -\xi_v u^2 + (\varphi_v - \xi_x)u + \varphi_x$$

and

$$\xi_v(n+2) = 0 \qquad \text{and} \qquad (n-1)\varphi_x = 0.$$

If $\xi=\xi(x,t)$ and $n=1$ the nonclassical method applied to (2) does not yield any new symmetry different from the ones obtained by the Lie classical method in Sec. II.

If $n=-2$ and $\varphi=\varphi(t,v)$, ξ and φ must satisfy the equation

$$\begin{array}{l} -2\left(2\xi^2\xi_{xx}+\varphi\xi_v\xi_x+\xi_t\xi_x-\varphi\xi\xi_{vx}-2\varphi\varphi_v\xi_v-\xi\xi_{tx}\right.\\ \left.-2\varphi_v\xi_t+2\varphi\varphi_{vv}\xi+2\varphi_{tv}\xi\right)-ku^{k+2}\left(-k\xi\xi_x-2\xi\xi_x\right.\\ \left.-\varphi\xi_v-\xi_t+k\varphi_v\xi+2\varphi_v\xi\right)+2u\left(\varphi\xi\xi_{vv}-2\xi^2\xi_{vx}\right.\\ \left.-\varphi\left(\xi_v\right)^2-\xi_t\xi_v-\varphi_v\xi\xi_v+\xi\xi_{tv}+3\varphi_{vv}\xi^2\right)\\ +k\left(k+1\right)u^{k+3}\xi\xi_v=0 \end{array} \tag{17}$$

From equation (17) if k is an arbitrary constant the nonclassical method applied to (2) does not yield any new symmetry different from the ones obtained by the Lie classical method in Sec. II. The same result we obtain if m is an arbitrary constant. But if $n=-2$, $k=-1$, $c\neq 0$ and $m=1$, ξ and φ must verify the following two equations

$$\begin{array}{l} -4c\xi^2\xi_{xx}+\varphi\left(2c\xi\xi_{vx}-2c\xi_v\xi_x\right)\\ +u\left(-c\xi\xi_x+\varphi\left(2c\xi\xi_{vv}-2c\left(\xi_v\right)^2-c\xi_v\right)-4c\xi^2\xi_{vx}\right.\\ \left.+\varphi_v\left(c\xi-2c\xi\xi_v\right)-2c\xi_t\xi_v+2c\xi\xi_{tv}-c\xi_t+6c\varphi_{vv}\xi^2\right)\\ -2c\xi_t\xi_x+\varphi_v\left(4c\varphi\xi_v+4c\xi_t\right)+2c\xi\xi_{tx}-4c\varphi\varphi_{vv}\xi\\ -4c\varphi_{tv}\xi=0, \end{array}$$

$$\begin{array}{l} u\left(\left(8c\xi\xi_v+c\xi\right)\xi_{xx}+\varphi_{vv}\left(4c\varphi_v\xi-2c\xi\xi_x\right)+4c\xi\xi_{vx}\xi_x\right.\\ +\varphi\left(2c\xi\xi_{vvx}-2c\xi_v\xi_{vx}\right)-4c\xi^2\xi_{vxx}-2c\xi_t\xi_{vx}-6c\varphi_v\xi\xi_{vx}\\ \left.+2c\xi\xi_{tvx}\right)+\varphi\left(2c\xi\xi_{vxx}-2c\xi_v\xi_{xx}\right)+\left(4c\xi\xi_x-2c\xi_t\right)\xi_{xx}\\ -4c\varphi_v\xi\xi_{xx}+u^2\left(\varphi\left(-\xi\xi_x-\xi_t\right)+2c\xi\xi_{vv}\xi_x-6c\xi^2\xi_{vvx}\right.\\ +\varphi_v\left(3\varphi\xi-4c\xi\xi_{vv}\right)+\left(8c\xi\xi_v+2c\xi\right)\xi_{vx}\\ \left.+\varphi_{vv}\left(-2c\xi\xi_v-c\xi\right)-\varphi^2\xi_v+2c\varphi_{vvv}\xi^2+\varphi_t\xi\right)\\ +u^3\left(\left(\xi^2+\left(-2b-2a\right)\xi\right)\xi_x-2c\xi^2\xi_{vvv}+\left(2c\xi\xi_v+c\xi\right)\xi_{vv}\right.\\ \left.+\varphi\left(\xi-b-a\right)\xi_v+\left(-b-a\right)\xi_t+\varphi_v\left(\left(3b+3a\right)\xi-3\xi^2\right)\right)\\ +2c\xi\xi_{txx}=0. \end{array} \tag{18}$$

Although equation (18) is too complicated to be solved in general, we deduce the infinitesimal generators

$$\xi=k_2\exp(k_1v)+\frac{1}{2k_1},\quad \tau=1,$$
$$\psi=-k_2k_1u^2\exp(k_1v),\quad \varphi=0.$$

If $n=k=-2$ and $m=1$, $\xi=\xi(v)$ and $\varphi=\varphi(v)$, ξ and φ must satisfy

$$\begin{array}{l} u\left(2c\varphi\xi\xi_{vv}-2c\varphi\left(\xi_v\right)^2+\left(2c-2c\varphi_v\right)\xi\xi_v+6c\varphi_{vv}\xi^2\right)\\ +\left(4c\varphi\varphi_v-2c\varphi\right)\xi_v-4c\varphi\varphi_{vv}\xi=0, \end{array}$$

$$\begin{array}{l} u^3\left(-2c\xi^2\xi_{vvv}+2c\xi\xi_v\xi_{vv}+\left(\varphi\xi+\left(-b-a\right)\varphi\right)\xi_v\right.\\ \left.-3\varphi_v\xi^2+\left(3b+3a\right)\varphi_v\xi\right)+u^2\left(\left(2c-4c\varphi_v\right)\xi\xi_{vv}\right.\\ \left.+\left(-2c\varphi_{vv}\xi-\varphi^2\right)\xi_v+2c\varphi_{vvv}\xi^2+3\varphi\varphi_v\xi\right)\\ +u\left(4c\varphi_v-2c\right)\varphi_{vv}\xi=0. \end{array} \tag{19}$$

From (19) for $c=0$ we get the infinitesimal generators:

$$\xi=c_0(k_1v+k_2)^3,\quad \tau=1,$$
$$\psi=-k_1u\left(3c_0{k_1}^2uv^2+6c_0k_1k_2uv+3c_0{k_2}^2u-1\right),$$
$$\varphi=k_1v+k_2.$$

IV. Some exact solutions

In this section we consider ODE (9) and look exact solutions of this equation in the form

$$h(z)=\frac{a_1\exp(z)}{1+a_2\exp(z)}. \tag{20}$$

Substituting (20) into (9), we get

$$\begin{array}{r} C_1+C_2\left(\frac{a_1e^z}{a_2e^z+1}\right)^m+C_3\left(\frac{a_1e^z}{a_2e^z+1}\right)^k\\ +C_4\left(\frac{a_1e^z}{a_2e^z+1}\right)^n=0 \end{array} \tag{21}$$

where

$$\begin{array}{rcl} C_1&=&-a_1\left(a_2e^z+1\right)\left[e^z(a_1\lambda-a_2\gamma-a_1b\mu)-\gamma\right]\\ C_2&=&a\mu\left(a_2e^z+1\right)^2\\ C_3&=&ck\mu^2\left(a_2e^z+1\right)\\ C_4&=&nn\mu^2\left(a_2e^z-n\right) \end{array}$$

From $C_1=0$ we obtain that $\gamma=0$ and $\lambda=b\mu$.
If $m=n=k$ equation (21) becomes

$$C_2+C_3+C_4=0. \tag{22}$$

By equating to zero the coefficients of $\exp(z)$, $\exp(2z)$ and the independent term, we get the following system

$$aa_2^2\;=0, \tag{23}$$

$$a_2\,\left(b\,k\,\mu^2+c\,k\,\mu+2\,a\right)\;=0, \tag{24}$$

$$-\left(b\,k^2\,\mu^2-c\,k\,\mu-a\right)\;=0. \tag{25}$$

From (23-25) we obtain

$$a=0,\quad m=n=k=-1,\quad c=-b\mu$$

which yields for these values of constants to the following exact solution

$$h(z)=\frac{a_1\exp(z)}{1+a_2\exp(z)}. \tag{26}$$

Back to the function u we have that

$$u(x,t)=\frac{a_1\exp(\mu x-b\,\mu\,t)}{1+a_2\exp(\mu x-b\,\mu\,t)} \tag{27}$$

is an exact solution for (1). For $a_1=a_2=1$, $\mu=\frac{1}{4}$ and $b=\frac{1}{2}$ and $t=0,5,10$ we plot the solution

$$u(x,t)=\frac{\exp\left(\frac{1}{4}(x-\frac{1}{2}t)\right)}{1+\exp\left(\frac{1}{4}(x-\frac{1}{2}t)\right)}. \tag{28}$$

We can observe that solution (27) describes a kink solution (see Figure 1).

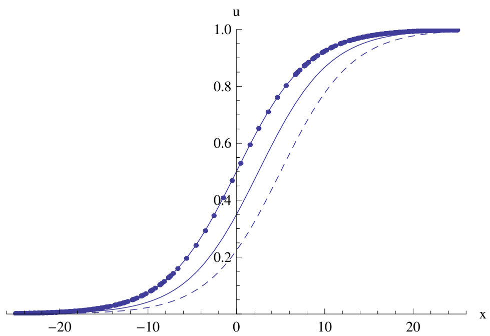

Fig. 1. Solution (28) for $t = 0, 5, 10$

V. Conclusions

In this paper we have studied the Lie symmetries of system (2), depending on the values of the constants a, b, c, n, k and m. By making use of the theory of symmetry reductions in differential equations. We have constructed all the invariant solutions with regard to the one-dimensional system of subalgebras. Besides the travelling wave solution, we find new similarity reductions for this system of equations. This system is a conservation law for the Generalized BBGB equation (1). We have proved that the symmetries of system (2) does not yield classical potential symmetries of equation (1). The ansatz to generate nonclassical solutions of the associated system (2) yields solutions of (1) which are not solutions arising from classical potential symmetries of (1).

Acknowledgment

The authors acknowledge the financial support from Junta de Andalucía group FQM–201 and from project MTM2009-11875.

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Appendix A
Determining equations

In Figure 2 we show the determining equations obtained from the program symmgrp2009.max by applying the classical potential symmetries

In Figure 3 we show the determining equations obtained from the program symmgrp2009.max by applying the non-classical potential symmetries

$[(eta2_{u})\, u^{n+1}, (eta2_{u})\, u^{k+1}, (eta2_{u})\, u^{n+3}, (eta1_{v})\, u^{2\,n+2}, (eta1_{v})$
$u^{n+k+2}, (eta1_{u})\, u^{n+3}, (eta2_{uv})\, u^{2\,n+2},\ u^{n+1}\,((eta2_{v})\, u+eta2_{x}),$
$(eta2_{u})(n-1),\ u^{n+3}\,((eta2_{v})\, u+eta2_{x}), (eta1_{v})(n-1)\, u^{2\,n+1}, (eta2_{uv})$
$(n-1)\, u^{2\,n+1},\ u^{n+3}\,(-a\,(eta1_{v})\, u^{s}-b\,(eta1_{v})\, u+eta1_{t}),\ u^{2\,n+1}$
$((eta1_{uv})\, u+(eta1_{v})\, n-eta1_{v}),\ u^{n+2}$
$(c\,(eta2_{vv})\, n\, u^{n+1}+c\,(eta2_{vx})\, n\, u^{n}-(eta2_{u})\, u),\ u^{n+k+1}$
$((eta1_{uv})\, u+(eta1_{v})\, n-eta1_{v}), (n-1)((eta2_{v})\, u+eta2_{x}), (n-1)$
$((eta1_{uv})\, u+(eta1_{v})\, n-eta1_{v}),\ u^{n+2}$
$((eta2_{uv})\, u^{2}+(eta2_{v})\, n\, u+(eta2_{ux})\, u+(eta2_{x})\, n-eta2_{x}),\ a\,(eta2_{u})\, u^{s+1}$
$+c\,(eta2_{v})\, k\, u^{k+1}+c\,(eta2_{x})\, k\, u^{k}+b\,(eta2_{u})\, u^{2}-(eta1_{u})\, u^{2}+(phi2_{u})\, u,$
$u^{n+1}\ (-c\,(eta1_{vv})\, n\, u^{n+2}+c\, n\,(phi1_{uv})\, u^{n+1}-c\,(eta1_{vx})\, n\, u^{n+1}+c\, n^{2}$
$(phi1_{v})\, u^{n}-c\, n\,(phi1_{v})\, u^{n}+c\,(eta2_{v})\, k\, u^{k+1}+2\,(eta1_{u})\, u^{2}),\ u^{n+1}$

$(-c\,(eta1_{vv})\, n\, u^{n+2}+c\, n\,(phi1_{uv})\, u^{n+1}-c\,(eta1_{vx})\, n\, u^{n+1}+c\, n^{2}$
$(phi1_{v})\, u^{n}-c\, n\,(phi1_{v})\, u^{n}+c\,(eta2_{v})\, k\, u^{k+1}+2\,(eta1_{u})\, u^{2}),\ u^{n+1}\ (c$
$(eta2_{vv})\, n^{2}\, u^{n+1}-c\,(eta2_{vv})\, n\, u^{n+1}+c\,(eta2_{vx})\, n^{2}\, u^{n}-c\,(eta2_{vx})\, n\, u^{n}$
$-(eta2_{uu})\, u^{2}-(eta2_{u})\, n\, u+(eta2_{u})\, u),\ a\,(eta2_{v})\, u^{s+1}+a\,(eta2_{x})\, u^{s}+b$
$(eta2_{v})\, u^{2}-(eta1_{v})\, u^{2}+(phi2_{v})\, u+b\,(eta2_{x})\, u-(eta1_{x})\, u+phi2_{x}-phi1$
$,\ u^{n+2}\ (a\,(eta2_{v})\, u^{s+1}+c\, n\,(phi1_{vv})\, u^{n+1}+c\, n\,(phi1_{vx})\, u^{n}+b\,(eta2_{v})$
$u^{2}+(phi2_{v})\, u-(phi1_{u})\, u+(eta1_{x})\, u-n\, phi1+phi1),\ -c\,(eta1_{vv})\, n^{2}$
$u^{n+2}+c\,(eta1_{vv})\, n\, u^{n+2}+c\, n^{2}\,(phi1_{uv})\, u^{n+1}-c\, n\,(phi1_{uv})\, u^{n+1}-c$
$(eta1_{vx})\, n^{2}\, u^{n+1}+c\,(eta1_{vx})\, n\, u^{n+1}+c\, n^{3}\,(phi1_{v})\, u^{n}-2\, c\, n^{2}\,(phi1_{v})\, u^{n}$
$+c\, n\,(phi1_{v})\, u^{n}-c\,(eta2_{uv})\, k\, u^{k+2}+c\,(eta2_{v})\, k\, n\, u^{k+1}-c\,(eta2_{v})\, k$
$u^{k+1}+(eta1_{uu})\, u^{3}+2\,(eta1_{u})\, n\, u^{2}-2\,(eta1_{u})\, u^{2},\ -a\,(eta1_{uv})\, n\, u^{s+n+2}-$
$a\,(eta1_{v})\, n^{2}\, u^{s+n+1}+a\,(eta1_{v})\, n\, u^{s+n+1}-c\,(eta1_{vv})\, k\, n\, u^{n+k+2}+c\, k\, n$
$(phi1_{uv})\, u^{n+k+1}-c\,(eta1_{vx})\, k\, n\, u^{n+k+1}+c\, k\, n^{2}\,(phi1_{v})\, u^{n+k}-c\, k\, n$
$(phi1_{v})\, u^{n+k}-b\,(eta1_{uv})\, n\, u^{n+3}-b\,(eta1_{v})\, n^{2}\, u^{n+2}+b\,(eta1_{v})\, n\, u^{n+2}+$
$(eta1_{tu})\, n\, u^{n+2}+(eta1_{t})\, n^{2}\, u^{n+1}-(eta1_{t})\, n\, u^{n+1}+c\,(eta2_{v})\, k^{2}\, u^{2\,k+1}+$

$(eta1_{u})\, k\, u^{k+2},\ a\, c\,(eta2_{vv})\, n\, u^{s+n+2}+a\, c\,(eta2_{vx})\, n\, u^{s+n+1}+a\,(eta2_{u})$
$u^{s+2}+b\, c\,(eta2_{vv})\, n\, u^{n+3}+c\, n\,(phi1_{uv})\, u^{n+2}+b\, c\,(eta2_{vx})\, n\, u^{n+2}-c$
$(eta2_{tv})\, n\, u^{n+2}+c\, n^{2}\,(phi1_{v})\, u^{n+1}+c\, n\,(phi1_{ux})\, u^{n+1}-c\,(eta2_{tx})\, n$
$u^{n+1}+c\, n^{2}\,(phi1_{x})\, u^{n}-c\, n\,(phi1_{x})\, u^{n}-c\,(eta2_{v})\, k\, u^{k+2}-c\,(eta2_{x})\, k$
$u^{k+1}+b\,(eta2_{u})\, u^{3}-(eta1_{u})\, u^{3}+(phi2_{u})\, u^{2},\ -a^{2}\,(eta2_{v})\, u^{2\,s+1}-a\, c\, n$
$(phi1_{vv})\, u^{s+n+1}-a\, c\, n\,(phi1_{vx})\, u^{s+n}-2\, a\, b\,(eta2_{v})\, u^{s+2}+a\,(eta1_{v})$
$u^{s+2}-a\,(phi2_{v})\, u^{s+1}+a\,(eta2_{t})\, u^{s+1}+a\, phi1\, s\, u^{s}-b\, c\, n\,(phi1_{vv})\, u^{n+2}$
$-b\, c\, n\,(phi1_{vx})\, u^{n+1}+c\, n\,(phi1_{tv})\, u^{n+1}+c\, n\,(phi1_{tx})\, u^{n}+c\, k\,(phi1_{v})$
$u^{k+1}+c\, k\,(phi1_{x})\, u^{k}-b^{2}\,(eta2_{v})\, u^{3}+b\,(eta1_{v})\, u^{3}-b\,(phi2_{v})\, u^{2}+b$
$(eta2_{t})\, u^{2}-(eta1_{t})\, u^{2}+(phi2_{t})\, u+b\, phi1\, u,\ -a\,(eta1_{vv})\, n\, u^{s+n+2}+a\, n$
$(phi1_{uv})\, u^{s+n+1}-a\,(eta1_{vx})\, n\, u^{s+n+1}$

$+a\, n^{2}\,(phi1_{v})\, u^{s+n}-a\, n\,(phi1_{v})\, u^{s+n}+2\, a\,(eta2_{v})\, k\, u^{s+k+1}+c\, k\, n$
$(phi1_{vv})\, u^{n+k+1}+c\, k\, n\,(phi1_{vx})\, u^{n+k}-b\,(eta1_{vv})\, n\, u^{n+3}+b\, n\,(phi1_{uv})$
$u^{n+2}-b\,(eta1_{vx})\, n\, u^{n+2}+(eta1_{tv})\, n\, u^{n+2}+b\, n^{2}\,(phi1_{v})\, u^{n+1}-b\, n$
$(phi1_{v})\, u^{n+1}-n\,(phi1_{tu})\, u^{n+1}+(eta1_{tx})\, n\, u^{n+1}-n^{2}\,(phi1_{t})\, u^{n}+n$
$(phi1_{t})\, u^{n}+2\, b\,(eta2_{v})\, k\, u^{k+2}+k\,(phi2_{v})\, u^{k+1}-k\,(phi1_{u})\, u^{k+1}-(eta2_{t})$
$k\, u^{k+1}+(eta1_{x})\, k\, u^{k+1}-k^{2}\, phi1\, u^{k}+k\, phi1\, u^{k},\ u^{n+1}\ (-a\,(eta2_{uv})\, u^{s+2}$
$+a\,(eta2_{v})\, n\, u^{s+1}-a\,(eta2_{v})\, u^{s+1}+c\, n^{2}\,(phi1_{vv})\, u^{n+1}-c\, n\,(phi1_{vv})$
$u^{n+1}+c\, n^{2}\,(phi1_{vx})\, u^{n}-c\, n\,(phi1_{vx})\, u^{n}-c\,(eta2_{vv})\, k\, u^{k+2}-c\,(eta2_{vx})$
$k\, u^{k+1}-b\,(eta2_{uv})\, u^{3}+(eta1_{uv})\, u^{3}-(phi1_{uu})\, u^{2}+b\,(eta2_{v})\, n\, u^{2}-b$
$(eta2_{v})\, u^{2}+(eta2_{tu})\, u^{2}+(eta1_{v})\, u^{2}+(eta1_{ux})\, u^{2}+n\,(phi2_{v})\, u-(phi2_{v})\, u$
$-2\, n\,(phi1_{u})\, u+2\,(phi1_{u})\, u+(eta1_{x})\, n\, u-(eta1_{x})\, u-n^{2}\, phi1+3\, n\, phi1-$
$2\, phi1)\ (-a\,(eta2_{uv})\, u^{s+2}+a\,(eta2_{v})\, n\, u^{s+1}-a\,(eta2_{v})\, u^{s+1}+c\, n^{2}$
$(phi1_{vv})\, u^{n+1}-c\, n\,(phi1_{vv})\, u^{n+1}+c\, n^{2}\,(phi1_{vx})\, u^{n}-c\, n\,(phi1_{vx})\, u^{n}-c$
$(eta2_{vv})\, k\, u^{k+2}-c\,(eta2_{vx})\, k\, u^{k+1}-b\,(eta2_{uv})\, u^{3}+(eta1_{uv})\, u^{3}-$
$(phi1_{uu})\, u^{2}+b\,(eta2_{v})\, n\, u^{2}-b\,(eta2_{v})\, u^{2}+(eta2_{tu})\, u^{2}+(eta1_{v})\, u^{2}+$
$(eta1_{ux})\, u^{2}+n\,(phi2_{v})\, u-(phi2_{v})\, u-2\, n\,(phi1_{u})\, u+2\,(phi1_{u})\, u+(eta1_{x})$
$n\, u-(eta1_{x})\, u-n^{2}\, phi1+3\, n\, phi1-2\, phi1)]$

Fig. 2. Determining equations obtained by applying the classical potential symmetries

$[phi2_{u}-(eta1_{u})\, u,\ eta1\, u^{n+1}$
$(-eta1\,(eta1_{uu})\, u+(eta1_{u})^{2}\, u+eta1\,(eta1_{u})\, n-eta1\,(eta1_{u})),\ -(eta1_{v})$
$u^{2}+(phi2_{v})\, u-(eta1_{x})\, u+phi2_{x}-phi1,\ 2\, eta1^{2}\,(eta1_{uv})\, n\, u^{n+3}+$
$(eta1_{u})(eta1_{v})\, n\, phi2\, u^{n+2}-eta1\,(eta1_{uv})\, n\, phi2\, u^{n+2}-eta1^{2}\, n$
$(phi1_{uu})\, u^{n+2}+eta1\,(eta1_{u})\, n\,(phi1_{u})\, u^{n+2}-eta1\,(eta1_{uu})\, n\, phi1\, u^{n+2}$
$+(eta1_{u})^{2}\, n\, phi1\, u^{n+2}-eta1\,(eta1_{u})(eta1_{x})\, n\, u^{n+2}+eta1^{2}\,(eta1_{v})\, n$
$u^{n+2}+eta1^{2}\,(eta1_{ux})\, n\, u^{n+2}+(eta1_{t})(eta1_{u})\, n\, u^{n+2}-eta1\,(eta1_{tu})\, n$
$u^{n+2}-eta1^{2}\, n^{2}\,(phi1_{u})\, u^{n+1}+eta1^{2}\, n\,(phi1_{u})\, u^{n+1}+2\, eta1\,(eta1_{u})\, n^{2}$
$phi1\, u^{n+1}-2\, eta1\,(eta1_{u})\, n\, phi1\, u^{n+1}+eta1^{2}\, n^{2}\, phi1\, u^{n}-eta1^{2}\, n$
$phi1\, u^{n}+2\, eta1\,(eta1_{u})\, k\, u^{k+2},\ -3\, a\, eta1\,(eta1_{u})\, u^{s+3}-c\, eta1^{2}$
$(eta1_{vv})\, n\, u^{n+4}+2\, c\, eta1\,(eta1_{u})\, n\,(phi1_{v})\, u^{n+3}+2\, c\, eta1^{2}\, n\,(phi1_{uv})$
$u^{n+3}-$

$c\, eta1\,(eta1_{v})\, n\,(phi1_{u})\, u^{n+3}+c\, eta1\,(eta1_{v})(eta1_{x})\, n\, u^{n+3}-2\, c$
$eta1^{2}\,(eta1_{vx})\, n\, u^{n+3}-c\, eta1\, n\,(phi1_{uv})\, phi2\, u^{n+2}+c\,(eta1_{v})\, n$
$(phi1_{u})\, phi2\, u^{n+2}-c\,(eta1_{v})(eta1_{x})\, n\, phi2\, u^{n+2}+c\, eta1\,(eta1_{vx})\, n$
$phi2\, u^{n+2}-c\, eta1\,(eta1_{u})\, n\,(phi1_{x})\, u^{n+2}+2\, c\, eta1^{2}\, n^{2}\,(phi1_{v})\, u^{n+2}-c$
$eta1^{2}\, n\,(phi1_{v})\, u^{n+2}-c\, eta1\, n\, phi1\,(phi1_{uu})\, u^{n+2}+c\, eta1^{2}\, n\,(phi1_{ux})$
$u^{n+2}+c\,(eta1_{u})\, n\, phi1\,(phi1_{u})\, u^{n+2}+c\,(eta1_{t})\, n\,(phi1_{u})\, u^{n+2}-c\, eta1$
$n\,(phi1_{tu})\, u^{n+2}-c\, eta1\,(eta1_{v})\, n^{2}\, phi1\, u^{n+2}-c\,(eta1_{u})(eta1_{x})\, n\, phi1$
$u^{n+2}+c\, eta1\,(eta1_{v})\, n\, phi1\, u^{n+2}+c\, eta1\,(eta1_{ux})\, n\, phi1\, u^{n+2}-$

$c(\mathit{eta1}_{t})(\mathit{eta1}_{x})\,n\,u^{n+2}+c\,\mathit{eta1}(\mathit{eta1}_{tx})\,n\,u^{n+2}-c\,\mathit{eta1}\,n^{2}(\mathit{phi1}_{v})$
$\mathit{phi2}\,u^{n+1}+c\,\mathit{eta1}\,n(\mathit{phi1}_{v})\mathit{phi2}\,u^{n+1}+c(\mathit{eta1}_{v})n^{2}\,\mathit{phi1}\,\mathit{phi2}\,u^{n+1}-$
$c(\mathit{eta1}_{v})\,n\,\mathit{phi1}\,\mathit{phi2}\,u^{n+1}+c\,\mathit{eta1}^{2}\,n^{2}(\mathit{phi1}_{x})\,u^{n+1}-c\,\mathit{eta1}^{2}\,n$
$(\mathit{phi1}_{x})\,u^{n+1}-c\,\mathit{eta1}\,n^{2}\,\mathit{phi1}(\mathit{phi1}_{u})\,u^{n+1}+c\,\mathit{eta1}\,n\,\mathit{phi1}(\mathit{phi1}_{u})\,u^{n+1}$
$-c\,\mathit{eta1}\,n^{2}(\mathit{phi1}_{t})\,u^{n+1}+c\,\mathit{eta1}\,n(\mathit{phi1}_{t})\,u^{n+1}+c(\mathit{eta1}_{u})n^{2}\,\mathit{phi1}^{2}$
$u^{n+1}-c(\mathit{eta1}_{u})\,n\,\mathit{phi1}^{2}\,u^{n+1}+c(\mathit{eta1}_{t})n^{2}\,\mathit{phi1}\,u^{n+1}-c(\mathit{eta1}_{t})\,n\,\mathit{phi1}$
$u^{n+1}+c\,\mathit{eta1}\,n^{2}\,\mathit{phi1}^{2}\,u^{n}-c\,\mathit{eta1}\,n\,\mathit{phi1}^{2}\,u^{n}-c\,\mathit{eta1}(\mathit{eta1}_{v})\,k\,u^{k+3}+c$
$(\mathit{eta1}_{v})\,k\,\mathit{phi2}\,u^{k+2}+c(\mathit{eta1}_{u})\,k\,\mathit{phi1}\,u^{k+2}+c(\mathit{eta1}_{t})\,k\,u^{k+2}+c\,\mathit{eta1}\,k$
$n\,\mathit{phi1}\,u^{k+1}-c\,\mathit{eta1}\,k^{2}\,\mathit{phi1}\,u^{k+1}+2\,\mathit{eta1}^{2}(\mathit{eta1}_{u})\,u^{4}-3\,b\,\mathit{eta1}$
$(\mathit{eta1}_{u})\,u^{4}+\mathit{eta1}^{2}(\mathit{phi2}_{u})\,u^{3}-3\,\mathit{eta1}(\mathit{eta1}_{u})\mathit{phi2}\,u^{3},2\,a\,\mathit{eta1}(\mathit{eta1}_{v})$
$u^{s+3}-a(\mathit{eta1}_{v})\mathit{phi2}\,u^{s+2}-a\,\mathit{eta1}(\mathit{phi1}_{u})\,u^{s+2}-a(\mathit{eta1}_{u})\mathit{phi1}\,u^{s+2}+$
$a\,\mathit{eta1}(\mathit{eta1}_{x})\,u^{s+2}-a(\mathit{eta1}_{t})\,u^{s+2}+a\,\mathit{eta1}\,\mathit{phi1}\,s\,u^{s+1}-a\,\mathit{eta1}\,n$
$\mathit{phi1}\,u^{s+1}+a\,\mathit{eta1}\,\mathit{phi1}\,u^{s+1}-c\,\mathit{eta1}^{2}\,n(\mathit{phi1}_{vv})\,u^{n+3}+2\,c\,\mathit{eta1}$
$(\mathit{eta1}_{v})\,n$

$(\mathit{phi1}_{x})\,u^{n+2}-2\,c\,\mathit{eta1}^{2}\,n(\mathit{phi1}_{vx})\,u^{n+2}+c\,\mathit{eta1}\,n(\mathit{phi1}_{u})(\mathit{phi1}_{v})\,u^{n+2}$
$-c\,\mathit{eta1}(\mathit{eta1}_{x})\,n(\mathit{phi1}_{v})\,u^{n+2}-c(\mathit{eta1}_{v})\,n(\mathit{phi1}_{x})\mathit{phi2}\,u^{n+1}+c\,\mathit{eta1}$
$n(\mathit{phi1}_{vx})\mathit{phi2}\,u^{n+1}-c(\mathit{eta1}_{u})\,n\,\mathit{phi1}(\mathit{phi1}_{x})\,u^{n+1}-c(\mathit{eta1}_{t})\,n$
$(\mathit{phi1}_{x})\,u^{n+1}+c\,\mathit{eta1}\,n^{2}\,\mathit{phi1}(\mathit{phi1}_{v})\,u^{n+1}-c\,\mathit{eta1}\,n\,\mathit{phi1}(\mathit{phi1}_{v})\,u^{n+1}$
$+c\,\mathit{eta1}\,n\,\mathit{phi1}(\mathit{phi1}_{ux})\,u^{n+1}+c\,\mathit{eta1}\,n(\mathit{phi1}_{tx})\,u^{n+1}+c\,\mathit{eta1}\,n^{2}\,\mathit{phi1}$
$(\mathit{phi1}_{x})\,u^{n}-c\,\mathit{eta1}\,n\,\mathit{phi1}(\mathit{phi1}_{x})\,u^{n}+c\,\mathit{eta1}\,k(\mathit{phi1}_{v})\,u^{k+2}+c\,\mathit{eta1}\,k$
$(\mathit{phi1}_{x})\,u^{k+1}-\mathit{eta1}^{2}(\mathit{eta1}_{v})\,u^{4}+2\,b\,\mathit{eta1}(\mathit{eta1}_{v})\,u^{4}-\mathit{eta1}^{2}(\mathit{phi2}_{v})\,u^{3}$
$+2\,\mathit{eta1}(\mathit{eta1}_{v})\mathit{phi2}\,u^{3}-b(\mathit{eta1}_{v})\mathit{phi2}\,u^{3}+\mathit{eta1}^{2}(\mathit{phi1}_{u})\,u^{3}-b\,\mathit{eta1}$
$(\mathit{phi1}_{u})\,u^{3}-b(\mathit{eta1}_{u})\mathit{phi1}\,u^{3}-\mathit{eta1}^{2}(\mathit{eta1}_{x})\,u^{3}+b\,\mathit{eta1}(\mathit{eta1}_{x})\,u^{3}-b$
$(\mathit{eta1}_{t})\,u^{3}+\mathit{eta1}\,\mathit{phi2}(\mathit{phi2}_{v})\,u^{2}+\mathit{eta1}\,\mathit{phi1}(\mathit{phi2}_{u})\,u^{2}+\mathit{eta1}(\mathit{phi2}_{t})$
$u^{2}-(\mathit{eta1}_{v})\mathit{phi2}^{2}\,u^{2}-\mathit{eta1}(\mathit{phi1}_{u})\mathit{phi2}\,u^{2}-(\mathit{eta1}_{u})\mathit{phi1}\,\mathit{phi2}\,u^{2}+$
$\mathit{eta1}(\mathit{eta1}_{x})\mathit{phi2}\,u^{2}-(\mathit{eta1}_{t})\mathit{phi2}\,u^{2}+\mathit{eta1}^{2}\,n\,\mathit{phi1}\,u^{2}-b\,\mathit{eta1}\,n\,\mathit{phi1}$
$u^{2}-\mathit{eta1}^{2}\,\mathit{phi1}\,u^{2}+2\,b\,\mathit{eta1}\,\mathit{phi1}\,u^{2}-\mathit{eta1}\,n\,\mathit{phi1}\,\mathit{phi2}\,u+\mathit{eta1}\,\mathit{phi1}$
$\mathit{phi2}\,u]$

Fig. 3. Determining equations obtained by applying the nonclassical potential symmetries